Hamiltonian orbit structure of the set of paraxial optical systems

Sameen Ahmed Khan and Kurt Bernardo Wolf

Centro de Ciencias Físicas, Universidad Nacional Autónoma de México, Apartado Postal 48-3, Cuernavaca, Morelos 62251, México

Received March 19, 2002; revised manuscript received June 19, 2002; accepted June 25, 2002

In the paraxial regime of three-dimensional optics, two evolution Hamiltonians are equivalent when one can be transformed to the other modulo scale by similarity through an optical system. To determine the equivalence sets of paraxial optical Hamiltonians one requires the orbit analysis of the algebra $\mathsf{sp}(4, \Re)$ of 4×4 real Hamiltonian matrices. Our strategy uses instead the isomorphic algebra $\mathsf{so}(3, 2)$ of 5×5 matrices with metric $(+1, +1, +1, -1, -1)$ to find four orbit regions (strata), six isolated orbits at their boundaries, and six degenerate orbits at their common point. We thus resolve the degeneracies of the eigenvalue classification.

OCIS codes: 080.2727, 080.2730, 000.3860.

1. INTRODUCTION: TWO-DIMENSIONAL SYSTEMS

The solution to the problem that we pose, i.e., the identification of the distinct equivalence classes of paraxial optical Hamiltonians, is well known in the case of 2-dim systems. There, $(p, q) \in \Re^2$ are the momentum and position coordinates of optical phase space,[1–5] and any quadratic Hamiltonian can be associated ($\leftrightarrow$) with a 2×2 traceless matrix,

$$H(\mathbf{m}; p, q) = \tfrac{1}{2} c p^2 - a q p - \tfrac{1}{2} b q^2 \leftrightarrow \mathbf{m} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}. \tag{1}$$

The generic Hamiltonian [Eq. (1)] is then equivalent to one of three orbit representative Hamiltonians $\mathbf{m}_\circ$, modulo similarity by an optical system $\mathbf{M}$ (represented by a 2×2 matrix of unit determinant) and a scale $\alpha \neq 0$, namely,

$$\mathbf{m} = \alpha \mathbf{M} \mathbf{m}_\circ \mathbf{M}^{-1},$$

$$\mathbf{m}_\circ = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}. \tag{2}$$

Each of the three orbit representatives in Eq. (2) is characterized by its eigenvalues $\{\lambda\}$, or the sign of the invariant determinant $\Delta = \det \mathbf{m} = -(a^2 + bc) = -\lambda^2$, as follows:

$$\text{H: Harmonic,} \qquad H\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \tfrac{1}{2} p^2 + \tfrac{1}{2} q^2,$$

$$\text{Oscillating trajectories,} \qquad \Delta > 0, \ \lambda = \pm i. \tag{3}$$

$$\text{R: Repulsive,} \qquad H\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \tfrac{1}{2} p^2 - \tfrac{1}{2} q^2,$$

$$\text{Hyperbolic trajectories,} \qquad \Delta < 0, \ \lambda = \pm 1. \tag{4}$$

$$\text{F: Free homogeneous,} \qquad H\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \tfrac{1}{2} p^2,$$

$$\text{Straight trajectories,} \qquad \Delta = 0, \ \lambda = 0 \ \text{(double)}. \tag{5}$$

In this paper we obtain the equivalence classes of paraxial optical Hamiltonians in three dimensions (generally astigmatic), and list their representatives. These Hamiltonians are quadratic in the—now four—coordinates of phase space, (p_x, p_y, q_x, q_y), and are represented by 4×4 (infinitesimal symplectic or) Hamiltonian matrices, whose properties will be collected in Section 2, where we also recall the concept of orbit within an algebra. The immediacy (and completeness) of the classification of two-dimensional orbits in Eqs. (3)–(5) is due to the accidental equality (isomorphism) $\mathsf{sp}(2, \Re) = \mathsf{sl}(2, \Re) = \mathsf{so}(2, 1)$ between two-dimensional symplectic, two-dimensional unimodular (unit determinant), and $(2 + 1)$-pseudo-orthogonal Lie algebras of matrices:

$$\begin{bmatrix} -\tfrac{1}{2} r_{1,3} & \tfrac{1}{2}(r_{2,3} - r_{1,2}) \\ \tfrac{1}{2}(r_{2,3} + r_{1,2}) & \tfrac{1}{2} r_{1,3} \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & r_{1,2} & r_{1,3} \\ -r_{1,2} & 0 & r_{2,3} \\ r_{1,3} & r_{2,3} & 0 \end{bmatrix}. \tag{6}$$

As we recall in Section 3, for 3-dimensional systems there is a second (and last) fortunate accidental equality: $\mathsf{sp}(4, \Re) = \mathsf{so}(3, 2)$, between the Lie algebras of 4×4 Hamiltonian matrices and 5×5 infinitesimal pseudo-orthogonal matrices with metric $(+1, +1, +1, -1, -1)$.[6] It turns out that the classification of three-dimensional Hamiltonian orbits is much facilitated by the use of these 5×5 matrices. In this way we separate rotations of phase space [U(2)-Fourier transforms[4,5]] Lorentzian boosts (repulsive waveguides or astigmatic imagers), and Euclidean Hamiltonians (free but astigmatic propagation, or lenses), as in Eqs. (3)–(5). In three dimensions, moreover, there are also rotations about the optical axis, generated by an angular momentum Hamiltonian; in mechanics, this is imparted to charged particles in an accelerator by a coaxial magnetic field.[7]

Under the name of anti-de Sitter algebra, $\mathsf{so}(3, 2)$ has also served as a model for field theories with a fundamental length.[8–10] The subject of equivalence classes in the symplectic algebra includes early investigations in celestial mechanics,[11] and three papers that address the more general problem of finding the maximal abelian subalgebras of the symplectic and pseudo-orthogonal Lie algebras.[12–14] In particular, based on Ref. 11, the paper by Moshinsky and Winternitz[13] enumerates essential equivalence classes of quantum-mechanical quadratic Hamiltonians in two dimensions [Ref. 13, Eqs. (38) and (44), number 11; cf. Ref. 12, Table IV, entries for $a_{1,1}$ to $\bar{a}_{1,12}$ imply 12, and Ref. 14, Subsection 5.2 with 10.] The analysis of $\mathsf{so}(3, 2)$ orbits has also been performed by Simon *et al.* in Ref. 15, Appendix C, who divide the orbits according to the dimension of their invariant subspaces in $\mathfrak{R}^5$, with the purpose of classifying anisotropic Gaussian Schell-model beams through their second-order moments.[16]

The strategy that we follow here is to reduce the ten-dimensional $\mathsf{so}(3, 2)$ orbit analysis to that of its subalgebras: $\mathsf{so}(2, 2)$ for $x-y$ separable Hamiltonians, $\mathsf{so}(3, 1)$ for Hamiltonians that we call Lorentzian, and the pseudo-Euclidean Hamiltonians in an $\mathsf{iso}(2, 1)$ subalgebra. In Section 4, by similarity with a $\mathsf{U}(2)$-Fourier transformer (of four parameters), we bring the generic Hamiltonian to a form where an isotropic imager will reduce it to lie entirely within each of the three six-dimensional subspaces. Then in Section 5 we indicate the tree of transformations to the inequivalent representative Hamiltonians. We deem this derivation to be more transparent than those in previous studies.

While in two-dimensional optics the eigenvalues provide the full orbit classification of Eqs. (3)–(5), in three-dimensions, since the conjugating matrix $\mathbf{M}$ in Eq. (2) is not general but only symplectic, some classes of Hamiltonians that are equivalent under the former will degenerate into separate subclasses. In Section 6 we recall the eigenvalue structure of Hamiltonian matrices, which was pivotal for previous studies,[11] and we resolve its degeneracies by knowing the orbit structure; fortunately, these turn out to be minor. We recapitulate results and offer some ensuing comments in Section 7.

