Fractional Fourier transforms in two dimensions

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We analyze the fractionalization of the Fourier transform (FT), starting from the minimal premise that repeated application of the fractional Fourier transform (FrFT) a sufficient number of times should give back the FT. There is a qualitative increase in the richness of the solution manifold, from U(1) (the circle $\mathcal{S}^1$) in the one-dimensional case to U(2) (the four-parameter group of 2×2 unitary matrices) in the two-dimensional case [rather than simply U(1) $\times$ U(1)]. Our treatment clarifies the situation in the N-dimensional case. The parameterization of this manifold (a fiber bundle) is accomplished through two powers running over the torus $\mathcal{T}^2 = \mathcal{S}^1 \times \mathcal{S}^1$ and two parameters running over the Fourier sphere $\mathcal{S}^2$. We detail the spectral representation of the FrFT: The eigenvalues are shown to depend only on the $\mathcal{T}^2$ coordinates; the eigenfunctions, only on the $\mathcal{S}^2$ coordinates. FrFT's corresponding to special points on the Fourier sphere have for eigenfunctions the Hermite–Gaussian beams and the Laguerre–Gaussian beams, while those corresponding to generic points are SU(2)-coherent states of these beams. Thus the integral transform produced by every Sp(4, $\Re$) first-order system is essentially a FrFT. [S0740-3232(00)00512-3]

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1. INTRODUCTION

The ease of producing a Fourier transform (FT) by optical means,[1] together with its inherent usefulness for signal analysis,[2,3] has spurred interest in its fractionalization. Most of the literature, however, is concerned with one-dimensional (1D) signals in two-dimensional (2D) optical systems, and there are statements to the effect that 2D Fourier analysis in 3D optics is the simple generalization of a direct product for the two coordinates.

In 2D optics (one transverse and one longitudinal dimension) the fractional Fourier transform (FrFT) is equivalent to evolution of the field along a parabolic-index fiber (or a quantum harmonic oscillator).[4] The manifold of these transforms is a circle, $\mathcal{S}^1$, and they form the 1D unitary group U(1) of phases. In the case of two transverse dimensions there is indeed a qualitative increase in the richness of the Fourier fractionalization problem. The manifold of fractionalizations does not become just U(1) $\times$ U(1) but rather grows to U(2), the group of 2×2 unitary matrices; while U(1) $\times$ U(1) is a 2D manifold, the manifold of U(2) has four dimensions. This shows the way for generalizations to N dimensions. There should be a corresponding diversification regarding the possibilities for application of these results in 2D image analysis. In this paper we offer a systematic study of both the manifold of the FrFT's and their optical implementation.

The 2D FT is an operator $\mathcal{F}$ that acts on square-integrable functions of the plane, $f \in \mathcal{L}^2(\Re^2)$. In Cartesian coordinates $\mathbf{q} = (q_x, q_y)^\top$ (a column vector), we write $\mathcal{F}$ in *official* form as

$$(\mathcal{F}f)(q_x, q_y) = \frac{1}{2\pi} \int_{\Re^2} \mathrm{d}q_x' \mathrm{d}q_y' \times \exp[-i(q_x q_x' + q_y q_y')] f(q_x', q_y'). \quad (1)$$

Its best-known (and basic) properties are linearity, unitarity $\mathcal{F}^\dagger = \mathcal{F}^{-1}$, and the fact that its fourth power is the unit operator $\mathcal{F}^4 = 1$ [whose integral kernel is $\delta(\mathbf{q} - \mathbf{q}')$]. Also noteworthy facts are that its square, $\mathcal{F}^2 = \mathcal{J}$, is a parity (or inversion) operator [of integral kernel $\delta(\mathbf{q} + \mathbf{q}')$, such that $\mathcal{J}^2 = 1$]. Equation (1) is actually only an integral transform realization of an abstract operator $\mathcal{F}$, which also has matrix representations that satisfy the same basic properties. The integral form of the FT (1) is in fact completely defined (up to a phase) in $\mathcal{L}^2(\Re^2)$ by its well-known intertwining with the Schrödinger operators of position and momentum, $\hat{\mathbf{q}} f(\mathbf{q}) = \mathbf{q} f(\mathbf{q})$ and $\hat{\mathbf{p}} f(\mathbf{q}) = -i\partial_{\mathbf{q}} f(\mathbf{q})$, given by[5]

$$\mathcal{F}\hat{\mathbf{p}} = \hat{\mathbf{q}}\mathcal{F}, \qquad \mathcal{F}\hat{\mathbf{q}} = -\hat{\mathbf{p}}\mathcal{F}. \quad (2)$$

In Section 2 we first analyze what the roots of $\mathcal{F}$ are independently of its realization. It is shown that there are sufficient roots to properly define the FrFT $\mathcal{F}^\alpha$ for any power α mod 4. The matrix representation of 3D paraxial optical systems (see, e.g., Refs. 6–8) is used in Section 3 to display the FrFT manifold as the unitary subgroup of the group of linear transformations by paraxial optical systems. The commuting center of this Fourier subgroup is the U(1) circle of *central* FrFT's, which is the naïve generalization of the 1D case. Two other FrFT's often found in the literature are the *separable* and the *gyrating* FrFT's; these are the subject of Section 4, where we prepare the analysis of the generic U(2) case presented in Section 5. The U(2)-FrFT's are characterized by two powers: α, β mod 4 on a torus (the order), and one axis $\vec{r}(\vartheta, \varphi)$ on the sphere (the type). In Section 6 we detail the action of U(2)-FrFT's on phase space and their composition.

The group of 2D FrFT's can be represented by complex unitary 2×2 matrices, by real symplectic 4×4 matrices, and as unitary integral transforms in a Hilbert space of optical (or quantum) wave fields. We use the standard generators of the Lie algebra,[9] which are quadratic functions of the phase-space coordinates and, in the wave/quantum representation, are second-order differential operators.[10,11] In Section 7 we identify the eigenfunctions of these generators; we thereby determine the U(2) Fourier integral kernel as their bilinear generating function. In Section 8 we discuss the Iwasawa decomposition. In Section 9 the U(2)-FrFT is explicitly displayed as a canonical transform[12]; the coefficients are determined from the torus and sphere parameters. In Section 10 we recapitulate the main results of the paper and indicate their generalization to N dimensions.

2. ROOTS OF THE FOURIER TRANSFORM

Properly speaking, a fractional FT is any operator $\mathcal{F}^{1/n}$ such that its nth power $(\mathcal{F}^{1/n})^n = \mathcal{F}$ is the FT. (In the preceding statement $1/n$ should be viewed simply as a superscript, and not as an exponent.) Thus $\mathcal{F}^{1/n}$ can be called a root or fraction of the FT. Even in one dimension we can expect that there will be more than one nth root for the FT, since in the case of complex numbers there are n of them. In higher dimensions there will be a corresponding increase in the number of transformations whose nth power is the official FT (1). And all of them qualify to be called FrFT's.

A. One-Dimensional Case

First we illustrate the issues, using the $N = 1$ case corresponding to one transverse dimension. This will indicate the nature of the generalization to two and higher dimensions. It is well known that the FT $\mathcal{F}$ is a $(1/2)\pi$ rigid rotation of the optical phase-space plane $(q, p)^{\mathsf{T}}$ (we use the superscript T to indicate transposition) around the origin. There it is represented ($\mapsto$) by the 2×2 matrix

$$\mathcal{F} \mapsto \mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3}$$

We may embed $\mathcal{F}$ as a particular element of the closed one-parameter Lie group

$$\mathrm{U}(1) = \left\{ \mathcal{F}^{\alpha} \mapsto \mathbf{F}^{\alpha} = \begin{bmatrix} \cos\frac{\pi}{2}\alpha & \sin\frac{\pi}{2}\alpha \\ -\sin\frac{\pi}{2}\alpha & \cos\frac{\pi}{2}\alpha \end{bmatrix} \middle| \, 0 \leqslant \alpha < 4 \right\}, \tag{4}$$

where it is clear that the nth roots of $\mathcal{F}$ are given by $\mathcal{F}^{\alpha_m}$, with

$$\alpha_m = \frac{1}{n} + 4\frac{m}{n}, \qquad m = 0, 1,..., n - 1. \tag{5}$$

The summand with the factor 4 arises because $\mathcal{F}^4 = \mathbf{1}$. There is a principal root $m = 0$ with a fractional power

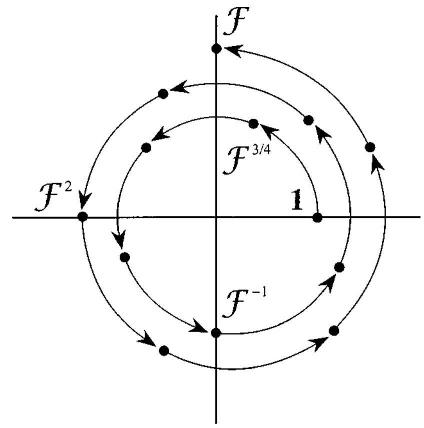

Fig. 1. Circle of FrFT's $\mathcal{F}^{\alpha}$, α counted modulo 4. Powers of $\mathcal{F}^{3/4}$ successively fall on $(\mathcal{F}^{3/4})^4 = \mathcal{F}^{-1}$, $(\mathcal{F}^{3/4})^8 = \mathcal{F}^2$, and $(\mathcal{F}^{3/4})^{12} = \mathcal{F}$. Next would come $(\mathcal{F}^{3/4})^{16} = \mathbf{1}$.

$\alpha_0 = 1/n$, while the roots $\alpha_m = (1 + 4m)/n$ require m windings around the circle $0 \leqslant (\pi/2)\alpha < 2\pi$ before falling on the FT $\mathcal{F}$. See Fig. 1.

We should now ask, For what values of the rational power n_1/n_2 with n_1, n_2 integers and $(n_1, n_2) = 1$ (i.e., n_1 prime relative to n_2) does $\mathcal{F}^{n_1/n_2}$ (where n_1/n_2 should be now viewed as an exponent, and not simply as a superscript) qualify to be called a FrFT? Examining Eqs. (5), we can see that there are four cases to be considered. If $n_1 \equiv 1 \bmod 4$, then $(\mathcal{F}^{n_1/n_2})^n$, the integer powers of $\mathcal{F}^{n_1/n_2}$ for $n = 1, 2,...,$ (eventually) pass through $\mathcal{F}$, $\mathcal{F}^2$, $\mathcal{F}^3 = \mathcal{F}^{-1}$, $\mathbf{1}$ at $n = n_2, 2n_2, 3n_2, 4n_2$, so that $\mathcal{F}^{n_1/n_2}$ is an (n_2)th root of $\mathcal{F}$. Similarly, if $n_1 \equiv 3 \bmod 4$, the sequence $(\mathcal{F}^{n_1/n_2})^n$ passes through $\mathcal{F}^3 = \mathcal{F}^{-1}$, $\mathcal{F}^2$, $\mathcal{F}$, $\mathbf{1}$ at $n = n_2, 2n_2, 3n_2, 4n_2$, so that $\mathcal{F}^{n_1/n_2}$ is a $(3n_2)$th root of $\mathcal{F}$. If $n_1 \equiv 2 \bmod 4$, the sequence $(\mathcal{F}^{n_1/n_2})^n$ passes through the inversion $\mathcal{F}^2$ at $n = n_2$ and through $\mathbf{1}$ at $n = 2n_2$ but never visits $\mathcal{F}$ or $\mathcal{F}^{-1}$. Finally, if $n_1 \equiv 4 \bmod 4$ (i.e., $n_1 \equiv 0 \bmod 4$), the sequence visits $\mathbf{1}$ at $n = n_2$ but never visits $\mathcal{F}$ or its second and third powers. Thus we conclude with the following theorem:

Theorem 1. Let α be a rational number of the form n_1/n_2, with $(n_1, n_2) = 1$. $\mathcal{F}^{\alpha}$ is a root of the FT (in the strict sense that some integer power of it yields $\mathcal{F}$) if and only if n_1 is odd.

