respectively. Then the integral (5.1) reads

$$\tilde{\phi}(y_1,y_2) = \int_0^\infty dx_{10}\, e^{ix_{10}y_{10}} \int_0^\infty dx_{20}\, e^{ix_{20}y_{20}} \times \int_0^\infty d|\mathbf{x}_1|\, |\mathbf{x}_1|^2 \int d\Omega_1 \int_0^\infty d|\mathbf{x}_2|\, |\mathbf{x}_2|^2 \times \int d\Omega_2\, e^{-i|\mathbf{x}_2||\mathbf{y}_2|\cos\theta_2} \delta(x_1^2)\delta(x_2^2) \times \delta[(x_1-x_2)^2]\phi(x_{10},x_{20},|\mathbf{x}_1|,|\mathbf{x}_2|,\Omega_1,\Omega_2). \tag{5.21}$$

Using the δ functions to perform some of the integrations and defining

$$\bar{\phi}(x_{10},x_{20},\cos\theta_2) = \frac{\pi}{2}\int_0^{2\pi} d\psi\, \phi(x_{i0}, |\mathbf{x}_i| = x_{i0}, \Omega_1, \Omega_2), \tag{5.22}$$

We get from Eq. (5.21)

$$\tilde{\phi}(y_1,y_2) = \int_0^\infty dx_{10}\, e^{ix_{10}y_{10}} \int_0^\infty dx_{20}\, e^{ix_{20}y_{20}} \int_{-1}^{+1} d\cos\theta_2 \times e^{-ix_{20}|\mathbf{y}_2|\cos\theta_2}\phi(x_{10},x_{20},\cos\theta_2). \tag{5.23}$$

With the substitutions $x_{10}=x_1, x_{20}=x_2$, and $x_{20}\times\cos\theta_2 = x_3$, it follows that

$$\tilde{\phi}(y_1,y_2) = \int_0^\infty dx_1\, e^{ix_1y_{10}} \int_0^\infty \frac{dx_2}{x_2} e^{ix_2y_{20}} \times \int_{-x_2}^{+x_2} dx_3 e^{-ix_3y_{23}} f(x_1,x_2,x_3), \tag{5.24}$$

where

$$f(x_1,x_2,x_3) = \bar{\phi}(x_{10},x_{20},\cos\theta_2). \tag{5.25}$$

The integral transform (5.24) is of the same type as Eq. (4.1). By using the results of Sec. 4 the lower terms of the asymptotic expansions of $\tilde{\phi}(y_1,y_2)$ are given in Table IV.

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The $U_{n,1}$ and IU_n Representation Matrix Elements

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We construct a realization of the $U_{n,1}$ and IU_n groups as multiplier representations of the space of functions on the U_n group manifold. Making use of the orthogonality and completeness of the U_n unitary irreducible representation matrix elements (UIRME's), we are able to express the $U_{n,1}$ boost and IU_n translation matrix elements (the generalized Wigner d-functions) of the principal series of UIR's as an integral over a compact domain (unit disc) of two U_n d-functions, phases, and the multiplier. This is an extension to the unitary groups of a method previously used [J. Math. Phys. **12**, 197 (1971)] to find the $SO_n, SO_{n,1}$, and ISO_n UIRME's in a recursive fashion. We establish a number of symmetry properties, the asymtotic (Regge-like) and contraction ($U_{n,1} \to IU_n$) behavior of these functions.

1. INTRODUCTION

The unitary and pseudo-unitary groups in nuclear and elementary-particle physics have been used mainly through the associated Lie algebra.[1] The states of a system are identified with the components of the bases for unitary irreducible representations (UIR's) classified in some mathematically convenient or physically relevant chain of subalgebras. Interactions are then represented by operators with either irreducible tensor properties under the group or constructible in some simple fashion out of the universal enveloping algebra. Thus, the Wigner coefficients and the matrix elements of the generators of the Lie algebra[2] have played the main role in the applications of unitary groups.