2. THREE-DIMENSIONAL SYSTEMS AND HAMILTONIANS

Optical systems transform phase space, conserving its Hamiltonian structure. The paraxial model consists of linear transformations, so it follows that the action of a system on phase space ($\mathbf{p}$, $\mathbf{q}$) (ray momentum and position referred to a standard screen[1–3]) is realized by a matrix on the vector $\binom{\mathbf{p}}{\mathbf{q}}$. Three-dimensional systems are represented by 4×4 real sympletic matrices:

$$\mathbf{M}\,\boldsymbol{\Omega}\,\mathbf{M}^{\mathsf{T}} = \boldsymbol{\Omega}, \qquad \boldsymbol{\Omega} = \begin{bmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \tag{7}$$

and form the group denoted $\mathsf{Sp}(4, \mathfrak{R})$, which has ten parameters.[6,17]

Symplectic matrices near the unit, $\mathbf{M} = \mathbf{1} + \epsilon\mathbf{m}$, $\epsilon^2 \approx 0$, define $\mathbf{m}$ as infinitesimal symplectic, or Hamiltonian, matrices,

$$\mathbf{m}\,\boldsymbol{\Omega} = -\boldsymbol{\Omega}\,\mathbf{m}^{\mathsf{T}}, \qquad \mathbf{m} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & -\mathbf{a} \end{bmatrix}, \qquad \begin{matrix} \mathbf{b} = \mathbf{b}^{\mathsf{T}}, \\ \mathbf{c} = \mathbf{c}^{\mathsf{T}}, \end{matrix} \tag{8}$$

form a linear vector space, and realize the Lie algebra $\mathsf{sp}(4, \mathfrak{R})$. For any symplectic $\mathbf{M}$, when $\mathbf{m}$ is Hamiltonian so is $\mathbf{m}' = \mathbf{M}\,\mathbf{m}\,\mathbf{M}^{-1}$ (the inverse $\mathbf{M}^{-1} = \boldsymbol{\Omega}\,\mathbf{M}^{\mathsf{T}}\,\boldsymbol{\Omega}$ is guaranteed to exist). Thus, given an element $\mathbf{m}_{\circ} \in \mathsf{sp}(4, \mathfrak{R})$, we define its

$$\mathrm{orbit}(\mathbf{m}_{\circ}) = \{\alpha \mathbf{M}\,\mathbf{m}_{\circ}\,\mathbf{M}^{-1} | \mathbf{M} \in \mathsf{Sp}(4, \mathfrak{R}), \quad 0 \neq \alpha \in \mathfrak{R}\}. \tag{9}$$

Orbits are equivalence classes, so they are disjoint, and their union is the original $\mathsf{sp}(4, \mathfrak{R})$.

The correspondence ($\leftrightarrow$) between quadratic $\mathsf{sp}(4, \mathfrak{R})$ Hamiltonian functions on phase space and Hamiltonian matrices [cf. Eq. (1) for $\mathsf{sp}(2, \mathfrak{R})$] is

$$\begin{aligned} H(\mathbf{m}; \mathbf{p}, \mathbf{q}) &= \tfrac{1}{2}c_x p_x^2 + \tfrac{1}{2}c_y p_y^2 + c_{\bowtie} p_x p_y \\ &\quad - \tfrac{1}{2}b_x q_x^2 - \tfrac{1}{2}b_y q_y^2 - b_{\bowtie} q_x q_y \\ &\quad - a_x q_x p_x - a_y q_y p_y \\ &\quad - a_{xy} q_x p_y - a_{yx} q_y p_x \\ \leftrightarrow \mathbf{m} &= \left[\begin{array}{cc|cc} a_x & a_{xy} & b_x & b_{\bowtie} \\ a_{yx} & a_y & b_{\bowtie} & b_y \\ \hline c_x & c_{\bowtie} & -a_x & -a_{yx} \\ c_{\bowtie} & c_y & -a_{xy} & -a_y \end{array}\right]. \end{aligned} \tag{10}$$

The set of Hamiltonian matrices $\mathbf{m}$ is closed under summation, multiplication by constants, and commutation (but not under product), they form the symplectic Lie algebra denoted $\mathsf{sp}(4, \mathfrak{R})$ and are vectors in a 10-dim space. Under Poisson brackets, the Hamiltonian functions close following the commutator of their matrices,[2,3]

$$H([\mathbf{m}_1, \mathbf{m}_2]; \mathbf{p}, \mathbf{q}) = \{H(\mathbf{m}_1; \mathbf{p}, \mathbf{q}), H(\mathbf{m}_2; \mathbf{p}, \mathbf{q})\}.$$

The symbol $\leftrightarrow$ is therefore an isomorphism between two different realizations of the same Lie algebra. The matrices $\mathbf{M}(z) = \exp(z\mathbf{m}) \in \mathsf{Sp}(4, \mathfrak{R})$ generated by Hamiltonian matrices $\mathbf{m} \in \mathsf{sp}(4, \mathfrak{R})$, form one-parameter subgroups; these may be seen as waveguides extending along the optical z axis. The equivalence relation between the orbits in the algebra extends naturally to an equivalence between the generated subgroups.

3. HAMILTONIANS AND so(3, 2) MATRICES

The orbit analysis of three-dimensional astigmatic Hamiltonians will be based on the homomorphism between the symplectic and pseudo-orthogonal groups $\mathrm{Sp}(4, \Re) \overset{2:1}{=} \mathrm{SO}(3, 2)$, and the corresponding isomorphism of algebras $\mathrm{sp}(4, \Re) = \mathrm{so}(3, 2)$.[6] Both have ten parameters, but the structure of the (pseudo-) orthogonal subgroups and algebras is simpler than that of the symplectic ones (the very root "-plectic" means "pleated, interwoven.") Pseudo-orthogonal matrices $\mathbf{R} \in \mathrm{SO}(3, 2)$ are those that satisfy $\mathbf{R}\,\mathbf{D}\,\mathbf{R}^{\top} = \mathbf{D}$, with the diagonal metric matrix $\mathbf{D} = \mathrm{diag}(+1, +1, +1, -1, -1)$ [cf. Eq. (7)]. Again, a pseudo-orthogonal matrix near the unit, $\mathbf{R} = \mathbf{1} + \epsilon \mathbf{r}$, $\epsilon^2 \approx 0$, defines $\mathbf{r}$ as an infinitesimal pseudo-orthogonal matrix (it is pseudo-skew-symmetric) which satisfies $\mathbf{r}\,\mathbf{D} = -\mathbf{D}\,\mathbf{r}^{\top}$ [cf. Eq. (8)]. These close under commutation into the Lie algebra denoted so(3,2).

As in Eq. (6), it is sufficient to present the following correspondence (also indicated $\leftrightarrow$) between the 4×4 and 5×5 matrices,

$$\frac{1}{2}\left[\begin{array}{cc|cc} r_{1,\bar{5}} + r_{3,\bar{4}} & r_{1,2} + r_{2,\bar{5}} & -r_{1,3} - r_{1,\bar{4}} - r_{3,\bar{5}} + r_{\bar{4},\bar{5}} & r_{2,3} + r_{2,\bar{4}} \\ -r_{1,2} + r_{2,\bar{5}} & -r_{1,\bar{5}} + r_{3,\bar{4}} & r_{2,3} + r_{2,\bar{4}} & r_{1,3} + r_{1\bar{4}} - r_{3,\bar{5}} + r_{\bar{4},\bar{5}} \\ \hline r_{1,3} - r_{1,\bar{4}} - r_{3,\bar{5}} - r_{\bar{4},\bar{5}} & -r_{2,3} + r_{2,\bar{4}} & -r_{1,\bar{5}} - r_{3,\bar{4}} & r_{1,2} - r_{2,\bar{5}} \\ -r_{2,3} + r_{2,\bar{4}} & -r_{1,3} + r_{1,\bar{4}} - r_{3,\bar{5}} - r_{\bar{4},\bar{5}} & -r_{1,2} - r_{2,\bar{5}} & r_{1,\bar{5}} - r_{3,\bar{4}} \end{array}\right] \tag{11}$$

$$\leftrightarrow \left[\begin{array}{ccc|cc} 0 & r_{1,2} & r_{1,3} & r_{1,\bar{4}} & r_{1,\bar{5}} \\ -r_{1,2} & 0 & r_{2,3} & r_{2,\bar{4}} & r_{2,\bar{5}} \\ -r_{1,3} & -r_{2,3} & 0 & r_{3,\bar{4}} & r_{3,\bar{5}} \\ \hline r_{1,\bar{4}} & r_{2,\bar{4}} & r_{3,\bar{4}} & 0 & r_{\bar{4},\bar{5}} \\ r_{1,\bar{5}} & r_{2,\bar{5}} & r_{3,\bar{5}} & -r_{\bar{4},\bar{5}} & 0 \end{array}\right], \tag{12}$$

where we stress by bars the coordinates with negative metric. The above correspondence can be verified to be an isomorphism under commutation.