The interesting point is that not all rational values of α qualify. Even so, the values of α that do qualify form a dense subset of the U(1) circle $0 \leqslant \alpha < 4$. With continuous representations in mind (by matrices or integral kernels), we make a deliberate logical jump and simply call $\mathcal{F}^{\alpha}$ the FrFT of order α, for every $0 \leqslant \alpha < 4$ (cf. Ref. 13). That such a logical jump is being undertaken should be appreciated, particularly in view of generalizations to higher dimensions.

B. Two-Dimensional Case

In the case $N = 2$ of two transverse dimensions we arrange the entries according to the phase-space coordinates $\mathbf{v} = (\mathbf{q}, \mathbf{p})^{\mathsf{T}}$. Clearly, the first-order or paraxial optical systems are then represented by 4×4 matrices acting as linear canonical transformations on the optical phase space or, equivalently, as operator kernels acting on the Hilbert space $\mathcal{L}^2(\Re^2)$ of field amplitudes that are square integrable over the transverse plane $\Re^2$.

One can naïvely generalize the known 1D results and define two separate 1D FrFT's acting on the two transverse directions, x and y, respectively. Such a 2D FrFT is a direct sum when viewed as a 4×4 matrix (but is a direct product when viewed as an operator kernel). With the phase-space coordinates arranged as $(q_x, q_y, p_x, p_y)^\mathsf{T}$, the official FT (1) is represented by the matrix

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{6}$$

As in the 1D case, we may embed the above matrix as a particular element of the two-parameter compact Abelian Lie group U(1) × U(1) comprising matrices of the form

$$\mathbf{F}^{\alpha,\beta} = \begin{bmatrix} \cos\frac{\pi}{2}\alpha & 0 & \sin\frac{\pi}{2}\alpha & 0 \\ 0 & \cos\frac{\pi}{2}\beta & 0 & \sin\frac{\pi}{2}\beta \\ -\sin\frac{\pi}{2}\alpha & 0 & \cos\frac{\pi}{2}\alpha & 0 \\ 0 & -\sin\frac{\pi}{2}\beta & 0 & \cos\frac{\pi}{2}\beta \end{bmatrix}. \tag{7}$$

The range of the angles is $0 \leqslant \alpha,\ \beta < 4$, consistent with the fact the group manifold of U(1) × U(1) is the 2D torus $\mathcal{T}^2$.

As in the 1D case, we may now ask, For the corresponding transformation $\mathcal{F}^{\alpha,\beta}$ with

$$\alpha = \frac{n_x}{n}, \qquad \beta = \frac{n_y}{n}, \qquad (n_x, n) = 1, \qquad (n_y, n) = 1, \tag{8}$$

for what (rational) values of α, β does $\mathcal{F}^{\alpha,\beta}$ qualify to be called a FrFT in the strict sense? It is clear from the 1D discussion that neither n_x nor n_y should be $0 \bmod 4$ or $2 \bmod 4$. Assume that, while one of n_x, n_y is $1 \bmod 4$, the other is $3 \bmod 4$. Then the sequence $(\mathcal{F}^{\alpha,\beta})^k$, $k = 1, 2, \ldots$ will visit the inverse FT in x when it visits the FT in y, and it will visit the inverse FT in y when it visits the FT in x, so the sequence never passes through the official FT. Thus we are led to the following characterization:

Theorem 2. Let α, β be rationals of the form $\alpha = n_x/n$, $\beta = n_y/n$, and $(n_x, n) = 1 = (n_y, n)$. $\mathcal{F}^{\alpha,\beta}$ is a root of the 2D FT (1) (in the strict sense that some integer power of it yields the 2D $\mathcal{F}$) if and only if both n_x and n_y equal $1 \bmod 4$ or both equal $3 \bmod 4$.

Having pinned down this restriction on α, β, we notice that the set of values permitted by the above criterion forms a dense subset of the torus $\mathcal{T}^2 = \{(\alpha, \beta) | 0 \leqslant \alpha, \beta < 4\}$. As in the 1D case, we may now make a logical jump and call $\mathcal{F}^{\alpha,\beta}$ a FrFT of order α, β for every $(\alpha, \beta) \in \mathcal{T}^2$.

It should be appreciated, however, that this toroidal generalization to the 2D case is naïve, for it is devoid of any genuinely 2D richness. The principal aim of this study is to go beyond this naïve generalization and to define the FrFT in a manner that captures the increasing richness of the higher-dimensional cases.

3. MATRIX AND INTEGRAL REPRESENTATIONS

In this section we shall work with matrices; their relationship to integral transforms will be resolved in Section 9. It is well known in the literature[14] that linear canonical transformations between N position coordinates and their conjugate N momentum coordinates are represented by $2N \times 2N$ real symplectic matrices $\mathbf{M}$, which satisfy

$$\mathbf{M}\mathbf{F}\mathbf{M}^\mathsf{T} = \mathbf{F}, \qquad \mathbf{F} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}, \tag{9}$$

where $\mathbf{F}$ is the symplectic metric matrix for $2N$ Cartesian coordinates, with $N \times N$ zero and unit blocks; for $N = 1$ and $N = 2$, $\mathbf{F}$ is the Fourier matrix (3) and (6), respectively. The Fourier matrix plays a fundamental dual role as the defining metric and as the symplectic transformation that rotates by $(1/2)\pi$ each one of the N canonical position–momentum pairs in phase space. Also, $\mathbf{F}$ represents $\mathcal{F}$, the official Fourier integral transform (1) on the Hilbert space $\mathcal{L}^2(\Re^N)$ of optical wave fields.

Through multiplication, the manifold of matrices that satisfy Eqs. (9) form the $2N$-dimensional real symplectic group, denoted by $\mathrm{Sp}(2N, \Re)$. When the matrix $\mathbf{M}$ is written in block form,

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \tag{10}$$

the $N \times N$ blocks obey the symplectic conditions

$$\mathbf{A}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{A}^\mathsf{T} = \mathbf{0}, \qquad \mathbf{A}\mathbf{D}^\mathsf{T} - \mathbf{B}\mathbf{C}^\mathsf{T} = \mathbf{1},$$
$$\mathbf{C}\mathbf{B}^\mathsf{T} - \mathbf{D}\mathbf{A}^\mathsf{T} = -\mathbf{1}, \qquad \mathbf{C}\mathbf{D}^\mathsf{T} - \mathbf{D}\mathbf{C}^\mathsf{T} = \mathbf{0}. \tag{11}$$

These conditions entail $2N^2 - N$ constraints and leave $2N^2 + N$ free parameters. In the case considered here, $N = 2$ for 3D optics (two transverse dimensions), the manifold of these matrices forms the real 10D symplectic group $\mathrm{Sp}(4, \Re)$ (Refs. 9, 15, and 16). From Eqs. (9) it follows that the determinant of $\mathbf{M}$ can be ± 1; further analysis (see, e.g., Ref. 6) shows that only $+1$ can occur.

Consider now the subset of symplectic matrices $\mathbf{O}$ that commute with $\mathbf{F}$, i.e., $\mathbf{FO} = \mathbf{OF}$. From Eqs. (9) we can see that $\mathbf{O}\mathbf{O}^\mathsf{T} = \mathbf{1}$ this characterizes $\mathbf{O}$ as a $2N \times 2N$ orthogonal matrix (of unit determinant), indicated as an element of the group $\mathrm{SO}(2N)$. When $\mathbf{O}$ is written in the block form (10), the blocks will satisfy

$$\mathbf{A}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T} = \mathbf{1}, \qquad \mathbf{A}\mathbf{C}^\mathsf{T} + \mathbf{B}\mathbf{D}^\mathsf{T} = \mathbf{0},$$
$$\mathbf{C}\mathbf{A}^\mathsf{T} + \mathbf{D}\mathbf{B}^\mathsf{T} = \mathbf{0}, \qquad \mathbf{C}\mathbf{C}^\mathsf{T} + \mathbf{D}\mathbf{D}^\mathsf{T} = \mathbf{1}. \tag{12}$$

The set of matrices that are both symplectic and orthogonal is also a group of matrices, whose $N \times N$ blocks satisfy both Eqs. (11) and (12). We can see the structure of this group by writing and verifying that

$$\mathbf{U}(\mathbf{O}) = \mathbf{A} + i\mathbf{B} = \mathbf{D} - i\mathbf{C}, \qquad \mathbf{U}\mathbf{U}^{\dagger} = \mathbf{1} = \mathbf{U}^{\dagger}\mathbf{U}, \tag{13}$$

$$\mathbf{O}(\mathbf{U}) = \begin{bmatrix} \mathrm{Re}\,\mathbf{U} & \mathrm{Im}\,\mathbf{U} \\ -\mathrm{Im}\,\mathbf{U} & \mathrm{Re}\,\mathbf{U} \end{bmatrix}, \qquad \mathbf{O}\mathbf{O}^{\mathsf{T}} = \mathbf{1} = \mathbf{O}^{\mathsf{T}}\mathbf{O}, \tag{14}$$

$$\mathbf{U}(\mathbf{O}_1)\mathbf{U}(\mathbf{O}_2) = \mathbf{U}(\mathbf{O}_1\mathbf{O}_2), \qquad \mathbf{O}(\mathbf{U}_1)\mathbf{O}(\mathbf{U}_2) = \mathbf{O}(\mathbf{U}_1\mathbf{U}_2). \tag{15}$$

The matrices $\mathbf{U}(\mathbf{O})$ and $\mathbf{O}(\mathbf{U})$ are N-dimensional and $2N$-dimensional representations, respectively, of one and the same group element. Therefore the intersection $\mathrm{Sp}(2N, \mathfrak{R}) \cap \mathrm{SO}(2N)$ is $\mathrm{U}(N)$ (the group of $N \times N$ unitary matrices), which has an $N \times N$ complex representation by unitary matrices and a $2N \times 2N$ real representation by symplectic orthogonal matrices.[11,16] This group $\mathrm{U}(N)$ is the commutant in $\mathrm{Sp}(2N, \mathfrak{R})$ of the official FT matrix $\mathbf{F}$.

It is important to appreciate that the $N \times N$ unitary matrix that corresponds, through Eqs. (13), to the FT $\mathbf{F}$ is $\mathbf{U}(\mathbf{F}) = i\mathbf{1}$, a scalar fourth root of the identity matrix $\mathbf{1}$ characterized by the phase $\exp(i\pi/2)$. This observation leads us to define a distinguished subgroup of central FrFT's by the matrices that are phase multiples of the unit: $\exp(i\pi\alpha/2)1 \in \mathrm{U}(1)$. These commute among themselves and with all $\mathrm{U}(N)$ matrices in Eqs. (13) to constitute the center of $\mathrm{U}(N)$. We also recall that the manifold decomposition of the unitary groups is $\mathrm{U}(N) = \mathrm{U}(1) \times \mathrm{SU}(N)/Z_N$, where $\mathrm{U}(1)$ is the commuting center of phases, $\mathrm{SU}(N)$ is the special unitary subgroup that contains all matrices of unit determinant, and we have divided the direct product manifold by $Z_N = \{\exp(2\pi ik/N)\mathbf{1}\}_{k=0}^{N-1}$, which is the subset of matrices common to both factors. The unitary and the symplectic representations of the central FrFT's (henceforth indicated by a subindex 0) are

$$\mathbf{U}(\mathbf{F}_0^{\alpha}) = \exp[i(\pi/2)\alpha]\mathbf{1},$$

$$\mathbf{F}_0^{\alpha} = \begin{bmatrix} \cos\frac{\pi}{2}\alpha\mathbf{1} & \sin\frac{\pi}{2}\alpha\mathbf{1} \\ -\sin\frac{\pi}{2}\alpha\mathbf{1} & \cos\frac{\pi}{2}\alpha\mathbf{1} \end{bmatrix}. \tag{16}$$

Evidently, $\mathbf{F}_0^{\alpha_1}\mathbf{F}_0^{\alpha_2} = \mathbf{F}_0^{\alpha_1+\alpha_2} = \mathbf{F}_0^{\alpha_2}\mathbf{F}_0^{\alpha_1}$, $\mathbf{F}_0^{\alpha} = \mathbf{F}$ for $\alpha \equiv 1 \bmod 4$, and $\mathbf{F}_0^{\alpha} = \mathbf{1}$ for $\alpha \equiv 0 \bmod 4$. We have detailed the matrix construction here, but the definition of the central FrFT is coordinate free.