The orthogonal, pseudo-, and inhomogeneous-orthogonal groups, on the other hand, have been widely used in connection with their finite transformations, either as a geometry group or in harmonic analysis on the $SO_{2,1}$ and $SO_{3,1}$ groups, whose UIR matrix elements (ME's) constitute a "best set" of functions in which to expand high-energy scattering data.[3] Also, a number of field theories have made use of the Poincaré group ($ISO_{3,1}$) manifold.[4]

There has been a corresponding increase of interest in considering the UIRME's—the generalized D and d-functions—as "special functions,"[5] that is, as orthogonal and complete[6] sets of functions in terms of which one can expand any well-behaved function on the group manifold which, furthermore, due to the group properties, exhibit summation and recursion formulae, the emphasis being placed not so much in their explicit expressions which, like the series expansion of a Bessel function, provides at best a limited insight into the aforementioned properties, but in the relations between functions which these properties imply.

It was in this spirit that we treated in Ref. 7 the generalized Wigner d-functions for the SO_n, $SO_{n,1}$, and ISO_n groups.[8] In the present paper we apply the techniques developed in Ref. 7 to the unitary (U_n), pseudo-unitary ($U_{n,1}$), and inhomogeneous unitary (IU_n) groups. As the method is essentially parallel, we shall skip most of the introductory material on multipliers as well as the detailed description of the U_n manifold and representation theory. In these, we use the concepts introduced in Ref. 9, giving a summary of notation in Sec. 2.

The U_2 UIRME's are essentially the classical Wigner d-functions. Bég and Ruegg[10] and T. J. Nelson[11] studied the U_3 harmonic functions, the analog of the SO_3 spherical harmonics on the (five-dimensional) manifold of the complex 3-sphere $C_3 \cong U_2\backslash U_3$, the eigenfunctions of the Laplace–Beltrami operators of the manifold. Using these techniques, Fischer and Rączka[12] gave explicit expressions for the U_n and $U_{p,q}$ harmonic functions. These can be used in order to find the UIRME's themselves, as was done by Holland[13] for SU_3 and by Delbourgo, Koller, and Williams[14] for SU_n. This technique, however, can only give the $[J_1 J \cdots J\, J_n]$ UIR's of the general U_n ($n > 3$) groups since[9] only these can be realized on the $C_n \cong U_{n-1}\backslash U_n$ homogeneous space.

A different line of approach was followed by Chacón and Moshinsky,[15] who expressed the general U_3 transformation as a product of several U_2 transformations and transpositions. This method was extended to U_n by Flores and Niederle.[16] Taking these d-functions as known, our approach hinges in defining the action of a $U_{n,1}$ group as a group of transformations of the U_n manifold such that, while the canonical U_n subgroup of $U_{n,1}$ produces "rigid" mappings (leaving the Haar measure invariant), the boosts of $U_{n,1}$ produce "deformations" of the manifold. This is detailed in Sec. 3. By considering those transformations which commute with the canonical U_{n-1} subgroup, it is sufficient to define the "deformation" on the $C_n \cong U_{n-1}\backslash U_n$ manifold. This leads to a multiplier representation and to the expression, in Sec. 4, of the $U_{n,1}$ d-functions of the principal series of UIR's[17,18] in terms of an integral over a compact domain of two U_n d-functions, phases, and a multiplier. Some properties of the d-functions are exhibited in Sec. 5. In Sec. 6, a similar procedure gives the IU_n d-functions. These are checked to correspond to contractions[19] of the $U_{n,1}$ d-functions.

The formalism works best when we use the unitary analog of the Euler angles,[15,20] the "last latitude" angle in $U_{n,1}$ being a boost and, in IU_n, a real translation. As for the orthogonal groups,[7] we want to emphasize that our procedure gives the $U_{n,1}$ principal series of UIRME's classified by the canonical chain of subgroups. Several properties are apparent from the integral form. This method seems to be extendable to other groups and manifolds in essentially the same form.