Collecting terms in the Hamiltonian according to the resulting so(3, 2) coefficients in relation (12), the generic quadratic Hamiltonian (10) can be written

$$H(\mathbf{r}) = \sum_{1 \leqslant m < n \leqslant 3} r_{m,n} j_{m,n} + \sum_{m=1,2,3} \sum_{\bar{n}=\bar{4},\bar{5}} r_{m,\bar{n}} j_{m,\bar{n}} + r_{\bar{4},\bar{5}} j_{\bar{4},\bar{5}}. \tag{13}$$

This introduces the basis of ten quadratic Hamiltonian functions, $j_{\alpha,\beta}$, $\alpha, \beta \in \{1, 2, 3, \bar{4}, \bar{5}\}$. From the symmetry of the coefficients $r_{\alpha,\beta}$ and $r_{\beta,\alpha}$ in relation (12), the functions with transposed indices are $j_{n,m} = -j_{m,n}$, $j_{\bar{n},\bar{m}} = -j_{\bar{m},\bar{n}}$, and $j_{\bar{n},m} = +j_{m,\bar{n}}$. Among these we distinguish harmonic and repulsive waveguide Hamiltonians as well as imager generators:

Harmonic generators:

$$j_{1,2} = \tfrac{1}{2}(q_x p_y - q_y p_x), \quad \tfrac{1}{2}\text{-angular momentum}; \tag{14}$$

$$j_{1,3} = \tfrac{1}{4}(-p_x^2 + p_y^2 - q_x^2 + q_y^2), \quad \text{counter-harmonic } \tfrac{1}{2}(-H_x + H_y), \tag{15}$$

$$j_{2,3} = \tfrac{1}{2}(p_x p_y + q_x q_y), \quad \text{cross-harmonic}, \tag{16}$$

$$j_{\bar{4},\bar{5}} = \tfrac{1}{4}(p_x^2 + p_y^2 + q_x^2 + q_y^2), \quad \text{isotropic harmonic } \tfrac{1}{2}(H_x + H_y). \tag{17}$$

Repulsive generators:

$$j_{1,\bar{4}} = \tfrac{1}{4}(p_x^2 - p_y^2 - q_x^2 + q_y^2), \quad \text{counter-repulsive } \tfrac{1}{2}(R_x - R_y), \tag{18}$$

$$j_{2,\bar{4}} = \tfrac{1}{2}(-p_x p_y + q_x q_y), \quad -\text{cross-repulsive}, \tag{19}$$

$$j_{3,\bar{5}} = \tfrac{1}{4}(|\mathbf{p}|^2 - |\mathbf{q}|^2), \quad \text{isotropic repulsive } \tfrac{1}{2}(R_x + R_y). \tag{20}$$

Imager generators:

$$j_{2,\bar{5}} = \tfrac{1}{2}(p_x q_x - p_y q_y), \quad \text{counter-imager } \tfrac{1}{2}(I_x - I_y), \tag{21}$$

$$j_{2,\bar{5}} = \tfrac{1}{2}(p_x q_y + p_y q_x), \quad \text{cross-imager}, \tag{22}$$

$$j_{3,\bar{4}} = \tfrac{1}{2}\,\mathbf{p} \cdot \mathbf{q}, \quad \text{isotropic imager } \tfrac{1}{2}(I_x + I_y). \tag{23}$$

Harmonic Hamiltonians are the four generators of rotations in the subalgebra $\mathrm{so}(3) \oplus \mathrm{so}(2) \subset \mathrm{so}(3, 2)$, and repulsive and imager generators are the six boosts. Sums and differences of harmonic and repulsive Hamiltonians yield the Hamiltonians of the free systems (or of lens generators); these are the three mutually commuting translation generators within the (pseudo-) Euclidean subalgebra $\mathrm{iso}(2, 1) \subset \mathrm{so}(3, 2)$, to be seen below.

We organize the previous list of Hamiltonians, displaying them as pseudo-orthogonal generators.[18] For systems G that can be harmonic (H), repulsive (R), free (F), or imagers (I), we place $j_{\alpha,\beta}$ ($\alpha < \beta$) in the α–β position of the pattern [cf. the upper-right triangle of the matrix (12)]:

	$2\downarrow$	$3\downarrow$	$\bar{4}\downarrow$	$\bar{5}\downarrow$	
$1\rightarrow$	angular momentum	counter harmonic	counter repulsive	counter imager	counter-G: $G_x - G_y$
	$2\rightarrow$	cross harmonic	cross repulsive	cross imager	cross terms!
		$3\rightarrow$	isotropic imager	isotropic repulsive	i.e., $G_x + G_y$
			$\bar{4}\rightarrow$	isotropic harmonic	central Fourier .

(24)

We shall handle the generic Hamiltonian generators, displaying their 10 parameters [Eq. (13)] instead of the 25 elements of a 5×5 matrix, by associating

$$H(\mathbf{m}; \mathbf{p}, \mathbf{q}) \;\leftrightarrow\; \mathcal{H}(\mathbf{m}(\mathbf{r})) = \begin{array}{|cc|cc|} \hline r_{1,2} & r_{1,3} & r_{1,\bar{4}} & r_{1,\bar{5}} \\ \hline & r_{2,3} & r_{2,\bar{4}} & r_{2,\bar{5}} \\ & & r_{3,\bar{4}} & r_{3,\bar{5}} \\ \hline & & & r_{\bar{4},\bar{5}} \\ \hline \end{array} \,. \tag{25}$$

The left and lower boxes house the 4 (compact) terms of harmonic motion, while the upper-right 3×2 rectangle displays the 6 (noncompact) counter and cross, repulsive and imager terms. The generic Hamiltonian [Eq. (13)] is a linear combination of the generators (17)–(23) with the coefficients in the pattern of relation (25); with SO(3, 2) transformations we shall bring this to a pattern with as many zeros as possible. The cross terms (with index 2) will be the first to be eliminated, because they do not correspond to any recognizable waveguide or magnetic system. Angular momentum is also a cross term, but it will serve to hinge the division of orbits into separable and nonseparable ones.

$$[j_{\alpha,\beta}, j_{\gamma,\delta}] = +g_{\alpha,\delta} j_{\beta,\gamma} + g_{\beta,\gamma} j_{\alpha,\delta} + g_{\gamma,\alpha} j_{\delta,\beta} + g_{\delta,\beta} j_{\gamma,\alpha}, \tag{27}$$

where $g_{\alpha,\beta} = \delta_{\alpha,\beta}$ when α and β are spacelike (unbarred) indices, or $g_{\alpha,\beta} = -\delta_{\alpha,\beta}$ when they are timelike (barred).

Seen in the pattern (24), a generator $j_{\alpha,\beta}$ commutes with all those outside its row α and column β (and the reflection of these across the diagonal as column α and row β). The commutator of two j's on the same row (column) yields the j in the intersection of their columns (rows), after one column (row) has been reflected into a row (column) by the diagonal. The action of the optical SO(3, 2) transformations on the so(3, 2) Hamiltonians can also be read off the pattern of coefficients (25). The one-parameter subgroups $T^{(\alpha,\beta)}(\tau) = \exp(\tau \hat{j}_{\alpha,\beta})$ produce covariant linear combinations of the Hamiltonians $\hat{j}_{\gamma,\delta}$ and contravariant ones for its coefficients $r_{\gamma,\delta}$ in relation (12). On a column $\kappa \in \{1, 2, 3, \bar{4}, \bar{5}\}$, the coefficients $r_{\alpha,\kappa}$ and $r_{\beta,\kappa}$ are linearly combined; on a row ρ, $r_{\rho,\alpha}$ and $r_{\rho,\beta}$ are so correspondingly. For two commuting SO(3, 2) one-parameter subgroups (for α, β, γ, δ all distinct), this action can be subsumed by

$$T^{(\alpha,\,\beta)}(\tau) T^{(\gamma,\,\delta)}(\tau'):\; \begin{bmatrix} r_{\alpha,\gamma} & r_{\alpha,\delta} \\ r_{\beta,\gamma} & r_{\beta,\delta} \end{bmatrix} = \mathbf{T}^{(\alpha,\,\beta)}(\tau) \begin{bmatrix} r_{\alpha,\gamma} & r_{\alpha,\delta} \\ r_{\beta,\gamma} & r_{\beta,\delta} \end{bmatrix} \mathbf{T}^{(\gamma,\,\delta)}(\tau')^{\mathsf{T}}, \tag{28}$$

with matrices $\mathbf{T}^{(\alpha,\,\beta)}(\tau)$ and $\mathbf{T}^{(\gamma,\,\delta)}(\tau')$, whose elements are trigonometric functions of $\tau \in \mathcal{S}_1$ (the circle), or hyperbolic functions of $\tau \in \Re$,

$$\mathbf{T}^{(\alpha,\,\beta)}(\tau) = \begin{cases} \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix} & \tau \in \mathcal{S}_1 \quad \begin{matrix} \text{when } \alpha, \beta \in \{1, 2, 3\} \\ \text{or } \alpha, \beta \in \{\bar{4}, \bar{5}\}; \end{matrix} \\[2ex] \begin{bmatrix} \cosh\tau & -\sinh\tau \\ -\sinh\tau & \cosh\tau \end{bmatrix} & \tau \in \Re \quad \begin{matrix} \text{when } \alpha \notin \{1, 2, 3\} \ni \beta \\ \text{or } \beta \notin \{\bar{4}, \bar{5}\} \ni \alpha \end{matrix} \end{cases} . \tag{29}$$