When we consider lenses and free spaces (positive displacements) as the basic building blocks of paraxial optical systems,[6,8,17] the assembly of central Fourier transformers is reduced to the problem of writing the fractional Fourier matrix (16) in terms of the triangular matrices

$$\mathbf{L}(\mathbf{g}) = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{g} & \mathbf{1} \end{bmatrix}, \qquad \mathbf{D}(z) = \begin{bmatrix} \mathbf{1} & z\mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \tag{17}$$

where $\mathbf{g}$ is the symmetric matrix of Gaussian astigmatic lenses, which is a multiple of the 2×2 unity only when the lens is axis symmetric, and $z \geqslant 0$ is the displacement distance. The central FrFT can be built with a symmetric system $F_0^{\alpha} = L_g D_z L_g$, which we obtain by equating the corresponding matrix product with Eq. (16); for $0 \leqslant \alpha < 2$, this results in $z = \sin(\pi/2)\alpha \in [0, 1]$ and $g = \tan(\pi/4)\alpha \in [0, \infty)$. Similarly, a symmetric system composed of a single lens within a total length z', $F_0^{\alpha} = D_{(1/2)z'}L_{g'}D_{(1/2)z'}$, will yield Eq. (16), resulting in $z' = 2\tan(\pi/4)\alpha \in [0, \infty)$ and $g' = \sin(\pi/2)\alpha \in [0, 1]$ for the same range of α. Concatenation of several such systems covers the full group of central FrFT's; this we denote by $\mathrm{U}_0(1)$.

In paraxial wave optics, lenses and displacements multiply the wave field by a quadratic phase and act with the Fresnel transform, respectively:

$$(\mathcal{L}_{\mathbf{g}}\psi)(\mathbf{q}) = \exp(i\,\tfrac{1}{2}\,\mathbf{q}^{\mathsf{T}}\mathbf{g}\mathbf{q})\psi(\mathbf{q}), \tag{18}$$

$$\begin{aligned}(\mathcal{D}_z\psi)(\mathbf{q}) &= \frac{\exp[-i(z + \frac{1}{2}\pi)]}{2\pi z}\int_{\mathfrak{R}^2}\mathrm{d}\mathbf{q}' \times \exp\left[\frac{i}{2z}(\mathbf{q} - \mathbf{q}')^2\right]\psi(\mathbf{q}') \\ &= \exp(-iz)\exp(i\mathbf{q}^2/2z)\frac{\exp(-i\pi/2)}{2\pi z}\int_{\mathfrak{R}^2}\mathrm{d}\mathbf{q}' \times \exp\left(-\frac{i}{z}\mathbf{q}\cdot\mathbf{q}'\right)\exp(i\mathbf{q}'^2/2z)\psi(\mathbf{q}').\end{aligned} \tag{19}$$

The integral transform produced by the $L_g D_z L_g$ system of the previous paragraph will then act [up to the phase $\exp(iz)$] as the 2D central integral FrFT, which we define also by using the index 0, as

$$(\mathcal{F}_0^{\alpha}\psi)(\mathbf{q}) = \exp(iz)\exp(i\,\tfrac{1}{2}\,\pi\alpha) \times [\mathcal{L}_{\tan(\pi/4)\alpha}\mathcal{D}_{\sin(\pi/2)\alpha}\mathcal{L}_{\tan(\pi/4)\alpha}\psi](\mathbf{q}) \tag{20}$$

$$= \int_{\mathfrak{R}^2}\mathrm{d}\mathbf{q}' F_0^{\alpha}(\mathbf{q}, \mathbf{q}')\psi(\mathbf{q}'), \tag{21}$$

$$F_0^{\alpha}(\mathbf{q}, \mathbf{q}') = \frac{\exp[i\,\frac{1}{2}\,\pi(\alpha - 1)]}{2\pi\sin(\pi/2)\alpha}\exp i\left[\frac{\mathbf{q}^2}{2\tan(\pi/2)\alpha} - \frac{\mathbf{q}\cdot\mathbf{q}'}{\sin(\pi/2)\alpha} + \frac{\mathbf{q}'^2}{2\tan(\pi/2)\alpha}\right]. \tag{22}$$

This integral kernel is well known for the 1D case,[18] where the phase in front is $\exp(-i\pi/4)$ and a square root appears in the denominator.[10] In the 2D case we obtain the official FT (1) when $\alpha = 1$.

4. SEPARABLE AND GYRATING FOURIER TRANSFORMS

In the case that interests us, $N = 2$, we define the x–y-separable FT's $\mathbf{F}_1^{\alpha,\beta}$ [cf. Eq. (1)] by

$$\mathbf{U}(\mathbf{F}_1^{\alpha,\beta}) = \begin{bmatrix} \exp\left(i\frac{\pi}{2}\alpha\right) & 0 \\ 0 & \exp\left(i\frac{\pi}{2}\beta\right) \end{bmatrix}$$

$$= \exp\left[i\frac{\pi}{4}(\alpha+\beta)\right]$$

$$\times \begin{bmatrix} \exp\left[i\frac{\pi}{4}(\alpha-\beta)\right] & 0 \\ 0 & \exp\left[-i\frac{\pi}{4}(\alpha-\beta)\right] \end{bmatrix}$$

$$= \exp\left\{i\frac{\pi}{4}(\alpha+\beta)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$\times \exp\left\{i\frac{\pi}{4}(\alpha-\beta)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}, \tag{23}$$

$$\mathbf{F}_1^{\alpha,\beta} = \begin{bmatrix} \cos\frac{\pi}{2}\alpha & 0 & \sin\frac{\pi}{2}\alpha & 0 \\ 0 & \cos\frac{\pi}{2}\beta & 0 & \sin\frac{\pi}{2}\beta \\ -\sin\frac{\pi}{2}\alpha & 0 & \cos\frac{\pi}{2}\alpha & 0 \\ 0 & -\sin\frac{\pi}{2}\beta & 0 & \cos\frac{\pi}{2}\beta \end{bmatrix}. \tag{24}$$

This is a 2D torus $\mathcal{T}^2$ of matrices; see Fig. 2. The x–y-separable fractional Fourier matrix (24) acting on 4D acting on 4D phase space $\mathbf{v} = (q_x, q_y, p_x, p_y)^\top$ is a rotation in the (q_x, p_x) plane by $(\pi/2)\alpha$ and a separate rotation in the (q_y, p_y) plane by $(\pi/2)\beta$. Separable FT's thus form a group that we denote by $\mathrm{U}_x(1) \times \mathrm{U}_y(1)$, with commutative product $\mathbf{F}_1^{\alpha_1,\beta_1}\mathbf{F}_1^{\alpha_2,\beta_2} = \mathbf{F}_1^{\alpha_1+\alpha_2,\beta_1+\beta_2}$ with exponents modulo 4, and the official FT is $\mathbf{F}_1^{\alpha,\beta} = \mathbf{F}$ for $\alpha, \beta \equiv 1 \bmod 4$. The factorization (23) shows that the manifold of separable $\mathbf{F}_1^{\alpha,\beta}$'s is a $(\pi/4)(\alpha+\beta)$ circle of central FT's, in direct product with a $(\pi/4)(\alpha-\beta)$ circle, where the rotations in the (q_x, p_x) and the (q_y, p_y) planes are equal and opposite. We can thus characterize the torus group $\mathrm{U}_x(1) \times \mathrm{U}_y(1)$ of separable FT's in the isomorphic form $\mathrm{U}_0(1) \times \mathrm{U}_1(1)$ also, which is better suited for further generalization.

To build x–y-separable Fourier transformers we can use $L_{g_x}D_zL_{g_x}$ systems with cylindrical lenses of powers g_x in the x direction, sharing the same total length z with a cylindrical lens system in the y direction, composed as $L_{g_y'}D_{1/2z}L_{g_y''}D_{1/2z}L_{g_y'}$, i.e., with two cylindrical lenses g_y' at the ends and one g_y'' in the middle.[1,6,8,17] In the x direction we can have a FrFT of order α, choosing $z = \sin(\pi/2)\alpha$ and $g_x = \tan(\pi/4)\alpha$. Then in the y direction we can choose another order β, provided that g_y' and g_y'' satisfy $g_y'\sin(\pi/2)\beta + (1/2)zg_y'' = 1 - \cos(\pi/2)\beta$, as can be seen through multiplication of the five matrices (17). Correspondingly, the integral transform and its kernel [cf. Eqs. (21) and (22), with a similar phase] will be

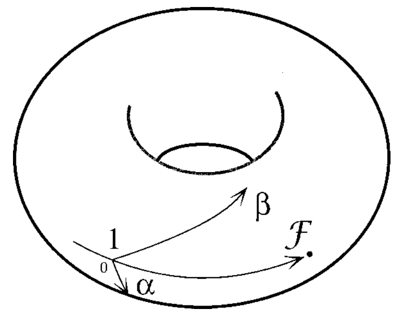

Fig. 2. Torus $\mathcal{T}^2$ of x–y-separable FT's $\mathbf{F}_1^{\alpha,\beta}$. The unit transform is at $\mathbf{1} = \mathbf{F}_1^{0,0}$, and the official FT is at $\mathbf{F} = \mathbf{F}_1^{n_1,n_2}$, with $n_1, n_2 \equiv 1 \bmod 4$. The $\alpha = \beta$ circle is composed of central FT's. For a given (α_0, β_0), the continuous powers $(t\alpha_0, t\beta_0)$ may or may not pass through the official transform.

$$(\mathcal{F}_1^{\alpha,\beta}\psi)(\mathbf{q}) = \int_{\mathfrak{R}^2} \mathrm{d}\mathbf{q}' F_1^{\alpha,\beta}(\mathbf{q},\mathbf{q}')\psi(\mathbf{q}'), \tag{25}$$

$$F_1^{\alpha,\beta}(\mathbf{q},\mathbf{q}') = \frac{\exp\{i(1/2)\pi[(1/2)(\alpha+\beta)-1]\}}{2\pi[\sin(\pi/2)\alpha\sin(\pi/2)\beta]^{1/2}} \times \exp i\left[\frac{q_x^2+q_x'^2}{2\tan(\pi/2)\alpha} - \frac{q_xq_x'}{\sin(\pi/2)\alpha} + \frac{q_y^2+q_y'^2}{2\tan(\pi/2)\beta} - \frac{q_yq_y'}{\sin(\pi/2)\beta}\right]. \tag{26}$$

It is evident in Eq. (23) that this transform is a central FrFT of power $(1/2)(\alpha+\beta)$ in the $\mathrm{U}_0(1)$ cycle, multiplied by an element of the $\mathrm{U}_1(1)$ cycle of FT's that act in the direction x with power $(1/2)(\alpha-\beta)$ and in the direction y with power $-(1/2)(\alpha-\beta)$.

Other kinds of FrFT's can be produced in $N = 2$ dimensions. We now define the gyrating FrFT's $\mathbf{F}_3^{\alpha,\beta}$, again labeled by the points of a 2D torus, by the matrices

$$\mathbf{U}(\mathbf{F}_3^{\alpha,\beta}) = \exp\left\{i\frac{\pi}{4}(\alpha+\beta)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \times \exp\left\{i\frac{\pi}{4}(\alpha-\beta)\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\right\}, \tag{27}$$

$$\mathbf{U}(\mathbf{F}_3^{\alpha,\beta}) = \exp\left[i\frac{\pi}{4}(\alpha+\beta)\right]\mathbf{R}\left[\frac{\pi}{4}(\alpha-\beta)\right],$$

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{28}$$

$$\mathbf{F}_3^{\alpha,\beta} = \begin{bmatrix} \cos\left[\frac{\pi}{4}(\alpha+\beta)\right]\mathbf{R}\left[\frac{\pi}{4}(\alpha-\beta)\right] & \sin\left[\frac{\pi}{4}(\alpha+\beta)\right]\mathbf{R}\left[\frac{\pi}{4}(\alpha-\beta)\right] \\ -\sin\left[\frac{\pi}{4}(\alpha+\beta)\right]\mathbf{R}\left[\frac{\pi}{4}(\alpha-\beta)\right] & \cos\left[\frac{\pi}{4}(\alpha+\beta)\right]\mathbf{R}\left[\frac{\pi}{4}(\alpha-\beta)\right] \end{bmatrix}. \quad (29)$$

Gyrating FrFT's are direct products of a central FrFT through $(\pi/4)(\alpha+\beta)$ and a joint rotation in the x–y planes of phase space through $(\pi/4)(\alpha-\beta)$; they form the group that we label $\mathrm{U}_0(1)\times\mathrm{U}_3(1)$. The official FT is again $\mathbf{F}=\mathbf{F}_3^{\alpha,\beta}$, for $\alpha,\beta\equiv 1 \bmod 4$. See Fig. 2 once again.