2. THE UNITARY GROUP MANIFOLDS AND REPRESENTATIONS

The Euler–angle parametrization[15,20] of U_n can be defined, enclosing collective variables in curly brackets:

$$u_n(\{\phi,\theta\}^{(n)}) = u_{n-1}(\{\phi,\theta\}^{(n-1)})\, c_n(\{\phi^{(n)},\theta^{(n)}\}), \tag{2.1a}$$

$$c_n(\{\phi^{(n)},\theta^{(n)}\}) = \Phi_n(\phi_n^{(n)})\, r_{n-1,n}(\theta_{n-1}^{(n)})\, c_{n-1}(\{\phi^{(n)},\theta^{(n)}\}), \tag{2.1b}$$

$$u_1(\{\phi,\cdot\}^{(1)}) = c_1(\{\phi^{(1)},\cdot\}) = \Phi_1(\phi_1^{(1)}), \tag{2.1c}$$

where $r_{pq}(\theta)$ are rotations by θ in the p-q plane of an n-dimensional complex coordinate space $Z^n \ni \mathbf{z}$, and $\Phi_k(\phi)$ are phase rotations by ϕ in the kth coordinate. Defining

$$\mathbf{z}(\{\phi,\theta\}) = c_n(\{\phi,\theta\})^{-1}\mathbf{z}_0$$

for a fixed $\mathbf{z}_0 \in Z^n$, we introduce complex-spherical coordinates in Z^n as

$$Z_k(\{\phi,\theta\}) \equiv r_k(\{\phi\})e^{-i\psi_k} = re^{-i(\phi_n+\cdots+\phi_k)} \times \sin\theta_{n-1}\cdots\sin\theta_k\cos\theta_{k-1}, \tag{2.2a}$$

for $k = 2,\ldots,n-1$. For $k=1$ we can put formally $\theta_0 \equiv 0$, while for $k=n$

$$Z_n(\{\phi,\theta\}) \equiv r_n(\{\theta\})e^{-i\psi_n} = re^{-i\phi_n}\cos\theta_{n-1}. \tag{2.2b}$$

Choosing the ranges[15] $\theta_i \in [0,\pi/2]$ ($i=1,\ldots,n-1$), $\phi_j \in [0,2\pi)$ ($j=1,\ldots,n$), we give to $r_k(\{\theta\})$ the meaning of the modulus of $\mathbf{z}$ and ψ_k as its phase.

For fixed r we have the $(2n-1)$-dimensional manifold of the complex n-sphere $C_n \cong U_{n-1}\backslash U_n$ with

$$dc_n(\{\phi,\theta\}) = d\mu_n(\phi_n,\theta_{n-1})\,dc_{n-1}(\{\phi,\theta\}), \tag{2.3a}$$

$$d\mu_k(\phi_k,\theta_{k-1}) = \sin^{2k-3}\theta_{k-1}\cos\theta_{k-1}\,d\phi_k\,d\theta_{k-1}, \tag{2.3b}$$

$$dc_1(\{\phi,\cdot\}) = d\mu_1(\phi_1,\cdot) = d\phi_1, \tag{2.3c}$$

and through (1.1) we construct the Haar measure for U_n. Integrating (2.3) over C_n we find its area to be $|C_n| = 2\pi^n/\Gamma(n)$. The volume of U_n is, from (2.1), $\mathrm{vol}U_n = \mathrm{vol}U_{n-1}\cdot|C_n|$, $\mathrm{vol}U_1 = 2\pi$.

For the $U_{n-1,1}$ group, the rotation angle in the $(n-1)$-n plane in (1.1b) is replaced by a boost $b_{n-1,n}(\zeta)$, $\zeta \in [0,\infty)$ in that plane, while for the IU_{n-1} group, it is replaced by a real translation $t_{n-1}(\xi)$, $\xi \in [0,\infty)$ in the $(n-1)$th direction.