4. U(2)-FOURIER REDUCTION AND BRANCHING OF so(3, 2) PATTERNS BY AN IMAGER

The simplest realization of the pseudo-orthogonal algebra so(3, 2) is by generators of rotations and boosts in a space of vectors $v = (v_1, v_2, v_3, v_{\bar{4}}, v_{\bar{5}}) \in \Re^5$ with the diagonal metric $\mathbf{D} = \|d_{\alpha,\beta}\| = \mathrm{diag}(+1, +1, +1, -1, -1)$ seen above. It is given by the linear differential operators

$$\hat{j}_{m,n} = v_m \partial_n - v_n \partial_m, \qquad \hat{j}_{m,\bar{n}} = v_m \partial_{\bar{n}} + v_{\bar{n}} \partial_m,$$

$$\hat{j}_{\bar{m},\bar{n}} = -v_{\bar{m}} \partial_{\bar{n}} + v_{\bar{n}} \partial_{\bar{m}}. \tag{26}$$

The Lie brackets of so(3, 2) are then found straightforwardly by commutation (the algebraic structure is independent of the realization):

The importance of the matrix form of Eq. (28) is that we can determine the values of τ and/or τ' to bring coefficients in the pattern to zero.

U(2)-Fourier transforms are finite rotations $T^{(1,2,3)}(\tau, \varphi, \chi) = T^{(1,2)}(\tau) \times T^{(2,3)}(\varphi) T^{(1,2)}(\chi) \in \mathrm{SO}(3)$ by three Euler angles and on the circle $T^{(\bar{4},\bar{5})}(\omega) \in \mathrm{SO}(2)$ of isotropic (central) fractional Fourier transforms.[4,5] The ten coefficients of so(3, 2) Hamiltonians (25) can be arranged into three 3-vectors under SO(3) rotations, $\mathbf{r} = (r_{2,3}, -r_{1,3}, r_{1,2})^{\mathsf{T}}$, $\mathbf{r}_{\cdot\bar{4}} = (r_{1,\bar{4}}, r_{2,\bar{4}}, r_{3,\bar{4}})^{\mathsf{T}}$, and $\mathbf{r}_{\cdot\bar{5}} = (r_{1,\bar{5}}, r_{2,\bar{5}}, r_{3,\bar{5}})^{\mathsf{T}}$. The last two vectors can be linearly combined by the isotropic fractional Fourier transform $T^{(\bar{4},\bar{5})}(\omega)$ with trigonometric coefficients, to make one of them (the new $\mathbf{r}_{\cdot\bar{4}}$) orthogonal to $\mathbf{r}$. Then, with SO(3) rotations we can align $\mathbf{r}_{\cdot\bar{4}}$ with the 1-axis (so $r_{2,\bar{4}} = r_{3,\bar{4}} = 0$), bring $\mathbf{r}_{\cdot\bar{5}}$ to the 1–3 plane (so $r_{2,\bar{5}} = 0$), and since $\mathbf{r}$ remains orthogonal to $\mathbf{r}_{\cdot\bar{4}}$, then $r_{2,3} = 0$. Thus we use the four parameters of U(2) Fourier transforms to reduce the Hamiltonian (25) to a Hamiltonian pattern with at most six nonzero coefficients,

$$\mathcal{H}(\mathbf{m}) \xrightarrow{T^{(\bar{4},\bar{5})}\,T^{(1,2,3)}} \mathcal{H}_{\mathrm{U}} = \begin{array}{|cc|cc|} \hline m & r & s & u \\ \hline & 0 & 0 & 0 \\ & & 0 & v \\ \hline & & & w \\ \hline \end{array}. \qquad (30)$$

We henceforth individualize the letters m for angular momentum, (r, s, u) for counter-(harmonic, repulsive, imager) terms, and (v, w) for isotropic (repulsive, harmonic) Hamiltonians.

To reduce further we must now choose a boost: the isotropic imager $T^{(3,\bar{4})}(\tau)$ [see Eq. (23)]. This will rescale $\mathbf{q} \mapsto \exp(-\frac{1}{2}\tau)\mathbf{q}$ versus $\mathbf{p} \mapsto \exp(\frac{1}{2}\tau)\mathbf{p}$ and thus mix the isotropic repulsive and harmonic Hamiltonian coefficients $v = r_{3,\bar{5}}$ and $w = r_{\bar{4},\bar{5}}$ (and separately the counterterms $r = r_{1,3}$ and $s = r_{1,\bar{4}}$) with hyperbolic functions of τ; the free Euclidean terms $\frac{1}{2}\,\mathbf{p}^{\mathsf{T}}\mathbf{c}\,\mathbf{p}$ and $\frac{1}{2}\,\mathbf{q}^{\mathsf{T}}\mathbf{b}\,\mathbf{q}$ $(r_{\rho,3} \pm r_{\rho,\bar{4}})$ are rescaled with $\exp(\pm\tau) > 0$. The magnification parameter of $T^{(3,\bar{4})}(\tau)$ can be chosen to reduce the two terms (v, w) of $\mathcal{H}_{\mathrm{U}}$ in Eq. (30) to a first branching into the three prototypical cases, whose reduction will be performed next:

H: Harmonic when $|w| > |v|$

x–y separable subalgebra $\mathrm{so}(2,2)$, $\mathcal{H}_{\mathrm{H}} = \begin{array}{|cc|cc|} \hline m & r & s & u \\ \hline & 0 & 0 & 0 \\ & & 0 & 0 \\ \hline & & & w \\ \hline \end{array},$ (31)

R: Repulsive when $|w| < |v|$

Lorentzian subalgebra $\mathrm{so}(3,1)$, $\mathcal{H}_{\mathrm{R}} = \begin{array}{|cc|cc|} \hline m & r & s & u \\ \hline & 0 & 0 & 0 \\ & & 0 & v \\ \hline & & & 0 \\ \hline \end{array},$ (32)

F: Free when $|w| = |v|$

Euclidean subalgebra $\mathrm{iso}(2,1)$, $\mathcal{H}_{\mathrm{F}}^{\pm} = \begin{array}{|cc|cc|} \hline m & r & s & u \\ \hline & 0 & 0 & 0 \\ & & 0 & 1 \\ \hline & & & \pm 1 \\ \hline \end{array}.$ (33)

(The other four parameters (m, r, s, u) will have been transformed also, but we do not distinguish their values by primes.)

5. SEPARABLE, LORENTZIAN, AND EUCLIDEAN HAMILTONIANS

Here we examine the further reduction of each of the three cases (31)–(33) to appropriate Hamiltonian orbit representatives. This will justify their names and the association with the pseudo-orthogonal and (pseudo-) Euclidean subalgebras $\mathrm{so}(2, 2)$, $\mathrm{so}(3, 1)$, and $\mathrm{iso}(2, 1)$.