To build gyrating FrFT's we need an optical device $\mathcal{R}_\theta$ that will perform the task of $\mathbf{R}(\theta)$ in Eqs. (27)–(29), namely, the joint rotation of position and momentum planes,

$$(\mathcal{R}_\theta\psi)(\mathbf{q}) = \psi[\mathbf{R}(\theta)^{-1}\mathbf{q}], \quad (30)$$

for $\theta=(\pi/4)(\alpha-\beta)$. This image gyrator was introduced in Ref. 6 (see also Ref. 17); it is built out of two image reflector devices, each of which inverts images across one coordinate axis, set at an angle 2θ. The gyrating FrFT is this gyrator concatenated with a central FrFT, in either order.

In paraxial wave optics the gyrating FrFT (27) is represented by the integral transform with the kernel

$$F_3^{\alpha,\beta}(\mathbf{q},\mathbf{q}') = \frac{\exp\{i(1/2)\pi[(1/2)(\alpha+\beta)-1]\}}{2\pi\sin(\pi/4)(\alpha+\beta)} \times \exp i\left\{\frac{\mathbf{q}^2+\mathbf{q}'^2}{2\tan(\pi/4)(\alpha+\beta)} - \frac{\mathbf{q}\cdot\mathbf{R}[(\pi/4)(\alpha-\beta)]\mathbf{q}'}{\sin(\pi/4)(\alpha+\beta)}\right\}. \quad (31)$$

We recall that, since $\mathbf{R}(\theta)$ is orthogonal, $\mathbf{q}\cdot\mathbf{R}(\theta)\mathbf{q}' = [\mathbf{R}(\theta)^{-1}\mathbf{q}]\cdot\mathbf{q}'$; thus, after integration by parts with the invariant measure $\mathrm{d}\mathbf{q}'$, the matrix $\mathbf{R}(\theta)^{-1}$ will act on the argument of the function that is subject to the central transformation.

5. U(2)-FRACTIONAL FOURIER TRANSFORMS

Separable and gyrating FrFT's are but two special cases within the manifold of all U(2)-FrFT's. Indeed, there is a very suggestive analogy between the 2D FrFT and the Poincaré sphere of polarization optics.[19,20] We recall that, in the latter, circular polarization is associated with the north and the south poles, whereas linear polarizations correspond to points on the equator. Here we show that a similar characterization can be made for FrFT's, inasmuch as the 4×4 symplectic matrices that represent the U(2) group on the coordinates of phase space $\mathbf{v} = (q_x, q_y, p_x, p_y)^\top$ are also orthogonal, so they will leave invariant the $\mathcal{S}^3$ sphere $\mathbf{v}^\top\mathbf{v} = q_x^2+q_y^2+p_x^2+p_y^2$ in phase space.

To keep a standard enumeration of axes, we introduce the $\boldsymbol{\tau}$ matrices (related to the Pauli $\boldsymbol{\sigma}$ matrices, below) and associate them ($\leftrightarrow$) with the well-known Schwinger generators of U(2) that are built from the creation and annihilation coordinates, $a_j=(q_j+ip_j)/\sqrt{2}$ and $a_j^\dagger=(q_j-ip_j)/\sqrt{2}$, respectively, for $j=x,y$. Commutators of operators are associated with the Poisson brackets of their classical functions on the phase-space coordinates.[21] We let

$$\boldsymbol{\tau}_0 = \boldsymbol{\sigma}_0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \leftrightarrow T_0 = a_x^\dagger a_x + a_y^\dagger a_y = \tfrac{1}{2}(p_x^2+p_y^2+q_x^2+q_y^2) - 1, \quad (32)$$

$$\boldsymbol{\tau}_1 = \boldsymbol{\sigma}_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \leftrightarrow T_1 = a_x^\dagger a_x - a_y^\dagger a_y = \tfrac{1}{2}(p_x^2-p_y^2+q_x^2-q_y^2), \quad (33)$$

$$\boldsymbol{\tau}_2 = \boldsymbol{\sigma}_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \leftrightarrow T_2 = a_x^\dagger a_y + a_y^\dagger a_x = p_x p_y + q_x q_y, \quad (34)$$

$$\boldsymbol{\tau}_3 = \boldsymbol{\sigma}_2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \leftrightarrow T_3 = i(a_x^\dagger a_y - a_y^\dagger a_x) = p_y q_x - p_x q_y. \quad (35)$$

Written in these terms, the x–y-separable and gyrating FrFT's, Eqs. (23) and (28), contain exponentials of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_3$, respectively; thus we attached the subindices 1 and 3. Now we generalize this subindex to a point on the sphere $\mathcal{S}^2$ that we shall call the Fourier sphere.

We parameterize each U(2)-FrFT by a fixed unit vector on the Fourier sphere, $\vec{r}(\vartheta,\varphi)\in\mathcal{S}^2$, with Cartesian coordinates $\vec{r}=(r_1,r_2,r_3)=(\sin\vartheta\cos\varphi,\sin\vartheta\sin\varphi,\cos\vartheta)$ and two powers (α,β) counted modulo 4. We let $\mu = (\pi/4)(\alpha+\beta)$ and $\nu=(\pi/4)(\alpha-\beta)$ and define the unitary matrices

$$\mathbf{U}_{\vec{r}}^{\alpha,\beta} = \exp(i\mu\boldsymbol{\tau}_0)\exp(i\nu\vec{r}\cdot\vec{\boldsymbol{\tau}}) = \exp(i\mu)(\mathbf{1}\cos\nu + i\vec{r}\cdot\vec{\boldsymbol{\tau}}\sin\nu) = \exp(i\mu)\begin{bmatrix}\cos\nu+ir_1\sin\nu & (r_3+ir_2)\sin\nu\\ (-r_3+ir_2)\sin\nu & \cos\nu-ir_1\sin\nu\end{bmatrix}. \quad (36)$$

Thus, for every axis $\vec{r}(\vartheta,\varphi)$ on the Fourier sphere, there is a corresponding torus $\mathcal{T}_{\vec{r}}^2$ (with coordinates α, β as in Fig. 2) of 2×2 unitary matrices that satisfy $\mathbf{U}_{\vec{r}}^{\alpha_1,\beta_1}\mathbf{U}_{\vec{r}}^{\alpha_2,\beta_2} = \mathbf{U}_{\vec{r}}^{\alpha_1+\alpha_2,\beta_1+\beta_2}$, $\mathbf{U}_{\vec{r}}^{0,0}=\mathbf{1}$, and $(\mathbf{U}_{\vec{r}}^{\alpha,\beta})^{-1} = \mathbf{U}_{\vec{r}}^{-\alpha,-\beta} = (\mathbf{U}_{\vec{r}}^{\alpha,\beta})^\dagger$. Hence they form a group that we denote by $\mathrm{U}_0(1)\times\mathrm{U}_{\vec{r}}(1)$. In particular, for $\alpha,\beta\equiv 1 \bmod 4$, $\mathbf{U}_{\vec{r}}^{\alpha,\beta}=\mathbf{U}(\mathbf{F})$.

U(2)-FrFT's in two dimensions are defined through Eqs. (14), (36), and

$$\mathbf{F}_{\vec{r}}^{\alpha,\beta} = \mathbf{O}(\mathbf{U}_{\vec{r}}^{\alpha,\beta}) = \mathbf{O}[\exp(i\mu)\mathbf{1}]\mathbf{O}[\exp(i\nu\vec{r}\cdot\vec{\boldsymbol{\tau}})]. \quad (37)$$

By construction, this manifold of U(2) transforms is $\mathcal{T}^2 \times \mathcal{S}^2$, the product of the torus with the Fourier sphere. Under U(2) group action this manifold is further revealed to be a fiber bundle,[22] whose base space is the sphere and whose fibers are the tori. We can call Eq. (37) a 2D fractional FT with the following justification:

Theorem 3. Let α,β be rationals of the form $\alpha = n_x/n$, $\beta = n_y/n$, and $(n_x, n) = 1 = (n_y, n)$. $\mathbf{F}_{\vec{r}}^{\alpha,\beta}$ is a root of the 2D FT matrix $\mathbf{F}$ (in the strict sense that some integer power of it yields $\mathbf{F}$) if and only if both n_x and n_y are equal to 1 mod 4 or both are equal to 3 mod 4. And this is true for every $\vec{r} \in \mathcal{S}^2$.

The proof is straightforward. Using 2×2 unitary matrices, for the separable case $\vec{r} = (1, 0, 0)$, we can see that $\mathbf{U}_1^{\alpha,\beta} = \mathbf{U}(\mathbf{F}_1^{\alpha,\beta})$, where α, β are rationals of the form specified in Theorem 2, is a root of the 2D FT. Now, in SU(2) any $\mathbf{U}_{\vec{r}}^{\alpha,\beta}$ is conjugate to $\mathbf{U}_1^{\alpha,\beta}$; i.e., given a $\mathbf{U}_{\vec{r}}^{\alpha,\beta}$, there exists a $\mathbf{U} \in \mathrm{SU}(2)$ such that $\mathbf{U}_{\vec{r}}^{\alpha,\beta} = \mathbf{U}\mathbf{U}_1^{\alpha,\beta}\mathbf{U}^{-1}$. The nth power of this equality (for n as in the theorem) shows that $(\mathbf{U}_{\vec{r}}^{\alpha,\beta})^n$ is conjugate by $\mathbf{U}$ to the nth power of the separable case, $(\mathbf{U}_1^{\alpha,\beta})^n = \mathbf{U}(\mathbf{F}^{1,1})$, which commutes (by definition) with any $\mathbf{U} \in \mathrm{SU}(2)$. Hence $(\mathbf{U}_{\vec{r}}^{\alpha,\beta})^n = \mathbf{U}(\mathbf{F})$, and the same equality follows for the symplectic orthogonal 4×4 matrices: $(\mathbf{F}_{\vec{r}}^{\alpha,\beta})^n = \mathbf{F}$. The restriction to some rational values applies only to the superscripts α, β, with no restriction whatsoever on $\vec{r}$. This completes the proof.

The restriction on α, β is of the same type as for $\mathcal{F}^{\alpha}$ in the 1D case of Theorem 1 and in the case of the earlier separable 2D generalization $\mathcal{F}^{\alpha,\beta}$ in Theorem 2. Therefore we may make the same logical jump as in the previous two cases and call $\mathbf{F}_{\vec{r}}^{\alpha,\beta}$, for every $0 \leqslant \alpha$, $\beta < 4$, a 2D FrFT matrix. It will be appreciated that the earlier, straightforward separable case corresponds in the U(2) picture to just one point on the Fourier sphere $\mathcal{S}^2$, namely, the point $\vec{r} = (1, 0, 0)$. We call $\mathbf{F}_{\vec{r}}^{\alpha,\beta}$ a 2D FrFT of order α, β (ranging over $\mathcal{T}^2$) and type $\vec{r}$ (ranging over $\mathcal{S}^2$). The straightforward treatment covered just one of an $\mathcal{S}^2$-worth continuum of types, all of which qualify as FrFT's. A difference between our treatment of nonseparable 2D FrFT's and the approach in Ref. 23 should be noted, since it appears that the values of their parameters are not bounded.