The U_n Gel'fand kets[2,21] will be abbreviated[7] $|J_n, \overline{J_{n-1}}\rangle$, where $J_n \equiv [J_{n,1}, J_{n,2},\ldots,J_{n,n}]$ labels the U_n UIR and $\overline{J_{n-1}}$ its row index: $\overline{J_{n-1}} \equiv \{J_{n-1},J_{n-2},\ldots,J_1\}$, where J_k denotes the UIR of the canonical U_k subgroup of U_n. The individual labels J_{km} obey the known "zig-zag" inequalities

$$J_{k,m-1} \geq J_{k-1,m-1} \geq J_{k,m}, \quad n \geq k \geq m \geq 2. \quad (2.4)$$

The U_n representation D-matrices are thus labeled as

$$D^{J_n}_{\overline{J_{n-1}},\overline{J'_{n-1}}}[u_n(\{\phi,\theta\}^{(n)})] \equiv \langle J_n\overline{J_{n-1}}|u_n(\{\phi,\theta\}^{(n)})|J_n\overline{J'_{n-1}}\rangle, \quad (2.5)$$

and can be decomposed through (1.1) into sums of products of the phase functions

$$p^{J_k}_{J_{k-1}}(\phi) \equiv \langle J_k\overline{J_{k-1}}|\Phi_k(\phi)|J_k\overline{J_{k-1}}\rangle, \quad (2.6)$$

which are diagonal and independent of the $U_m(m > k$ and $m < k-1)$ labels, and the generalized Wigner d-functions

$$d^{J_k}_{J_{k-1},J_{k-2},J'_{k-1}}(\theta) \equiv \langle J_kJ_{k-1}\overline{J_{k-2}}|r_{k-1,k}(\theta)|J_kJ'_{k-1}\overline{J_{k-2}}\rangle, \quad (2.7)$$

diagonal in the U_k and U_{k-2} UIR labels and independent of the $U_m(m > k$ and $m < k-2)$ labels. The U_n D-functions (2.5) are orthogonal and complete on the U_n manifold with the U_n Haar measure and the Plancherel weight $\dim J_n/\mathrm{vol}U_n$.

For the $U_{n-1,1}$ and IU_{n-1} groups, the Gel'fand patterns[17,18] are similar to the U_n ones, except for the labels J_{n1} and J_{nn} which are, in general, complex and do not abide (1.4). The representations are thus infinite-dimensional. The d-functions we want to calculate, which we shall denote by ${}^Pd$ and ${}^Id$ for the pseudo- and inhomogeneous-unitary groups, are the matrix elements respectively, of the boost $b(\zeta)$ and the real translation $t(\xi)$ in the corresponding Euler-angle parametrization.

3. THE $U_{n,1}$ ALGEBRA AND MULTIPLIER REPRESENTATION

The set of operators on the complex n-space Z^n

$$\mathcal{C}_j{}^k \equiv z_j\frac{\partial}{\partial z_k} - z^k\frac{\partial}{\partial z^j} \quad (3.1)$$

with $\overline{z_k} = z^k$ (complex conjugation), have the well-known commutation relations of the generators of the u_n algebra.[21] They leave the n-sphere $C_n \cong U_{n-1}\backslash U_n$ invariant. If we add the z_k and z^k $(k = 1, \ldots, n)$ to the set (3.1), we have the generators of an iu_n algebra. Using the second-order Casimir operator

$$\Psi^{(n)} \equiv \mathcal{C}_j{}^k\mathcal{C}_k{}^j \quad (3.2)$$

(sum over repeated indices, unless otherwise indicated), we can construct, out of the universal enveloping algebra of iu_n, the operators

$${}^{(\sigma)}\mathcal{C}_k^{n+1} \equiv \tfrac{1}{2}[\Psi^{(n)}, z_k] + \sigma z_k = z_l\mathcal{C}^l_k + (\tfrac{1}{2}n + \sigma)z_k, \quad (3.3a)$$

$${}^{(\sigma)}\mathcal{C}^k_{n+1} \equiv \tfrac{1}{2}[\Psi^{(n)}, z^k] + \bar{\sigma} z^k = z^l\mathcal{C}_l^k + (-\tfrac{1}{2}n + \bar{\sigma})z^k, \quad (3.3b)$$

$${}^{(\sigma)}\mathcal{C}^{n+1}_{n+1} \equiv [{}^{(\sigma)}\mathcal{C}^k_{n+1}, {}^{(\sigma)}\mathcal{C}_k^{n+1}] - \mathcal{C}^k_k = z_lz^j\mathcal{C}^l_j + (\sigma + \bar{\sigma}) \quad (3.3c)$$