A. Separable Hamiltonians

We rotate the 2–3 plane of the pattern $\mathcal{H}_{\mathrm{H}}$ in Eq. (31) to annul the angular momentum coefficient $m = 0$ and thus eliminate entirely the index 2 of cross systems. The remaining six generators $j_{\alpha,\beta}$, $\alpha, \beta \in \{1, 3, \bar{4}, \bar{5}\}$ close into a subalgebra $\mathrm{so}(2, 2) \subset \mathrm{so}(3, 2)$ and can be arranged in a similar subpattern,

$$\mathcal{H}_{\mathrm{H}} \xrightarrow{T^{(2,3)}} \begin{array}{|cc|cc|} \hline 0 & r & s & u \\ \hline & 0 & 0 & 0 \\ & & 0 & 0 \\ \hline & & & w \\ \hline \end{array} \supset \begin{array}{|c|cc|} \hline r & s & u \\ \hline & 0 & 0 \\ \hline & & w \\ \hline \end{array}. \qquad (34)$$

And the accident occurs[6] that $\mathrm{so}(2, 2)$ is a direct sum of two $\mathrm{so}(2, 1)$ algebras [this is similar to the better-known accident $\mathrm{so}(4) = \mathrm{so}(3) \oplus \mathrm{so}(3)$], which can be written in terms of the patterns of $\mathrm{so}(2, 1)$ harmonic $(h = j_{1,2})$, repulsive $(r = j_{1,\bar{3}})$ and imaging $(i = j_{2,\bar{3}})$ generators, as

$$\begin{array}{|c|c|} \hline h_x & r_x \\ \hline & i_x \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline h_y & r_y \\ \hline & i_y \\ \hline \end{array} = \begin{array}{|c|cc|} \hline -h_x + h_y & r_x - r_y & i_x - i_y \\ \hline & i_x + i_y & r_x + r_y \\ \hline & & h_x + h_y \\ \hline \end{array}. \qquad (35)$$

Since each $\mathrm{so}(2, 1)$ has three orbits (3)–(5) of Hamiltonians $G = H, R, F$ (harmonic, repulsive, free) their direct sum will contain all and only *x–y separable* Hamiltonians:

$$G_{\Theta}(\mathbf{p}, \mathbf{q}) = G_x(p_x, q_x)\cos\Theta + G'_y(p_y, q_y)\sin\Theta,$$
$$0 \leqslant \Theta < \pi, \qquad (36)$$

(The range of Θ is the half-circle $\mathcal{S}_1/\mathcal{Z}_2$ because G_0 and $G_{\pi} = -G_0$ belong to the same orbit). In principle there are thus $3 \times 3 = 9$ circles of Hamiltonian orbits indicated G–G'_{Θ} and parameterized by Θ. Each such manifold of parameter-related orbits is called a stratum. But notice that further equivalences occur between the nine strata within the larger mother group of $\mathrm{SO}(3, 2)$ matrices: With rotations of the screen $T^{(1,2)}(\pi)$ we can change the signs of all pattern entries with indices 1 and 2 (and those of index 2 are zero) and thus exchange the two direct summand algebras $\mathrm{so}(2, 1)_x \oplus \mathrm{so}(2, 1)_y$ in Eq. (35). Hence the Hamiltonians G_x–G'_y and G'_x–G_y are $\mathrm{SO}(3, 2)$ equivalent. The original nine strata therefore coalesce to six cases of separable Hamiltonians, indicated H–H_{Θ}, H–R_{Θ}, R–R_{Θ}, H–F_{Θ}, R–F_{Θ}, and F–F_{Θ}. In each stratum there will be still further equivalences.

In the H–H_{Θ} Hamiltonian stratum, since we can exchange the two coefficients, Θ and $\frac{1}{2}\pi - \Theta$ are also equivalent, so the range of Θ can be restricted to $-\frac{1}{4}\pi < \Theta \leqslant \frac{1}{4}\pi$. In the stratum of H–$\mathrm{R}_{\Theta}$ Hamiltonians, the Fourier transform $T^{(\bar{4},\bar{5})}(\frac{1}{2}\pi)$ inverts the sign of the repulsive part, so the orbits Θ and $-\Theta$ are the same; thus we restrict $0 < \Theta < \frac{1}{2}\pi$, setting apart the $\Theta = 0$ orbit because H_x properly belongs to the H–H_{Θ} stratum. In the R–R_{Θ} stratum the two previous equivalences hold, so the range of the stratum parameter reduces to $0 \leqslant \Theta \leqslant \frac{1}{4}\pi$. The three separable Hamiltonians containing a free summand, H–F, R–F, and F–F, are actually point orbits (and not one-parameter strata), because with an imager in y we can multiply this term by any positive factor and thus the ratio of the x and y coefficients can be brought to $1{:}\sigma$, with $\sigma \in \{+1, 0, -1\}$; in the H and R cases, the value $\sigma = 0$ yields the previous orbits H–H_0 and R–R_0.

We thus collect the following six inequivalent stratum and orbit representatives of separable Hamiltonians and compare them with those defined in Ref. 13 (where the authors chose the scale $\mathcal{H}_i^{\lambda} = G_x + \lambda G_y$):

Separable	Hamiltonians	Range	Notation[13]	
H–H_Θ stratum:	$H_x \cos\Theta + H_y \sin\Theta$,	$-\frac{1}{4}\pi < \Theta \leqslant \frac{1}{4}\pi$,	$\mathcal{H}_3^{-1<\lambda\leqslant 1}$,	
H–R_Θ stratum:	$H_x \cos\Theta + R_y \sin\Theta$,	$0 < \Theta < \frac{1}{2}\pi$,	$\mathcal{H}_2^{0<\lambda<\infty}$,	
R–R_Θ stratum:	$R_x \cos\Theta + R_y \sin\Theta$,	$0 \leqslant \Theta \leqslant \frac{1}{4}\pi$,	$\mathcal{H}_1^{0\leqslant\lambda\leqslant 1}$,	
H–F_σ orbits:	$H_x + \sigma F_y$,	$\sigma \in \{-1, +1\}$,	$\mathcal{H}_5^{\sigma=\pm}$,	
R–F orbit:	$R_x + F_y$,		$\mathcal{H}_4$,	
F–F_σ orbits:	$F_x + \sigma F_y$,	$\sigma \in \{-1, 0, +1\}$,	$\mathcal{H}_6^{\sigma=0,\pm}$.	(37)

B. Lorentzian Hamiltonians

We now follow the second, repulsive branch of patterns $\mathcal{H}_{\mathrm{R}}$ in Eq. (32) and try to eliminate the Hamiltonian coefficients with the (timelike) index $\bar{4}$, thus reducing it to the Lorentz subalgebra $\mathsf{so}(3,1) \subset \mathsf{so}(3,2)$ with generators $j_{\alpha,\beta}$, $\alpha, \beta \in \{1, 2, 3, \bar{5}\}$.

First we bring v in $\mathcal{H}_{\mathrm{R}}$ to the 2–$\bar{5}$ position with $T^{(2,3)}(\frac{1}{2}\pi)$ and then use an imager boost $T^{(3,\bar{4})}(\tau)$ to produce hyperbolic linear combinations between the counterharmonic and counter-repulsive coefficients, r and s, leaving the zeros in their places. As with all boosts, there are three outcomes according to the ratio $|r| : |s|$, only one of which is of interest in this subsection:

$$\mathcal{H}_{\mathrm{R}} \xrightarrow{T^{(2,3)}(\frac{1}{2}\pi)} \begin{array}{|cc|cc|} m & r & s & u \\ & 0 & 0 & v \\ & & 0 & 0 \\ & & & 0 \end{array} \xrightarrow{T^{(3,\bar{4})}} \begin{cases} |r| < |s|,\ r \mapsto 0,\ \text{index 2 out}, & \Rightarrow\ \text{separable } \mathsf{so}(2,2) \text{ Hamiltonians} \\ |r| > |s|,\ s \mapsto 0,\ \text{index 4 out}, & \Rightarrow\ \begin{array}{|cc|cc|} m & r & 0 & u \\ & 0 & 0 & v \\ & & 0 & 0 \\ & & & 0 \end{array} \supset \begin{array}{|cc|c|} m & r & u \\ & 0 & v \\ & & 0 \end{array} \\ |r| = |s| \mapsto 1,\ \text{(next subsection)} & \Rightarrow\ \text{Euclidean } \mathsf{iso}(2,1) \text{ Hamiltonians} \end{cases} \quad . \qquad (38)$$