6. FOURIER SPHERE

We analyze now the rotations of 4D phase space produced by the 4×4 symplectic and orthogonal FrFT matrices, which leave invariant the phase-space three-sphere $\mathcal{S}^3$ of squared radius $\mathbf{v}^{\mathrm{T}}\mathbf{v} = q_x^2 + q_y^2 + p_x^2 + p_y^2 = \text{constant}$. From the results given in Section 5 through the map (14), using μ, ν as in Eqs. (36) and (37) and indicating $s_\mu = \sin\mu, \ldots, c_\nu = \cos\nu$ for brevity, we write the 4×4 U(2)-FrFT matrix factored into a central and an SU(2)-FrFT:

$$\mathbf{F}_{\vec{r}}^{\alpha,\beta} = \mathbf{F}_0^{\frac{1}{2}(\alpha+\beta)}\mathbf{F}_{\vec{r}}^{\frac{1}{2}(\alpha-\beta),-\frac{1}{2}(\alpha-\beta)}$$
$$= \begin{bmatrix} c_\mu\mathbf{1} & s_\mu\mathbf{1} \\ -s_\mu\mathbf{1} & c_\mu\mathbf{1} \end{bmatrix}\begin{bmatrix} c_\nu & r_3 s_\nu & r_1 s_\nu & r_2 s_\nu \\ -r_3 s_\nu & c_\nu & r_2 s_\nu & -r_1 s_\nu \\ -r_1 s_\nu & -r_2 s_\nu & c_\nu & r_3 s_\nu \\ -r_2 s_\nu & r_1 s_\nu & -r_3 s_\nu & c_\nu \end{bmatrix}. \quad (38)$$

Consider first the case in which $\vec{r}$ is in the 1–2 equatorial plane of the Fourier sphere in Fig. 3, where $r_3 = 0$ and $r_1^2 + r_2^2 = 1$. Much insight is gained about the transformations of type $\vec{r} = (\cos\varphi, \sin\varphi, 0)$ by definition of new Cartesian coordinates of phase space that are rotated from the x–y axes by $(1/2)\varphi$, and distinguished by the superscript φ, as follows:

$$\begin{bmatrix} q_x^\varphi \\ q_y^\varphi \\ p_x^\varphi \\ p_y^\varphi \end{bmatrix} = \begin{bmatrix} c_{\varphi/2} & s_{\varphi/2} & 0 & 0 \\ -s_{\varphi/2} & c_{\varphi/2} & 0 & 0 \\ 0 & 0 & c_{\varphi/2} & s_{\varphi/2} \\ 0 & 0 & -s_{\varphi/2} & c_{\varphi/2} \end{bmatrix}\begin{bmatrix} q_x \\ q_y \\ p_x \\ p_y \end{bmatrix}. \quad (39)$$

In the new coordinates the action of the SU(2)-FrFT in the right-hand side of Eq. (38) is

$$\mathbf{F}_{\vec{r}}^{\alpha,-\alpha}: \begin{bmatrix} q_x^\varphi \\ q_y^\varphi \\ p_x^\varphi \\ p_y^\varphi \end{bmatrix} \mapsto \begin{bmatrix} c_\nu & 0 & s_\nu & 0 \\ 0 & c_\nu & 0 & -s_\nu \\ -s_\nu & 0 & c_\nu & 0 \\ 0 & s_\nu & 0 & c_\nu \end{bmatrix}\begin{bmatrix} q_x^\varphi \\ q_y^\varphi \\ p_x^\varphi \\ p_y^\varphi \end{bmatrix}, \quad (40)$$

with the rotation angle $\nu = (1/2)\pi\alpha$ simultaneously in the $(q_x^\varphi, p_x^\varphi)$ and the $(q_y^\varphi, p_y^\varphi)$ planes, but in opposite senses. Thus we find that the action of $\mathbf{F}_{\vec{r}}^{\alpha,-\alpha}$ [with $\vec{r} = (\cos\varphi, \sin\varphi, 0)$ in the equatorial plane of Fig. 3] on the rotated Cartesian coordinates is precisely the separable action of $F_1^{\alpha,-\alpha}$ in the original x–y coordinates. These are the most general separable FrFT's.

On the $\varphi = 0$ meridian of the Fourier sphere, for $\vec{r} = (\sin\vartheta, 0, \cos\vartheta)$, the analog of Eq. (38) shows that rotations take place in the orthogonal planes of q_x and $r_3 q_y$

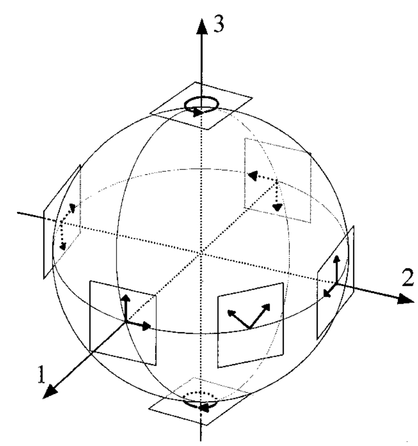

Fig. 3. The Fourier sphere serves to classify U(2) FT's. The 1 axis (on the equator) corresponds to FT's separable in the x–y coordinates. The 3 axis (north pole) corresponds to central FT's that also rotate the image (gyrators). The 2 axis corresponds to cross gyrators in the planes (q_x, p_y) and (q_y, p_x). Around the equator the FT is separable in rotated coordinates.

$+ r_1 p_x$ and of q_y and $-(r_3 q_x + r_1 p_y)$ or, equivalently, of p_x and $-(r_1 q_x - r_3 p_y)$ and of p_y and $r_1 q_y - r_3 p_x$. When $\vartheta = 0$ the north pole of the Fourier sphere represents the gyrating FrFT's in Eq. (29), where the position coordinates rotate among each other and the momentum coordinates do the same accordingly. When $\vartheta = (1/2)\pi$ we are back at the x–y-separable case (24).

Since the product of two U(2)-FrFT's is again a transform of the same group, it is straightforward to prove for Eq. (38) that

$$\mathbf{F}_{\vec{r}}^{\alpha,\beta}\mathbf{F}_{\vec{r}'}^{\alpha',\beta'} = \mathbf{F}_{\vec{r}''}^{\alpha+\alpha',\beta+\beta'}, \tag{41}$$

where the type axes compose through

$$\vec{r}'' s_{\nu+\nu'} = \vec{r} s_\nu c_{\nu'} + \vec{r}' s_{\nu'} c_\nu - (\vec{r} \times \vec{r}') s_\nu s_{\nu'}, \tag{42}$$

with (as before) $\nu = (\pi/4)(\alpha - \beta)$, $s_\nu = \sin\nu$, $c_\nu = \cos\nu$, and similarly for the primed ν's. This composition law may appear reminiscent of the composition of nonparallel Lorentz boosts and the consequent Wigner rotation[24] or the Thomas precession.[25] In the same way, it can be seen that

$$(\mathbf{F}_{\vec{r}}^{\alpha,\beta})^{-1} = \mathbf{F}_{\vec{r}}^{-\alpha,-\beta} = \mathbf{F}_{-\vec{r}}^{-\beta,-\alpha}. \tag{43}$$

The SU(2) axis $\vec{r}$ of a U(2) FrFT can be rotated on the Fourier sphere by means of a similarity transformation of the same kind. From Eq. (36) it follows that

$$\mathbf{F}_{\vec{r}'}^{\gamma,\delta}\mathbf{F}_{\vec{r}}^{\alpha,\beta}(\mathbf{F}_{\vec{r}'}^{\gamma,\delta})^{-1} = \mathbf{F}_{\mathbf{D}(\vec{r}',\theta)\vec{r}}^{\alpha,\beta}, \qquad \theta = (\pi/4)(\gamma - \delta), \tag{44}$$

where $\mathbf{D}$ is a 3×3 representation of the SU(2)-FrFT of axis $\vec{r}'$ and angle θ:

$$\mathbf{D}(\vec{r}', \theta)\vec{r} \cdot \vec{\boldsymbol{\tau}} = \vec{r} \cdot [\mathbf{U}(\mathbf{F}_{\vec{r}'}^{\gamma,\delta})\vec{\boldsymbol{\tau}}\,\mathbf{U}(\mathbf{F}_{\vec{r}'}^{\gamma,\delta})^{-1}]. \tag{45}$$

The order α, β of the FrFT $\mathbf{F}_{\vec{r}}^{\alpha,\beta}$ thus characterizes conjugation classes[9] of U(2). Each conjugation class consists of the Fourier sphere of transforms with all axes $\vec{r}(\vartheta, \varphi) \in \mathcal{S}^2$. These $\mathbf{F}_{\vec{r}}^{\alpha,\beta}$'s can be realized, for every value of α, β, and $\vec{r}$, by a paraxial system consisting of free spaces and thin astigmatic lenses.[6]

7. SPECTRAL REPRESENTATION

We have considered three realizations of the U(2) group of 2D FrFT's: (i) in terms of complex 2×2 unitary matrices; (ii) in terms of real 4×4 matrices; and, in less detail so far, (iii) in terms of integral transform operators acting on wave fields (Sections 3 and 4). Clearly, for complex-valued field amplitudes of light beams, it is realization (iii) that is of direct relevance. We now study the integral operators in some detail.

A. Generators of U(2) Fourier Transforms

A canonical way to describe an operator (and its domain) is to exhibit its eigenfunctions and the associated eigenvalues; this is called the spectral representation of the operator. For the 2D FrFT operators, the spectral representation domain is the Hilbert space $\mathcal{L}^2(\mathfrak{R}^2)$ of complex-valued square-integrable field amplitude functions over $\mathfrak{R}^2$. For clarity we carry out this task first for x–y-separable transforms and then present the general case on the Fourier sphere. We work within the context of coherent-mode decomposition of partially coherent light[26,27] and of two-mode squeezed states.[28]

Of central importance is the isomorphism (32)–(35) between the $\boldsymbol{\tau}$ matrices and the T functions and between the commutators of the former and the Poisson brackets of the latter. The $\boldsymbol{\tau}$ matrices are a set of generators for the Lie algebra[9] of the group U(2) because their commutators are

$$[\boldsymbol{\tau}_0, \boldsymbol{\tau}_k] = 0, \qquad [\boldsymbol{\tau}_j, \boldsymbol{\tau}_k] = 2i\epsilon_{jkl}\boldsymbol{\tau}_l, \qquad j, k, l = 1, 2, 3. \tag{46}$$

Now, from the basic Schrödinger commutators $[\hat{q}_j, \hat{p}_k] = i\delta_{j,k}$, $j, k = x, y$, and $[\hat{q}_j, \hat{q}_k] = 0 = [\hat{p}_j, \hat{p}_k]$, the quadratic phase-space functions T_0, T_1, T_2, T_3 in relations (32)–(35) can be quantized to unique operators[5]; following Ref. 26, we denote these four operators by $\hat{T}_0$, $\hat{T}_1$, $\hat{T}_2$, $\hat{T}_3$ in the Hilbert space $\mathcal{L}^2(\mathfrak{R}^2)$. We first note that $\hat{T}_0$ is the Hamiltonian operator of an isotropic 2D oscillator with the zero-point energy suppressed; it generates joint rotations in the $(\hat{q}_x, \hat{p}_x)$ and the $(\hat{q}_y, \hat{p}_y)$ planes of Schrödinger operators. The additive constant -1 is in principle arbitrary; our choice will be shown below to generate $\mathcal{F}^\alpha$ from $\exp[-i(1/2)\pi\alpha\hat{T}_3]$ without any intervening phase. Also, we recognize $\hat{T}_3$ as the angular-momentum operator,[9] which generates joint rotations in x–y planes of both position and momentum. The four $\hat{T}_\mu$'s generate rotations in the 4D phase space of Schrödinger operators and are self-adjoint in $\mathcal{L}^2(\mathfrak{R}^2)$. They obey the same commutation relations (46) as the $\boldsymbol{\tau}$ matrices[26]:

$$[\hat{T}_0, \hat{T}_k] = 0, \qquad [\hat{T}_j, \hat{T}_k] = 2i\epsilon_{jkl}\hat{T}_l, \qquad j, k, l = 1, 2, 3. \tag{47}$$

Furthermore, their commutators with the Schrödinger operators of position and momentum are defined naturally and will include Eqs. (2). Therefore the group of Hilbert-space transformations generated by these operators is locally isomorphic to the group U(2).