(no sum over k), where σ is an (as yet) arbitrary complex number. As the notation suggests, (3.3) together with (3.1) generate a $u_{n,1}$ algebra which leaves C_n invariant with $r = 1$. Each value of σ gives a different set of generators which will produce a correspondingly distinct UIR, as can be seen from the $u_{n,1}$ Casimir operator

$$\Psi^{(n,1)}(\sigma) = \Psi^{(n,1)}(0) + \sigma^2 + \bar{\sigma}^2 \quad (3.4a)$$

and the unitary invariant

$$\Omega^{(n,1)} \equiv \sum_{k=1}^{n} \mathcal{C}^k_k - \mathcal{C}^{n+1}_{n+1} = -\sigma - \bar{\sigma}. \quad (3.4b)$$

We can build an $so_{n,1} \subset u_{n,1}$ subalgebra generated by

$$M_{jk} \equiv \mathcal{C}_j{}^k - \mathcal{C}_j{}^k \quad (1 \leq j < k \leq n) \quad (3.5a)$$

and

$${}^{(\sigma)}M_{k,n+1} \equiv {}^{(\sigma)}\mathcal{C}^{n+1}_k - {}^{(\sigma)}\mathcal{C}^k_{n+1} \quad (k = 1, \ldots, n), \quad (3.5b)$$

anti-Hermitean under the measure (2.3). The operators (3.5a) will generate boosts in the kth direction. Now, since ${}^{(\sigma)}M_{n,n+1}$ commutes with the generators of u_{n-1}, its action on U_n can be fully studied as the action on $C_n \cong U_{n-1}\backslash U_n$. It is sufficient, therefore, to construct ${}^{(\sigma)}M_{n,n+1}$ in terms of the complex-spherical coordinates (2.2). Direct calculation through (2.2), (3.1), (3.3), and (3.5b) yields

$$\begin{aligned}{}^{(\sigma)}M_{n,n+1} = {} & \sin\theta_{n-1}\cos\phi_n\frac{\partial}{\partial\theta_{n-1}} \\ & + (\sec\theta_{n-1} + \cos\theta_{n-1})\sin\phi_n\frac{\partial}{\partial\phi_n} \\ & - \sec\theta_{n-1}\sin\phi_n\frac{\partial}{\partial\phi_{n-1}} + \cos\theta_{n-1} \\ & [(n + 2i\,\mathrm{Im}\sigma)\cos\phi_n - 2i\,\mathrm{Re}\sigma\sin\phi_n]. \end{aligned} \quad (3.6)$$

The exponentiation of (3.6), for $\sigma = 0$ yields the action of $b_{n,n+1}(\zeta)$ on C_n and can be found from the action of $U_{n,1}$ on itself (in the Iwasawa decomposition $U_{n,1} = U_n \cdot A \cdot N$) modulo N, in the same fashion as was done in Ref. 7, generating the following transformation of $\mathbf{z} \in C_n$: the unit disc $|z_n| \leq 1$,

$$z_n \to z'_n = \frac{z_n\cosh\zeta - \sinh\zeta}{\cosh\zeta - z_n\sinh\zeta}, \quad (3.7a)$$

which defines $\phi_n \to \phi'_n$ and $\theta_n \to \theta'_n$, and

$$\begin{aligned}\phi_{n-1} \to \phi'_{n-1} & = \phi_{n-1} + \arg(\cos\theta_{n-1}\cosh\zeta - \exp i\phi_n\sinh\zeta) \\ & \equiv \phi_{n-1} + \chi(\phi_n, \theta_{n-1}, \zeta), \end{aligned} \quad (3.7b)$$