Lorentzian Hamiltonians are general linear combinations of $\mathsf{so}(3,1)$ rotation and boost generators, whose coefficients belong to the two 3-vectors $\mathbf{r}$ and $\mathbf{r}_{\cdot\bar{5}}$ discussed for the $\mathsf{U}(2)$ reduction above Eq. (30). Under Lorentz transformations there are two (Casimir) invariants: $C_1 = |\mathbf{r}|^2 - |\mathbf{r}_{\cdot\bar{5}}|^2$ and $C_2 = \mathbf{r} \cdot \mathbf{r}_{\cdot\bar{5}}$. By means of boosts—provided that $C_2 \neq 0$ ($\mathbf{r}$ and $\mathbf{r}_{\cdot\bar{5}}$ not orthogonal)—we can always bring both vectors to be parallel (boosting boost generators linearly combines them with generators of rotation), and with rotations we can align them with the 3-axis, so that only $m = r_{1,2}$ and $v = r_{3,\bar{5}}$ are nonzero in the pattern of expression (38). The representatives of the generic stratum of Lorentzian Hamiltonians are thus linear combinations (by $\cos\Theta$ and $\sin\Theta$, $\Theta \in \mathcal{S}_1/\mathcal{Z}_2$ as before) of an isotropic repulsive term $R = j_{3,\bar{5}}$ and angular momentum $M = j_{1,2}$ [see expression (39) below]; their relative signs can be flipped by means of a Fourier transform $T^{(\bar{4},\bar{5})}(\frac{1}{2}\pi)$ [which is outside $\mathsf{so}(3,1)$], so the orbit range of Θ reduces to $0 \leqslant \Theta < \frac{1}{2}\pi$.

In the special case when $C_2 = 0$ ($\mathbf{r} \perp \mathbf{r}_{\cdot\bar{5}}$), we can arrange that only $r_{1,2} \neq 0$ and $r_{2,\bar{5}} \neq 0$. Then by means of a Lorentz transformation we can bring the Hamiltonian (38) to one of three forms according to the sign of C_1: pure angular momentum ($C_1 > 0$, only $r_{1,2} \neq 0$); a pure boost generator ($C_1 < 0$, so that only $r_{2,\bar{5}} \neq 0$), or a Euclidean translation ($C_1 = 0$, where $\pm r_{1,2} = r_{2,\bar{5}} = 1$ are the only nonzero coefficients). The pure angular momentum Hamiltonian lies in an orbit that we should consider apart from the generic stratum above, because it alone generates bounded trajectories. The pure boost can be rotated to $R = j_{3,\bar{5}}$, which belongs to the R–R_Θ stratum of separable Hamiltonians in expressions (37) for $\Theta = \frac{1}{4}\pi$ (as well as to the previous generic Lorentzian stratum). The most degenerate $C_1 = C_2 = 0$ translation Hamiltonians are also separate from the Lorentzian stratum, and the two signs are again equivalent under a Fourier transform. We conclude that there are three new distinct

Lorentzian	Hamiltonians	Range	Notation[13]
R–M$_\Theta$ stratum:	$R\cos\Theta + M\sin\Theta$,	$0 < \Theta < \frac{1}{2}\pi$,	$\mathcal{H}_9^{\lambda>0}$,
M orbit:	M,		$\mathcal{H}_7$,
X orbit:	$q_x p_y$,		—. (39)

We denote the last orbit by X because it is a double cross in x and y and in p and q.

C. Euclidean Hamiltonians

Lastly, the pseudo-Euclidean branch in Eq. (33) and the Hamiltonians remaining in expression (38) require careful treatment even though they give rise to only one more physically relevant Hamiltonian and two others of marginal interest.

Within the algebra $\mathrm{so}(3, 2)$ there are the following two sets of elements,

$$t_\alpha^\pm = j_{\alpha,3} \pm j_{\alpha,\bar{4}}, \qquad \alpha \in \{1, 2, \bar{5}\},$$
$$\mathbf{t}^\pm = (t_1^\pm, t_2^\pm, t_{\bar{5}}^\pm)^\top, \tag{40}$$

$$\mathbf{t}^- = (-\tfrac{1}{2}(p_x^2 - p_y^2),\ p_x p_y,\ \tfrac{1}{2}(p_x^2 + p_y^2))^\top,$$
$$\mathbf{t}^+ = \mathbf{t}^-(p \mapsto q). \tag{41}$$

These are generators of astigmatic-free propagation ($\mathbf{t}^-$) and of lenses ($\mathbf{t}^+$). They behave as (pseudo-) Euclidean translations in the (2 + 1)-Minkowski vector subspace $\{1, 2, \bar{5}\}$. The elements of each set commute among themselves, but the two sets do not commute between each other. We rewrite the generic Hamiltonian (13) in the light-cone coordinates $\{1, 2, \bar{5}, +, -\}$, where the basis generators (41) have the coefficients $e_\rho^\pm = 1/2(r_{\rho,3} \pm r_{\rho,\bar{4}})$, at rows $\rho \in \{1, 2, \bar{5}\}$. In particular, the Fourier-reduced pattern $\mathcal{H}_\mathrm{F}^-$ in Eq. (33) has $e_{\bar{5}}^- = v$, $e_{\bar{5}}^+ = 0$ (for $\mathcal{H}_\mathrm{F}^+$ we change signs), and $e_2^- = 0 = e_2^+$.

Translations $T_\mathrm{E}^{(2,-)}(\tau) = \exp(\tau \hat{t}_2^-)$ of rotations in the 1–2 plane spawn translations along the 1-axis:

$$T_\mathrm{E}^{(2,-)}(\tau): j_{1,2} = j_{1,2} - \tau t_1^-,$$
$$T_\mathrm{E}^{(2,-)}(\tau): t_1^+ = t_1^+ + \tau j_{1,2} - \tfrac{1}{2}\tau^2 t_1^-,$$
$$m \mapsto m + \tau e_1^+,$$
$$e_1^- \mapsto e_1^- - \tau m - \tfrac{1}{2}\tau^2 e_1^+. \tag{42}$$

We use these results on the Hamiltonian (33) to bring the angular momentum coefficient m to zero (with $\tau = -m/e_1^+$—whenever $e_1^+ \neq 0$), leaving the previous zeros on the 2 row in their places. Having eliminated the index 2 however, we recognize that we are back among the $\mathrm{so}(2, 2)$ separable Hamiltonian strata described above. Euclidean Hamiltonians are therefore also characterized by $e_1^+ = 0$, which means that no q^2 terms can be present. For these Hamiltonians, we can cross out the row + and remain with the six generators in the indices $\{1, 2, \bar{5}, -\}$, which form the (pseudo-) Euclidean algebra $\mathrm{iso}(2,1) \subset \mathrm{so}(3,2)$.

To examine the Euclidean orbits, it is convenient to build again two (2 + 1)-vector arrays, $\mathbf{c} = (r_{2,\bar{5}}, -r_{1,\bar{5}}, r_{1,2})^\top$ and $\mathbf{e} = (e_1^-, e_2^-, e_{\bar{5}}^-)^\top$, containing the coefficients of the generators indexed similarly so that the Euclidean Hamiltonians are $H = \mathbf{c} \cdot \mathbf{j} + \mathbf{e} \cdot \mathbf{t}^-$. With the metric $\mathbf{E} = \mathrm{diag}(+1, +1, -1)$, one has the invariant $-C_\mathrm{E} = \mathbf{e} \cdot \mathbf{E}\mathbf{e} = e_1^2 + e_2^2 - e_{\bar{5}}^2$. Then it is straightforward to show that under $T_\mathrm{E}^-(\boldsymbol{\tau}) = \exp(\boldsymbol{\tau} \cdot \mathbf{t}^-)$, the Hamiltonians transform as

$$T_\mathrm{E}(\boldsymbol{\tau}): (\mathbf{c} \cdot \mathbf{j} + \mathbf{e} \cdot \mathbf{t}) = \mathbf{c} \cdot \mathbf{j} + [\mathbf{e} + \mathbf{E}(\boldsymbol{\tau} \times \mathbf{c})] \cdot \mathbf{t}. \tag{43}$$

And so according to the sign of C_E, there is again the trichotomy into harmonic, repulsive, and free cases ($\mathbf{c}$ time-, space-, and light-like—and the trivial fourth: $\mathbf{c} = \mathbf{0}$). For $C_\mathrm{E} \neq 0$, from Eq. (43) one can find a $\boldsymbol{\tau}$ such that $\mathbf{e} + \mathbf{E}(\boldsymbol{\tau} \times \mathbf{c})$ is parallel to $\mathbf{c}$, and also in the free case $C_\mathrm{E} = 0$, $\mathbf{c} = (0, 1, \pm 1)^\top$, we can bring $\mathbf{e}$ to $(0, 1, \pm 1)^\top$. Within $\mathrm{iso}(2, 1)$, therefore, we can choose the representatives of the strata parameterized by $j_\alpha \cos\Theta + t_\alpha^- \sin\Theta$, $0 \leqslant \Theta < \pi$ as before, with (j_α, t_α^-) given in the three cases [see expressions (44) below] by $(j_{1,2}, t_{\bar{5}}^-)$ for $C_\mathrm{E} > 0$ (harmonic), $(j_{2,\bar{5}}, t_1^-)$ [or $(-j_{1,\bar{5}}, t_2^-)$] for $C_\mathrm{E} > 0$ (repulsive), and $(j_{1,2} \pm j_{1,\bar{5}}, t_{\bar{5}}^- \pm t_1^-)$ [or $1 \leftrightarrow 2$] for $C_\mathrm{E} = 0$ (free). The trivial null case ($\mathbf{c} = \mathbf{0}$), still allows $\mathbf{e}$ to be time-, space-, or light-like; but these points have already been counted before as the separable F–F$_\sigma$ orbits.