B. Central Transforms

Note that the 4×4 matrices act on the column vector of classical coordinates $\mathbf{v} = (\mathbf{q}, \mathbf{p})^\top$, whereas the transformations generated by relations (32)–(35) act adjointly [see Eqs. (2) and discussion below] on the vector of Schrödinger operators $\hat{\mathbf{v}} = (\hat{\mathbf{q}}, \hat{\mathbf{p}})^\top$.

In particular, for the central fractional transforms (16), we can obtain the integral transforms $\mathcal{F}_0^\alpha$ (21) and (22) by expanding and summing the full series that we indicate below:

$$\begin{bmatrix}\hat{\mathbf{q}}(\alpha)\\ \hat{\mathbf{p}}(\alpha)\end{bmatrix} = \exp\left(-i\frac{1}{2}\pi\alpha\hat{T}_0\right)\begin{bmatrix}\hat{\mathbf{q}}\\ \hat{\mathbf{p}}\end{bmatrix}\exp\left(i\frac{1}{2}\pi\alpha\hat{T}_0\right)$$
$$= \begin{bmatrix}\cos\frac{1}{2}\pi\alpha & \sin\frac{1}{2}\pi\alpha\\ -\sin\frac{1}{2}\pi\alpha & \cos\frac{1}{2}\pi\alpha\end{bmatrix}^{-1}\begin{bmatrix}\hat{\mathbf{q}}\\ \hat{\mathbf{p}}\end{bmatrix} = \mathcal{F}_0^\alpha\begin{bmatrix}\hat{\mathbf{q}}\\ \hat{\mathbf{p}}\end{bmatrix}\mathcal{F}_0^{-\alpha}. \tag{48}$$

The isomorphism among the matrices $\mathbf{F}_0^\alpha$ that act on 4D phase-space coordinate vectors, the unitary matrices $\mathbf{U}(\mathbf{F}_0^\alpha)$, the integral transform operators that act on wave fields, and the exponentials of generators, is thus

$$\mathbf{U}(\mathbf{F}_0^\alpha) = \exp(i\,\tfrac{1}{2}\,\pi\alpha\boldsymbol{\tau}_0) \leftrightarrow \mathcal{F}_0^\alpha = \exp(-i\,\tfrac{1}{2}\,\pi\alpha\hat{T}_0). \quad (49)$$

C. Separable Transforms

Separable FrFT matrices $\mathbf{F}_1^{\alpha,\beta}$ and integral transforms were introduced in Section 4. Their isomorphism can be similarly established from Eqs. (23), (24) and (25), (26) to be

$$\mathbf{U}(\mathbf{F}_1^{\alpha,\beta}) = \exp[i(\pi/4)(\alpha+\beta)\boldsymbol{\tau}_0 + i(\pi/4)(\alpha-\beta)\boldsymbol{\tau}_1]$$
$$\leftrightarrow \mathcal{F}_1^{\alpha,\beta} = \exp[-i(\pi/4)(\alpha+\beta)\hat{T}_0 - i(\pi/4)(\alpha-\beta)\hat{T}_1]. \quad (50)$$

We have immediate use for these relations because the eigenfunctions of the separable FrFT's $\mathcal{F}_1^\alpha$ are easily found by inspection. First, since $\hat{T}_0$ and $\hat{T}_1$ commute, they have a complete set of simultaneous eigenfunctions in $L^2(\Re^2)$. Second, since both $\hat{T}_0$ and $\hat{T}_1$ are linear combinations of $a_x^\dagger a_x$ and $a_y^\dagger a_y$, the required eigenfunctions are products of the well-known real Hermite–Gaussian modes for $n_x, n_y = 0, 1, 2, \ldots$,

$$a_x^\dagger a_x \Psi_{n_x,n_y}(\mathbf{q}) = n_x \Psi_{n_x,n_y}(\mathbf{q}),$$
$$a_y^\dagger a_y \Psi_{n_x,n_y}(\mathbf{q}) = n_y \Psi_{n_x,n_y}(\mathbf{q}), \quad (51)$$

$$[\tfrac{1}{4}(\alpha+\beta)\hat{T}_0 + \tfrac{1}{4}(\alpha-\beta)\hat{T}_1]\Psi_{n_x,n_y}(\mathbf{q}) = \tfrac{1}{2}(n_x\alpha + n_y\beta)\Psi_{n_x,n_y}(\mathbf{q}), \quad (52)$$

$$\Psi_{n_x,n_y}(\mathbf{q}) = \frac{H_{n_x}(q_x)H_{n_y}(q_y)}{[2^{n_x+n_y} n_x! n_y! \pi]^{1/2}} \exp\left(-\frac{1}{2}\mathbf{q}^2\right). \quad (53)$$

These are the eigenvectors of $\mathcal{F}_1^{\alpha,\beta}$ in relation (50) with eigenvalues $\exp[-i(1/2)\pi(n_x\alpha + n_y\beta)]$. Therefore we have the following spectral representation for the kernel (26) as a bilinear generating function,[29]

$$F_1^{\alpha,\beta}(\mathbf{q},\mathbf{q}') = \sum_{n_x,n_y} \Psi_{n_x,n_y}(\mathbf{q}) \times \exp[-i\,\tfrac{1}{2}\,\pi(n_x\alpha + n_y\beta)]\Psi_{n_x,n_y}(\mathbf{q}'). \quad (54)$$

D. General Case

To handle the general U(2) fractional transform $\mathcal{F}_{\vec{r}}^{\alpha,\beta}$, it will be convenient to label the unit vector $\vec{r} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \in \mathcal{S}^2$ by its polar coordinates (ϑ,φ), as $\mathcal{F}_{(\vartheta,\varphi)}^{\alpha,\beta}$. For each type $(\vartheta,\varphi) \in \mathcal{S}^2$, we define the matrix $\mathbf{U}_{(\vartheta,\varphi)}$ and the transform $\mathcal{U}_{(\vartheta,\varphi)}$ [both forming groups U(2)] by[26]

$$\mathbf{U}_{(\vartheta,\varphi)} = \exp(-i\varphi\boldsymbol{\tau}_1)\exp(-i\vartheta\boldsymbol{\tau}_3)\exp(i\varphi\boldsymbol{\tau}_1) = \begin{bmatrix} \cos\frac{1}{2}\vartheta & -\sin\frac{1}{2}\vartheta\exp(-i\varphi) \\ \sin\frac{1}{2}\vartheta\exp(i\varphi) & \cos\frac{1}{2}\vartheta \end{bmatrix} \quad (55)$$

$$\mathcal{U}_{(\vartheta,\varphi)} = \exp(-i\varphi\hat{T}_1)\exp(-i\vartheta\hat{T}_3)\exp(i\varphi\hat{T}_1). \quad (56)$$

The relationship between the 2D FrFT of general type $\mathcal{F}_{(\vartheta,\varphi)}^{\alpha,\beta}$ and the separable one of the same order $\mathcal{F}_1^{\alpha,\beta}$ follows from the isomorphism between the matrices generated by the $\boldsymbol{\tau}_\mu$'s and the transformations generated by the $\hat{T}_j$'s. We let

$$\mathbf{U}(\mathbf{F}_{(\vartheta,\varphi)}^{\alpha,\beta}) = \mathbf{U}_{(\vartheta,\varphi)}\mathbf{U}(\mathbf{F}_1^{\alpha,\beta})\mathbf{U}_{(\vartheta,\varphi)}^{-1}$$
$$\leftrightarrow \mathcal{F}_{(\vartheta,\varphi)}^{\alpha,\beta} = \mathcal{U}_{(\vartheta,\varphi)}\mathcal{F}_1^{\alpha,\beta}\mathcal{U}_{(\vartheta,\varphi)}^{-1}. \quad (57)$$

The detailed discussion of the action of $\mathcal{U}_{(\vartheta,\varphi)}$ on the Hilbert space $\mathcal{L}^2(\Re^2)$ is facilitated by the introduction of the usual angular-momentum index combinations $j = (1/2) \times (n_x + n_y)$, $m = (1/2)(n_x - n_y)$ in place of n_x, n_y, and $\Phi_{j,m}(\mathbf{q}) = \Psi_{j+m,j-m}(\mathbf{q})$. Then

$$\hat{T}_0\Phi_{j,m}(\mathbf{q}) = j\Phi_{j,m}(\mathbf{q}), \qquad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \quad (58)$$

$$\hat{T}_1\Phi_{j,m}(\mathbf{q}) = m\Phi_{j,m}(\mathbf{q}), \qquad m = -j, -j+1, \ldots, j. \quad (59)$$

With this relabeling of the Hermite–Gaussian modes, the mode functions associated with each point $(\vartheta,\varphi) \in \mathcal{S}^2$ are defined by

$$\Phi_{j,m}^{(\vartheta,\varphi)}(\mathbf{q}) = \mathcal{U}_{(\vartheta,\varphi)}\Phi_{j,m}(\mathbf{q}). \quad (60)$$

Since $\Phi_{j,m}(\mathbf{q})$ are the eigenvectors of $\mathcal{F}_1^{\alpha,\beta}$ in relation (50), and since $\mathcal{F}_{(\vartheta,\varphi)}^{\alpha,\beta}$ and $\mathcal{F}_1^{\alpha,\beta}$ are related through a similarity transformation (57) by $\mathcal{U}_{(\vartheta,\varphi)}$, we can conclude that $\Phi_{j,m}^{(\vartheta,\varphi)}(\mathbf{q})$ are the eigenvectors of $\mathcal{F}_{(\vartheta,\varphi)}^{\alpha,\beta}$.

For a unit vector $\vec{r}$ in the direction (ϑ,φ), we use the linear combination of generators $\hat{T}_{(\vartheta,\varphi)} = \vec{r}\cdot\hat{\vec{T}} = r_1\hat{T}_1 + r_2\hat{T}_2 + r_3\hat{T}_3$ that generates rotations about the direction (ϑ,φ):

$$\mathcal{U}_{(\vartheta,\varphi)}\hat{T}_0\mathcal{U}_{(\vartheta,\varphi)}^{-1} = \hat{T}_0, \qquad \hat{T}_0\Phi_{j,m}^{(\vartheta,\varphi)}(\mathbf{q}) = j\Phi_{j,m}^{(\vartheta,\varphi)}(\mathbf{q}),$$
$$\mathcal{U}_{(\vartheta,\varphi)}\hat{T}_1\mathcal{U}_{(\vartheta,\varphi)}^{-1} = \hat{T}_{(\vartheta,\varphi)},$$
$$\hat{T}_{(\vartheta,\varphi)}\Phi_{j,m}^{(\vartheta,\varphi)}(\mathbf{q}) = m\Phi_{j,m}^{(\vartheta,\varphi)}(\mathbf{q}). \quad (61)$$

The operator pair $(\hat{T}_0, \hat{T}_{(\vartheta,\varphi)})$ determines the new modes $\Phi_{j,m}^{(\vartheta,\varphi)}(\mathbf{q})$ in exactly the manner in which the pair $(\hat{T}_0, \hat{T}_1)$ determined the original Hermite–Gauss modes $\Phi_{j,m}(\mathbf{q})$ in Eq. (53).