all other coordinates of $\mathbf{z}$ remaining unaffected. This can be seen as the "complexification" of the more familiar transformation $\tan\frac{1}{2}\theta \to \tan\frac{1}{2}\theta' = e^\zeta\tan\frac{1}{2}\theta$ which appears in connection with the pseudo-orthogonal groups[3,7,8,22]. The Jacobian of the transformation (3.7) is

$$\frac{dc_n(\{\phi', \theta'\})}{dc_n(\{\phi, \theta\})} = \frac{d\mu_n(\phi'_n, \theta'_{n-1})}{d\mu_n(\phi_n, \theta_{n-1})} = \left(\frac{\sin\theta'_{n-1}}{\sin\theta_{n-1}}\right)^{2n}. \quad (3.8)$$

We have thus, for $\sigma = i\tau$ (τ real), a unitary multiplier representation[23] of $b_{n,n+1}(\zeta)$ on the space of functions f on C_n (and therefore on U_n) as

$$T^{(\sigma)}(b_{n,n+1}(\zeta))f(\mathbf{z}) = \exp[\zeta^{(\sigma)}M_{n,n+1}]f(\mathbf{z}) = [\sin\theta'_{n-1}/\sin\theta_{n-1}]^{n+\sigma} f(\mathbf{z}'). \quad (3.9)$$

4. THE U_{n-1} MATRIX ELEMENTS

The phase functions (2.6) are the matrix elements of transformations generated by $\mathcal{C}_k{}^k$ (no sum). As[21] $\mathcal{C}_k{}^k|J_{n+1}\overline{J_n}\rangle = \omega_k|J_{n+1}\overline{J_n}\rangle$ with $\omega_k = \sum_{l=1}^{k} J_{kl} - \sum_{l=1}^{k-1} J_{k-1,l}$

$$p^{J}_{J}{}^{k}_{k-1}(\phi) = \exp(i\omega_k\phi), \quad k = j, \ldots, n+1. \quad (4.1)$$

The eigenvalue of the unitary invariant (3.4b) is $\omega_1 + \omega_2 + \cdots + \omega_n - \omega_{n+1}$. For σ pure imaginary, (3.4b) is zero and hence $\omega_{n+1} = \sum_{k=1}^{n} J_{nk}$.

The calculation of the $U_{n,1}$ ^Pd-functions, however, will require the multiplier representation (3.9). Given a set $\{\phi_k^{(\mu)}\}$ $k \in N$ (N an index set determined by μ) of orthogonal functions on a manifold M, a representation of a group of transformations $G \ni g$ of M can be constructed as[7]

$$D^{(\lambda,\mu)}_{k,k'}(g) = [\omega(k)\omega(k')]^{1/2}(\phi_k^{(\mu)}, T^{(\lambda)}(g)\phi_{k'}^{(\mu)})_M, \quad (4.2)$$

where ω is the Plancherel weight of N. Using for M the U_n manifold and $D^{J_n}_{\overline{J_{n-1}}, \overline{J_{n-1}}}$ as the set of orthogonal functions, we proceed to prove that, in close analogy with the orthogonal groups[7], the ^Pd-functions can be found as

$$^Pd^{\{\alpha, J'_{n-1}, \beta\}}_{J_n J_{n-1} J'_n}(\zeta) = [\dim J_n \dim J'_n)^{1/2}/\mathrm{vol}U_n] \times \left(D^{J_n}_{\overline{J'_{n-1}}, \overline{J_{n-1}}}, T^{(\sigma)}(b_{n,n+1}(\zeta))D^{J'_n}_{\overline{J'_{n-1}}, \overline{J_{n-1}}}\right), \quad (4.3)$$

where the connection between α, J'_{n-1}, β, and the $U_{n,1}$ UIR labels J_{n+1} will be clarified below.