Under similarity by the full $\mathrm{SO}(3, 2)$ group, orbits of Euclidean Hamiltonians will coalesce. From Eq. (41) we see that the Fourier transform relates $\mathbf{t}^-$ with $\mathbf{t}^+$, so no separate treatment is necessary for the + case. The isotropic imager $T^{(3,\bar{4})}(\tau)$ will scale the generators $t_\alpha^\pm$ by $\exp(\mp 2\tau) > 0$ and leave the $\mathrm{so}(2, 1)$ generators j_α invariant; the Θ stratum ($\Theta \neq 0, \frac{1}{2}\pi$) will then collapse to $j_\alpha \pm t_\alpha^-$; in the repulsive and free cases a rotation $T^{(1,2)}(\pi)$ will bridge the two signs. We can thus choose the following representatives of

Euclidean	Hamiltonians	Notation[13]
F–M$_\pm$ orbits:	$F \pm M = \frac{1}{2}\lvert\mathbf{p}\rvert^2 \pm \frac{1}{2}(q_x p_y - q_y p_x)$,	$\mathcal{H}_8^{\epsilon=\pm}$,
F–I orbit:	$\frac{1}{2}(p_x^2 - p_y^2 + p_x q_y + p_y q_x)$	
	[or $p_x p_y + \frac{1}{2}(p_x q_x - p_y q_y)$],	$\equiv\mathcal{H}_{10}$,
F–X orbit:	$\frac{1}{2}p_x^2 + p_y q_x$,	$\equiv\mathcal{H}_{11}$. (44)

6. EIGENVALUE STRUCTURE AND THE RESOLUTION OF DEGENERACIES

We commented before that orbits of matrices defined by general similarity are characterized by their eigenvalues, but when the similarity is restricted to be symplectic, some values may be degenerate. Still, the eigenvalue structure is very informative and provides a good criterion for *in*equivalence. Moreover, the eigenvalues can be found directly from the 4×4 matrix representation of Hamiltonian matrices (10).

The eigenvalue equation for Hamiltonian matrices (10) is biquadratic; i.e.,

$$\det(\mathbf{m} - \lambda \mathbf{1}) = \lambda^4 + \Gamma\lambda^2 + \Delta = 0, \tag{45}$$

$$\Gamma = \begin{vmatrix} a_x & b_x \\ c_x & -a_x \end{vmatrix} + \begin{vmatrix} a_y & b_y \\ c_y & -a_y \end{vmatrix} + 2\begin{vmatrix} a_{xy} & b_{\bowtie} \\ c_{\bowtie} & -a_{yx} \end{vmatrix}, \tag{46}$$

$$\Delta = \det \mathbf{m} = \Delta^{\text{non}}_{\text{cross}} + \Delta_{\text{cross}}, \tag{47}$$

$$\Delta^{\text{non}}_{\text{cross}} = \begin{vmatrix} a_x & b_x \\ c_x & -a_x \end{vmatrix}\begin{vmatrix} a_y & b_y \\ c_y & -a_y \end{vmatrix} - 2a_x a_y a_{xy} a_{yx} + a_{xy}^2 a_{yx}^2 + a_{xy}^2 b_y c_x + a_{yx}^2 b_x c_y, \tag{48}$$

$$\Delta_{\text{cross}} = b_{\bowtie}^2 c_{\bowtie}^2 - b_{\bowtie}^2 c_x c_y - c_{\bowtie}^2 b_x b_y + 2b_{\bowtie} c_{\bowtie}(a_x a_y + a_{xy} a_{yx}) - 2b_{\bowtie}(a_x a_{yx} c_y + a_y a_{xy} c_x) - 2c_{\bowtie}(a_x a_{xy} b_y + a_y a_{yx} b_x). \tag{49}$$

We have grouped the summands into noncross and cross summands, according to the absence or presence of the cross-term coefficients $b_{\bowtie}$ and $c_{\bowtie}$. The four solutions to Eq. (45) are the eigenvalues

$$\lambda = \pm\left(-\tfrac{1}{2}\Gamma \pm \sqrt{(\tfrac{1}{2}\Gamma)^2 - \Delta}\right)^{1/2}. \tag{50}$$

When λ is an eigenvalue, then so are $-\lambda$, λ^*, and $-\lambda^*$ if distinct.[15]

The two radicands in Eq. (50) can be positive, negative, or zero; there are thus nine eigenvalue patterns in the Γ–Δ plane. These are shown in expression (51) below and in Fig. 1. There are four regions, four boundaries (two branches of the line $\Delta = 0$ and two branches of the parabola $\Delta = \frac{1}{4}\Gamma^2$), and one osculation point ($\Gamma = \Delta = 0$—very degenerate). Under scaling $H \mapsto \alpha H$ ($\alpha \neq 0$), the eigenvalues multiply by α and the points of the plane will shift on the parabolas ($\alpha^2\Gamma$, $\alpha^4\Delta$). The eigenvalue patterns thus correspond to points of the circle and the origin, where the Hamiltonian orbits are as follows:

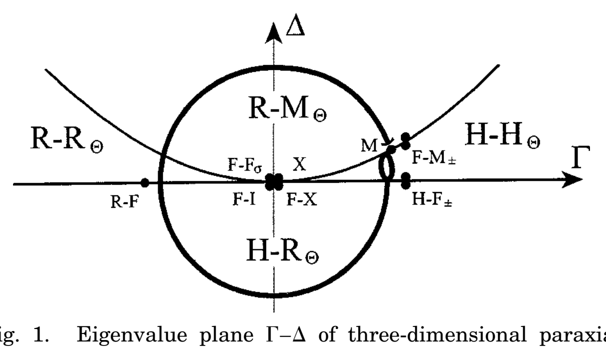

Fig. 1. Eigenvalue plane Γ–Δ of three-dimensional paraxial Hamiltonians. The Hamiltonian orbits are parabolas $\Delta = \frac{1}{4}\alpha^2\Gamma^2$, $0 \neq \alpha \in \Re$, that we project on a circle, and degenerate points at the origin. There are four strata (H–H$_\Theta$, H–R$_\Theta$, R–R$_\Theta$, and R–M$_\Theta$). On their boundaries we find six isolated orbits (H–F$_\pm$, R–F, F–M$_\pm$, and M), and six orbits coexist at the origin (F–F$_{0,\pm}$, X, F–I and F–X).

$\Gamma<0,\ \Delta=\frac{1}{4}\Gamma^2$ R orbit,	$\Gamma \in \Re,\ \Delta>\frac{1}{4}\Gamma^2$ R-M$_\Theta$ stratum,	$\Gamma>0,\ \Delta=\frac{1}{4}\Gamma^2$ H, M, F-M$_\pm$ orbit,
$\Gamma<0,\ 0\leq\Delta\leq\frac{1}{4}\Gamma^2$ R-R$_\Theta$ stratum,	$\Gamma=0,\ \Delta=0$ center point F-F$_\sigma$, X, F-I, F-X,	$\Gamma>0,\ 0<\Delta<\frac{1}{4}\Gamma^2$ doubly degenerate H-H$_\Theta$ stratum,
$\Gamma<0,\ \Delta=0$ R$_y$, R-F orbits,	$\Gamma \in \Re,\ \Delta<0$ H-R$_\Theta$ stratum,	$\Gamma>0,\ \Delta=0$ H$_x$, H-F$_\pm$ orbits.