The mode functions $\Phi_{j,m}^{(\vartheta,\varphi)}(\mathbf{q})$ are linear combinations of the more familiar Hermite–Gaussian modes $\Phi_{j,m}(\mathbf{q})$ through the Wigner formula.[30] We use Dirac's notation $|j, m\rangle$ for the basis of vectors (61), and the Dirac eigenbasis of transverse position $|\mathbf{q}\rangle$, $\mathbf{q} \in \Re^2$. The Hermite–Gaussian modes are thus $\Phi_{j,m}(\mathbf{q}) = \langle\mathbf{q}|j, m\rangle$, and then[26]

$$\Phi_{j,m}^{(\vartheta,\varphi)}(\mathbf{q}) = \sum_{m'=-j}^{j} D_{m,m'}^{j}(\vartheta,\varphi)\Phi_{j,m'}(\mathbf{q}), \quad (62)$$

$$D^j_{m,m'}(\vartheta, \varphi) = \langle j, m|\mathcal{U}_{(\vartheta, \varphi)}|j, m'\rangle = \exp[-i\varphi(m - m')]d^j_{m,m'}(\vartheta), \tag{63}$$

$$d^j_{m,m'}(\vartheta) = \langle j, m|\exp(-i\vartheta \hat{T}_3)|j, m'\rangle = (-1)^{m'-m}\sum_\nu \times \frac{(-1)^\nu[(j+m)!(j-m)!(j+m')!(j-m')!]^{1/2}}{(j-m-\nu)!(j+m'-\nu)!\nu!(m-m'+\nu)!} \times (\cos\tfrac{1}{2}\vartheta)^{2j+m'-m-2\nu}(\sin\tfrac{1}{2}\vartheta)^{m-m'+2\nu}. \tag{64}$$

The spectral representation of the U(2)-FrFT's is the integral transform kernel given by $F^{\alpha,\beta}_{(\vartheta,\varphi)}(\mathbf{q}, \mathbf{q}') = \langle \mathbf{q}|\mathcal{F}^{\alpha,\beta}_{(\vartheta,\varphi)}|\mathbf{q}'\rangle$. [Cf. Eqs. (22), (26), and (31).] The result on bilinear generating functions that generalizes Eq. (54) is Theorem 4.

Theorem 4. The integral kernel of the 2D FrFT of order α, β and type (ϑ, φ) is

$$F^{\alpha,\beta}_{(\vartheta,\varphi)}(\mathbf{q}, \mathbf{q}') = \sum_{j=0,1/2,\ldots}\sum_{m=-j}^{j}\Phi^{(\vartheta,\varphi)}_{j,m}(\mathbf{q}) \times \exp\{-i\tfrac{1}{2}\pi[(j+m)\alpha + (j-m)\beta]\}\Phi^{(\vartheta,\varphi)}_{j,m}(\mathbf{q}')^*, \tag{65}$$

where $\Phi^{(\vartheta,\varphi)}_{j,m}(\mathbf{q})$ are the general mode functions in Eq. (62).

E. Eigenstates of Fourier Transforms

Several comments are required regarding the spectral representation given by Eq. (65), where the order $(\alpha, \beta) \in \mathcal{T}^2$ enters only in the eigenvalues and is nicely separated from the type $(\vartheta, \varphi) \in \mathcal{S}^2$, which enters only in the mode functions.

For any fixed type (ϑ, φ), the mode functions $\Phi^{(\vartheta,\varphi)}_{j,m}(\mathbf{q})$ form a complete and orthonormal basis in $\mathcal{L}^2(\Re^2)$:

$$\int_{\Re^2} d^2\mathbf{q}\,\Phi^{(\vartheta,\varphi)}_{j,m}(\mathbf{q})^*\Phi^{(\vartheta,\varphi)}_{j',m'}(\mathbf{q}) = \delta_{jj'}\delta_{mm'}, \tag{66}$$

$$\sum_{j=0,1/2,\ldots}\sum_{m=-j}^{j}\Phi^{(\vartheta,\varphi)}_{j,m}(\mathbf{q})\Phi^{(\vartheta,\varphi)}_{j,m}(\mathbf{q}')^* = \delta(\mathbf{q} - \mathbf{q}'). \tag{67}$$

Indeed, this property, combined with the fact that the complex eigenvalues in Eq. (65) all have unit magnitude, is equivalent to the unitarity of the operators $\mathcal{F}^{\alpha,\beta}_{(\vartheta,\varphi)}$. On the other hand, for a fixed mode j, m, there is a sphere manifold of types (ϑ, φ), all obtained from the fiducial separable Hermite–Gaussian mode $\Phi_{j,m}(\mathbf{q})$ by means of the transformation $\mathcal{U}_{(\vartheta,\varphi)} \in$ SU(2) in Eq. (60). Thus, for fixed j, m, the $\Phi^{(\vartheta,\varphi)}_{j,m}(\mathbf{q})$'s are overcomplete for $(\vartheta, \varphi) \in \mathcal{S}^2$ and constitute the system of SU(2) coherent states.[26,31]

For select values of (ϑ, φ), the modes $\Phi^{(\vartheta,\varphi)}_{j,m}(\mathbf{q})$ will be familiar to the reader, and SU(2) transformations will express one mode in terms of the others as linear combinations. The $x-y$-separable fiducial Hermite–Gaussian mode $\Psi_{n_x,n_y}(\mathbf{q}) = \Phi_{j,m}(\mathbf{q})$ in Eq. (53) corresponds to the point on the 1 axis because $\hat{T}_1 = \hat{T}_{((1/2)\pi,0)}$. The equator of the sphere $((1/2)\pi, \varphi)$ indicates the separable modes that are eigenstates of

$$\hat{T}_{(\frac{1}{2}\pi,\varphi)} = \cos\varphi\hat{T}_1 + \sin\varphi\hat{T}_2 = \tfrac{1}{2}[(\widehat{p^\varphi_x})^2 + (\widehat{q^\varphi_x})^2] - \tfrac{1}{2}[(\widehat{p^\varphi_y})^2 + (\widehat{q^\varphi_y})^2], \tag{68}$$

where the classical φ-rotated coordinates were defined as in Eq. (39). It follows that the modes $\Phi^{((1/2)\pi,\varphi)}_{j,m}(\mathbf{q})$ are precisely the Hermite–Gaussian modes $\Phi_{j,m}(\mathbf{q})$ in the rotated coordinate system:

$$\Phi^{((1/2)\pi,\varphi)}_{j,m}(q_x, q_y) = \Phi_{j,m}(q^\varphi_x, q^\varphi_y) = \frac{H_{j+m}(q_x\cos\tfrac{1}{2}\varphi + q_y\sin\tfrac{1}{2}\varphi)H_{j-m}(q_y\cos\tfrac{1}{2}\varphi - q_x\sin\tfrac{1}{2}\varphi)}{[n^{2j}(j+m)!(j-m)!\pi]^{1/2}}\exp\left(-\frac{1}{2}\mathbf{q}^2\right). \tag{69}$$

Equation (62) thus reduces to an interesting linear combination of $H_{j+m}(q^\varphi_x) \times H_{j-m}(q^\varphi_y)$ in terms of $H_{j+m'}(q_x)H_{j-m'}(q_y)$ for fixed j and $m' \in \{-j, -j+1, \ldots, j\}$. This expansion is known for Hermite polynomials, but here we have derived it from the requirement of separability under FrFT's.

Another interesting case corresponds to the north pole $\vartheta = 0$ of the sphere in Fig. 3. In this case $\hat{T}_{(0,\varphi)} = \hat{T}_3$ is the angular-momentum operator (35). There $\Phi^{(0,\varphi)}_{j,m}$ are eigenfunctions of both $\hat{T}_0$ and $\hat{T}_3$, and, because of Eqs. (63) and (64), their dependence on φ is only through a phase $\exp(-im\varphi)$; we shall set $\varphi = 0$ for simplicity. The simultaneous eigenfunctions of the Hamiltonian and angular-momentum operators of an isotropic 2D oscillator are precisely the Laguerre–Gaussian modes that are separable in polar coordinates. For $\mathbf{q} = (q\sin\kappa, q\cos\kappa)^{\mathsf{T}}$, we can conclude that

$$\Phi^{(0,0)}_{j,m}(\mathbf{q}) = \left[\frac{(j-|m|)!}{\pi(j+|m|)!}\right]^{1/2}\exp(2im\kappa)q^{2|m|} \times \exp(-\tfrac{1}{2}q^2)L^{2|m|}_{j-|m|}(q^2). \tag{70}$$

Thus in the special case (0, 0), Eq. (62) is the expansion of the Laguerre–Gaussian modes in terms of the Hermite–Gaussian modes.[26–28,32] Following our discussion in Section 6, on the $\varphi = 0$ meridian and elsewhere on the Fourier sphere of Fig. 3, we can see that, since the rotations of phase space take place in planes that mix the position and the momentum coordinates, the corresponding eigenmodes will not be separable in the screen coordinates alone (cf. Ref. 23).

8. IWASAWA DECOMPOSITION

Given its ray-transfer matrix $\mathbf{M} \in \mathrm{Sp}(4, \Re)$, we decompose the optical system into simpler factors[26]:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{g} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & \mathbf{X} \end{bmatrix}, \quad (71)$$

where the factor submatrices are given (uniquely) by

$$\mathbf{S} = \mathbf{S}^{\mathsf{T}} = (\mathbf{A}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathbf{B}^{\mathsf{T}})^{1/2} \quad \text{positive definite,}$$

$$\mathbf{X} + i\mathbf{Y} = (\mathbf{A}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathbf{B}^{\mathsf{T}})^{-1/2}(\mathbf{A} + i\mathbf{B}) \in \mathrm{U}(2),$$

$$\mathbf{g} = \mathbf{g}^{\mathsf{T}} = -(\mathbf{C}\mathbf{A}^{\mathsf{T}} + \mathbf{D}\mathbf{B}^{\mathsf{T}})(\mathbf{A}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathbf{B}^{\mathsf{T}})^{-1}. \quad (72)$$

This slight modification of the well-known Iwasawa decomposition is valid for any semisimple Lie group,[6,26] which suits our purposes.

The three factors on the right-hand side of Eq. (71) (read from right to left) represent a FrFT $\mathbf{X} + i\mathbf{Y} \in \mathrm{U}(2)$, a 2D symmetric scaling $\mathbf{S}$, and an astigmatic lens of Gaussian matrix $\mathbf{g}$, respectively. Let (ϑ, φ) be the type and (α, β) the order associated with this FrFT, and let $\psi^{\alpha,\beta}_{(\vartheta,\varphi)}(\mathbf{q})$ be the result of its action on $\psi(\mathbf{q})$. We also know that the scaling takes an input $\psi(\mathbf{q})$ to $(\det \mathbf{S})^{-1/2}\psi(\mathbf{S}^{-1}\mathbf{q})$ and that the Gaussian lens factor takes $\psi(\mathbf{q})$ to $\exp[-i(1/2)\mathbf{q}^{\mathsf{T}}\mathbf{g}\mathbf{q}]\psi(\mathbf{q})$. It follows that the first-order system $\mathbf{M} \in \mathrm{Sp}(4, \Re)$ produces the following effect on any input field amplitude $\psi(q)$:

$$\mathbf{M} : \psi(q) \mapsto \psi^{M}(q)$$

$$= \frac{1}{\sqrt{\det \mathbf{S}}} \exp\left(-i\frac{1}{2}\mathbf{q}^{\mathsf{T}}\mathbf{g}\mathbf{q}\right)\psi^{\alpha,\beta}_{(\vartheta,\varphi)}(\mathbf{S}^{-1}\mathbf{q}). \quad (73)$$

We have thus established the following result:

Theorem 5. Every first-order system $\mathbf{M} \in \mathrm{Sp}(4, \Re)$ is a 2D FrFT modulo a symmetric scaling and a (generally astigmatic) phase curvature.

For the simpler 1D case, the corresponding result was presented in Ref. 4, and the corresponding result for free flight in particular was presented in Ref. 33.

9. CANONICAL AND FOURIER TRANSFORMS

The group U(2) of 2D FrFT's will be now placed back as a subgroup of the group $\mathrm{Sp}(4, \Re)$ of all 3D paraxial optical systems. We follow the notions and notation presented in Section 3. The integral representation (of the twofold cover) of the $2N$-dimensional symplectic groups was found by Moshinsky and Quesne[12] in 1971. With each 4×4 symplectic matrix $\mathbf{M}$ we associate (two) integral transforms $\pm\mathcal{C}(\mathbf{M})$ such that

$$\mathcal{C}(\mathbf{M})\begin{bmatrix} \hat{\mathbf{q}} \\ \hat{\mathbf{p}} \end{bmatrix}\mathcal{C}(\mathbf{M})^{-1} = \mathbf{M}^{-1}\begin{bmatrix} \hat{\mathbf{q}} \\ \hat{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{D}}^{\mathsf{T}} & -\mathbf{B}^{\mathsf{T}} \\ -\mathbf{C}^{\mathsf{T}} & \mathbf{A}^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} \hat{\mathbf{q}} \\ \hat{\mathbf{p}} \end{bmatrix}, \quad (74)$$

$$[\mathcal{C}(\mathbf{M})f](\mathbf{q}) = \int_{\Re^2} \mathrm{d}\mathbf{q}' \, C(\mathbf{M})(\mathbf{q}, \mathbf{q}')f(\mathbf{q}'), \quad (75)$$

[cf. Eq. (48)] with an integral kernel[12,34]

$$C(\mathbf{M})(\mathbf{q}, \mathbf{q}') = \frac{-i}{2\pi\sqrt{\det \mathbf{B}}} \times \exp i\left(\tfrac{1}{2}\mathbf{q}^{\mathsf{T}}\mathbf{D}\mathbf{B}^{-1}\mathbf{q} - \mathbf{q}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}-1}\mathbf{q}' + \tfrac{1}{2}\mathbf{q}'^{\mathsf{T}}\mathbf{B}^{-1}\mathbf{A}\mathbf{q}'\right). \quad (76)$$

(The generic N-dimensional case has a factor of $\exp(-i\pi/4)/\sqrt{2\pi}$ for each dimension.)

The operators $\mathcal{C}(\mathbf{M})$ are unitary in $\mathcal{L}^2(\Re^2)$, and they compose as the matrices up to a sign: $\mathcal{C}(\mathbf{M}_1)\mathcal{C}(\mathbf{M}_2) = \pm\mathcal{C}(\mathbf{M}_1\mathbf{M}_2)$, $\mathcal{C}(\mathbf{1}) = \pm 1$ and $\mathcal{C}(\mathbf{M})^{-1} = \pm\mathcal{C}(\mathbf{M}^{-1})$. The two signs $\pm$ appear because $\mathcal{C}$ represents faithfully the double-cover group of $\mathrm{Sp}(4, \Re)$, called the metaplectic group.[6,34,35] Two elements of the metaplectic group correspond to one 4×4 matrix.[6] Because the matrix parameters are much more convenient, here we need mention only that it is the U(1) subgroup of central transforms that bears the onus of this bivaluation. The symplectic matrices (74) with $\mathbf{B} = \mathbf{0}$ form a seven-parameter subgroup, whose integral transform kernels (77) collapse to Dirac δ's with scaling and Gaussian phase.[10] They are

$$\left(\mathcal{C}\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{A}^{\mathsf{T}-1} \end{bmatrix} f\right)(\mathbf{q}) = \frac{1}{\sqrt{\det \mathbf{A}}} \exp\left(i\frac{1}{2}\mathbf{q}^{\mathsf{T}}\mathbf{C}\mathbf{A}^{\mathsf{T}-1}\mathbf{q}\right) f(\mathbf{A}^{-1}\mathbf{q}). \quad (77)$$

This is the geometric action of imaging optical devices, and it includes the one-parameter subgroup of pure rotations in the x–y planes, $\mathbf{M} = \mathrm{diag}[\mathbf{R}(\theta), \mathbf{R}(\theta)]$ [cf. Eqs. (28) and (39)]. These rotations simultaneously belong to the group $\mathrm{U}(2) = \mathrm{Sp}(4, \Re) \cap \mathrm{SO}(4)$ of 2D FrFT's that we defined in Section 3. Since one can factorize any real symplectic matrix into a solvable times an orthogonal matrix,[6] the action of a 3D paraxial optical system factorizes into a purely geometric action (77) times the more properly integral transform action of the U(2) Fourier factor of the system [cf. Eqs. (71) and (73)].

In particular, the official FT $\mathcal{F}$ in Eq. (1) corresponds—up to an important phase—to the symplectic metric matrix $\mathbf{F}$ in Eq. (6):

$$\mathcal{F} = i\mathcal{C}(\mathbf{F}),$$

$$F(\mathbf{q}, \mathbf{q}') = iC(\mathbf{F})(\mathbf{q}, \mathbf{q}') = \frac{1}{2\pi}\exp(-i\mathbf{q}^{\mathsf{T}}\mathbf{q}'), \quad (78)$$

Next, the central FrFT's correspond to $\mathbf{F}_0^{\alpha}$ in Eq. (16):

$$\mathcal{F}_0^{\alpha} = \exp(i\tfrac{1}{2}\pi\alpha)\mathcal{C}(\mathbf{F}_0^{\alpha}),$$

$$F_0^{\alpha}(\mathbf{q}, \mathbf{q}') = \exp(i\tfrac{1}{2}\pi\alpha)C(\mathbf{F}_0^{\alpha})(\mathbf{q}, \mathbf{q}'). \quad (79)$$

The x–y-separable Fourier matrices $\mathbf{F}_1^{\alpha,\beta}$ in Eq. (24) lead to the separable FrFT $\mathcal{F}_1^{\alpha,\beta}$ with kernel Eq. (26), and from Eq. (77) we have

$$\mathcal{F}_1^{\alpha,\beta} = \exp[i\pi(\alpha + \beta)/4]\mathcal{C}(\mathbf{F}_1^{\alpha,\beta}),$$

$$F_1^{\alpha,\beta}(\mathbf{q}, \mathbf{q}') = \exp[i\pi(\alpha + \beta)/4]C(\mathbf{F}_1^{\alpha,\beta})(\mathbf{q}, \mathbf{q}'). \quad (80)$$

For the gyrating transforms, Eqs. (27)–(29) lead to the integral kernel (31); the phase between $\mathcal{F}_3^{\alpha,\beta}$ and $\mathcal{C}(\mathbf{F}_3^{\alpha,\beta})$ is the same as in Eq. (80), $\exp[i(1/2)\pi\mu]$, with $\mu = (1/2)(\alpha$

$+\ \beta)$. These transforms are particular cases of the separation of any U(2)-FrFT's into a U(1) central transformand and an SU(2) transform of power $\nu = (1/2)(\alpha - \beta)$ with axis $\vec{r}(\vartheta, \varphi) \in \mathcal{S}^2$ [cf. Eqs. (36)–(38)], namely,

$$\mathcal{F}_{\vec{r}}^{\alpha,\beta} = \exp(i\tfrac{1}{2}\pi\mu)\mathcal{C}(\mathbf{F}_0^{\mu}\mathbf{F}_{\vec{r}}^{\nu,-\nu}). \tag{81}$$

This phase relation is a consequence of having added -1 to the generator $\hat{T}_0$ of the central FrFT's [see relation (32) and (49)], while the corresponding canonical transforms are generated by the true harmonic oscillator Hamiltonian.[10,35]

Consider now the SU(2)-FrFT's $\mathcal{F}_{\vec{r}}^{\alpha,-\alpha}$ given by Eqs. (36)–(38). When $\mu = 0$ and $\nu = (1/2)\pi\alpha \neq 0,\ \pi$, the 2×2 submatrices are

$$\mathbf{A}_0 = \mathbf{D}_0 = \mathrm{Re}\,\mathbf{U}_{\vec{r}}^{\alpha,-\alpha} = \begin{bmatrix} \cos\nu & r_3\sin\nu \\ -r_3\sin\nu & \cos\nu \end{bmatrix}. \tag{82}$$

$$\mathbf{B}_0 = -\mathbf{C}_0 = \mathrm{Im}\,\mathbf{U}_{\vec{r}}^{\alpha,-\alpha} = \begin{bmatrix} r_1 & r_2 \\ r_2 & -r_1 \end{bmatrix}\sin\nu. \tag{83}$$

Since $\det\mathbf{B}_0 = -r_{12}^2\sin^2\nu \neq 0$, with $r_{12}^2 = r_1^2 + r_2^2$, $\mathbf{B}_0$ can be inverted for all but the purely gyrating transforms ($r_3 = \pm 1$) at the north and the south poles of the Fourier sphere of Fig. 3. Thus we find the three symmetric matrices in the exponent of the SU(2)-FrFT kernel (77):

$$\mathbf{B}_0^{-1} = \frac{1}{r_{12}^2\sin\nu}\begin{bmatrix} r_1 & r_2 \\ r_2 & -r_1 \end{bmatrix} = \frac{\mathbf{B}_0}{r_{12}^2\sin^2\nu} = \mathbf{B}_0^{\top -1}, \tag{84}$$

$$\mathbf{A}_0\mathbf{B}_0^{-1} = \frac{1}{r_{12}^2\sin^2\nu}(\mathbf{B}_0\cos\nu + \mathbf{B}_{\perp 0}r_3\sin\nu), \tag{85}$$

$$\mathbf{B}_0^{-1}\mathbf{A}_0 = \frac{1}{r_{12}^2\sin^2\nu}(\mathbf{B}_0\cos\nu - \mathbf{B}_{\perp 0}r_3\sin\nu), \tag{86}$$

$$\text{where}\quad \mathbf{B}_{\perp 0} = \begin{bmatrix} r_2 & -r_1 \\ -r_1 & -r_2 \end{bmatrix}\sin\nu \tag{87}$$

is obtained from $\mathbf{B}_0$ by a $(1/2)\pi$ rotation about the r_3 axis [i.e., $(r_1, r_2) \leftrightarrow (r_2, -r_1)$ carries $\mathbf{B}_0 \leftrightarrow \mathbf{B}_{\perp 0}$]. For separable transforms ($r_3 = 0$) on the equator of Fig. 3, all quadratic forms are built with matrices proportional to $\mathbf{B}_0$.

For $\mu \neq 0,\ \pi$, the generic U(2)-FrFT $\mathcal{F}_{\vec{r}}^{\alpha,\beta}$ will be characterized by submatrices (36) that are rotated combinations of the previous ones:

$$\mathbf{A}_\mu = \mathbf{A}_0\cos\mu - \mathbf{B}_0\sin\mu, \qquad \mathbf{B}_\mu = \mathbf{A}_0\sin\mu + \mathbf{B}_0\cos\mu. \tag{88}$$

The determinant in Eq. (77) is now $\Delta_\mu = \det\mathbf{B}_\mu = \sin^2\mu - r_{12}^2\sin^2\nu = \Delta_0 + \sin^2\mu$, and

$$\mathbf{B}_\mu^{-1} = \Delta_\mu^{-1}(-\cos\mu\,\mathbf{B}_0 + \sin\mu\mathbf{A}_0^{\top}), \tag{89}$$

$$\mathbf{A}_\mu\mathbf{B}_\mu^{-1} = \Delta_\mu^{-1}(\mathbf{1}\cos\mu\sin\mu + \Delta_0\mathbf{A}_0\mathbf{B}_0^{-1}). \tag{90}$$

$$\mathbf{B}_\mu^{-1}\mathbf{A}_\mu = \Delta_\mu^{-1}(\mathbf{1}\cos\mu\sin\mu + \Delta_0\mathbf{B}_0^{-1}\mathbf{A}_0), \tag{91}$$

Thus we build the general fractional U(2) Fourier integral transform kernel.

10. CONCLUSIONS

We have shown that the fractional Fourier transforms (FrFT's) in two (and more) dimensions have a considerably richer structure than what is apparent in one dimension. Their manifold is U(2) rather than simply a circle or a torus, and they are classified by order (points of a torus) and type (points of a sphere). Our Fourier sphere of types, the spheres recently studied by Padgett and Courtial[36] and by Agarwal[37] in connection with the orbital angular momentum of light beams, and the sphere of U(2) coherent states of the Hermite–Gaussian modes studied earlier in Ref. 26 are essentially equivalent to one another: All of them are the coset space of $\mathrm{U}(2) = \mathrm{Sp}(4, \Re) \cap \mathrm{SO}(4)$ with respect to $\mathrm{U}(1) \times \mathrm{U}(1)$—different incarnations of one and the same structure.

One of the main applications of FrFT's has been in the optical processing of 1D images. Two-dimensional images are richer, and a wider choice of transformations is now available. The analysis of N-dimensional Fourier transformers will lead to N-dimensional tori—because there are N commuting U(1) subgroups—which will be fibers on more-complicated $N(N-1)$-dimensional coset base manifolds of SU(N). All definitions can be made coordinate free.

Finally, FrFT's have been revealed to be the integral part of paraxial optical systems. An interesting challenge would be to build a paraxial optical device to implement U(2)-FrFT's so that every point of their compact 4D manifold can be realized, preferably by the smooth rotation of a few astigmatic elements around a common axis, as was done for the Poincaré sphere of polarization in Ref. 8.

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