(51)

The degeneracies are thus as follows (clockwise from the H–H$_\Theta$ stratum): The separable harmonic waveguides (37), $H_x \cos\Theta + H_y \sin\Theta\,(-\frac{1}{4}\pi < \Theta \leqslant \frac{1}{4}\pi)$, have $\Gamma = 1$, $\Delta = \frac{1}{4}\sin^2 2\Theta$; hence *two* distinct orbits $\pm\Theta$ correspond to each point on that arc of circle in Fig. 1. The upper boundary in this region $\Delta = \frac{1}{4}\Gamma^2$ is the isotropic harmonic orbit $\Theta = \frac{1}{4}\pi$ of $H = H_x + H_y$; the lower boundary $\Delta = 0$ is the orbit $\Theta = 0$ of H_x (or equivalently of H_y) which joins smoothly with the neighboring H–R$_\Theta$ stratum. Superposed on the boundary are the two H–F$_\pm$ orbits.

No degeneracy occurs in the H–R$_\Theta$ stratum ($\Gamma = \cos 2\Theta$, $\Delta = -\frac{1}{4}\sin^2 2\Theta < 0$ for $0 < \Theta < \frac{1}{2}\pi$), which also joins smoothly with the R–R$_\Theta$ stratum at R_y; on the same boundary is the R–F orbit. The R–R$_\Theta$ and R–M$_\Theta$ strata are nondegenerate, and their common boundary (isotropic repulsive R) is uneventful. However, the boundary of the R–M$_\Theta$ region of Lorentzian Hamiltonians (39), $R\cos\Theta + M\sin\Theta$ ($\Gamma = -2\cos 2\Theta$, $\Delta = 1$) is open for $\Theta \to \frac{1}{2}\pi^+$. The limit excludes the boundary where lies the separate compact Lorentzian orbit M of angular momentum, which is degenerate with the two Euclidean F–M$_\pm$ orbits, and with the isotropic harmonic waveguide Hamiltonian H in the H–H$_\Theta$ stratum seen above.

Besides the previous four Hamiltonian strata and the six isolated degenerate orbits on the circle of Fig. 1, there remains the origin where all remaining degeneracy lies. There are the three separable free Hamiltonians (37) in the F–F$_\sigma$ orbits, the Lorentzian orbit X of $q_x p_y$ in expression (39), and the two pseudo-Euclidean orbits F–I and F–X in expressions (44), adding to a total of six distinct Hamiltonian orbits at the origin.

7. CONCLUDING REMARKS

Whereas two-dimensional paraxial optical Hamiltonians belong to one of three orbits, harmonic (H), repulsive (R), or free (F), we have seen here that in three-dimensional systems there are four strata and six isolated orbits that we can arrange in a circle and six isolated orbits at zero, divided into separable, Lorentzian, and (pseudo-) Euclidean. The generators of U(2)-Fourier transforms encompass the H–H$_\Theta$ separable stratum and the M Lorentzian orbit and form the maximal compact subalgebra of such systems. By means of fractional Fourier transformers and pure imagers (i.e., phase-space rotations and isotropic scalings) we obtained the chosen representative Hamiltonians for each orbit. Their parameters are found at each step by solving pairs of linear equations that are determined by the coefficients of the original Hamiltonian and that for reasons of space we have not detailed here. The eigenvalue degeneracy is solved.

The classification of the Hamiltonian generators for higher-order astigmatic aberrations has been based on their transformation properties under the paraxial subgroup;[19] it appears that each of the three Hamiltonian subalgebras can carry its own disjoint retinue of subaberrations, whose classification should be investigated elsewhere. Hamiltonians in $\mathsf{sp}(4, \mathfrak{R})$ generate one-parameter subgroups within the exponential part of the group $\mathsf{Sp}(4, \mathfrak{R})$ of paraxial optical systems.[4] The resolution of $\mathsf{sp}(4, \mathfrak{R})$ into orbits of equivalent Hamiltonians implies the foliation of $\mathsf{Sp}(4, \mathfrak{R})$ systems into one-parameter subgroups—waveguides of z-independent media. Results will apply not only in the context of geometric optics or classical mechanics but also in the isomorphic theories of 3-dim linear wave optics and 2-dim quantum oscillators and Kepler–Coulomb systems.[18]

ACKNOWLEDGMENTS

The authors acknowledge the support of the Universidad Nacional Autónoma de México (UNAM) project DGAPA-IN-112300 Optica Matemática and the help of Anahit Balabekian and Guillermo Krötzsch (Centro de Ciencias Físicas, UNAM) with the macros and figures. Sameen Ahmed Khan acknowledges a postdoctoral schollarship from the Consejo Nacional de Ciencia y Tecnología and UNAM, which made this collaboration possible.

Corresponding author K. B. Wolf can be reached by e-mail at bwolf@fis.unam.mx.

REFERENCES

1. H. M. Ozaktas, Z. Zalevsky, and M. Alper Kutay, *The Fractional Fourier Transform* (Wiley, Chichester, UK, 2001).
2. S. Steinberg, "Lie series, Lie transformations, and their applications," in *Lie Methods in Optics*, J. Sánchez-Mondragón and K. B. Wolf, eds., Vol. 250 of Lecture Notes in Physics (Springer-Verlag, Heidelberg, 1986), Chap. 3, pp. 45–102.
3. A. J. Dragt, E. Forest, and K. B. Wolf, "Foundations of a Lie algebraic theory of geometrical optics," in *Lie Methods in Optics*, J. Sánchez-Mondragón and K. B. Wolf, eds., Vol. 250 of Lecture Notes in Physics (Springer-Verlag, Heidelberg, 1986) Chap. 4, pp. 105–158.
4. R. Simon and K. B. Wolf, "Structure of the set of paraxial optical systems," J. Opt. Soc. Am. A **17**, 342–355 (2000).
5. R. Simon and K. B. Wolf, "Fractional Fourier transforms in two dimensions," J. Opt. Soc. Am. A **17**, 2368–2381 (2000).
6. R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974).
7. A. J. Dragt, "Elementary and advanced Lie algebraic methods with applications to accelerator design, electron microscopes, and light optics," Nucl. Instrum. Methods Phys. Res. A **258**, 339–354 (1987).
8. P. A. M. Dirac, "A remarkable representation of the 3 + 2 de Sitter group," J. Math. Phys. **4**, 901–909 (1963).
9. C. Fronsdal, "Elementary particles in a curved space," Rev. Mod. Phys. **37**, 201–223 (1965).
10. T. O. Philips and E. P. Wigner, "De Sitter space and positive energy," in *Group Theory and Its Applications*, E. M. Loebl, ed. (Academic, New York, 1968), pp. 631–676.
11. A. J. Laub and K. Meyer, "Canonical forms for symplectic and Hamiltonian matrices," Celest. Mech. **9**, 213–238 (1974).
12. J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, "Continuous subgroups of the fundamental groups of physics. III. The de Sitter groups," J. Math. Phys. **18**, 2259–2288 (1977).
13. M. Moshinsky and P. Winternitz, "Quadratic Hamiltonians in phase space and their eigenstates," J. Math. Phys. **21**, 1667–1682 (1980).
14. J. Patera, P. Winternitz, and H. Zassenhaus, "Maximal abelian subalgebras of real and complex symplectic Lie algebras," J. Math. Phys. **24**, 1973–1985 (1983).
15. R. Simon, E. C. G. Sudarshan, and N. Mukunda, "Anisotropic Gaussian Schell-model beams: passage through optical systems and associated invariants," Phys. Rev. A **31**, 2419–2434 (1985).
16. R. Simon and N. Mukunda, "Optical phase space, Wigner representation, and invariant quality parameters," J. Opt. Soc. Am. A **17**, 342–355 (2000).
17. M. Kauderer, *Symplectic Matrices, First Order Systems and Special Relativity* (World Scientific, Singapore, 1994).
18. K. B. Wolf, "Dynamical groups for the point rotor and the hydrogen atom," Nuovo Cimento Suppl. **5**, 1041–1050 (1967).
19. K. B. Wolf, "The group-theoretical treatment of aberrating systems. III. The classification of asymmetric aberrations," J. Math. Phys. **28**, 2498–2507 (1987).