At $\zeta = 0$, the orthogonality of the D's insures that $^Pd^{J_{n+1}}_{J_n J_{n-1} J'_n}(0) = \delta_{J_n, J'_n}$ (the Kronecker δ in the collective indices J_n and J'_n stands for a product of δ's in the individual indices J_{nk} and J'_{nk}, $k = 1, \ldots, n$). The completeness of the D's gives the addition formula

$$\sum_{J''_n} {}^Pd^{J_{n+1}}_{J_n J_{n-1} J''_n}(\zeta_1)\, {}^Pd^{J_{n+1}}_{J''_n J_{n-1} J'_n}(\zeta_2) = d^{J_{n+1}}_{J_n J_{n-1} J'_n}(\zeta_1 + \zeta_2), \quad (4.4)$$

hence (4.3) together with (4.1) and (4.2) for $g \in U_n$ provide us with a representation $U_{n,1}$. There is no invariant subspace. This construction gives us the classification through the Gel'fand patterns of the $U_{n,1}$ UIR $J_{n+1} = \{\alpha, J'_{n-1}, \beta\}$ since the individual indices $J_{n+1,1} = \alpha$, $J_{n+1,k+1} = J'_{n-1,k}$ $(k = 1, \ldots, n)$, $J_{n+1,n+1} = \beta$ restricted through the zig-zag inequalities (2.4) for $U_n \supset U_{n-1}$, when taken as the $U_{n,1}$ UIR's restrict in turn the UIR labels of $U_n \subset U_{n,1}$. The "end point" labels α and β will now be related to σ when we identify them as the continuation of the values of $J_{n+1,1}$ and $J_{n+1,n+1}$ entering into the expressions for (i) the unitary invariant (3.4b) eigenvalue

$$0 = \sum J_{n-1,k} = \alpha + \beta + \sum J'_{n-1,k} \quad (4.5a)$$

(the sum extending over the allowed values of the free index) and (ii) the second-order Casimir operator (3.4a) eigenvalue

$$\sum J_{n+1,k}(J_{n+1,k} - 2k + n + 2) = \alpha(\alpha + n) + \beta(\beta - n) + \sum J'_{n-1,k}(J'_{n-1,k} - 2k + n), \quad (4.5b)$$

which, if the representation is to be unitary, (iii) has to be real. Lastly, (iv) the dependence of (4.5b) on $\sigma = i\tau$ must be that given by (3.4a).

All four conditions (i)–(iv) can be satisfied by the choice $\alpha = -\frac{1}{2}(n + \sum J'_{n-1,k}) + i\tau$ and $\beta = \frac{1}{2}(n - \sum J'_{n-1,k}) - i\tau$. The parameter τ can be identified with Chakrabarti's[17] parameter ϵ, and seen to label the continuum of principal series UIR's of $U_{n,1}$. Values of τ and $-\tau$ give equivalent UIR's.

The integral over U_n in (4.3) can be simplified when the D's are written in terms of p's, d's, and the U_{n-1} D's as in (2.1). Orthogonality relations can be used to yield Kronecker δ's in the corresponding labels, and the multiple integral reduces to an integral over the unit disc:

$$^Pd^{\{\alpha, J'_{n-1}, \beta\}}_{J_n J_{n-1} J'_n}(\zeta) = \frac{(\dim J_n \dim J'_n)^{1/2}}{\dim J_{n-1} \dim J'_{n-1}} \times \frac{(\mathrm{vol}U_{n-1})^2}{\mathrm{vol}U_n \,\mathrm{vol}U_{n-2}} \times \sum_{J_{n-2}} \dim J_{n-2} \int d\mu_n(\phi, \theta)\, \overline{p^{J_n}_{J_{n-1}}(\phi)}\; \overline{d^{J_n}_{J'_{n-1} J_{n-2} J_{n-1}}(\theta)} \times \left(\frac{\sin\theta'}{\sin\theta}\right)^{n+i\tau} \exp\left[i\left(\sum J_{n-1} - \sum J_{n-2}\right)\chi(\phi, \theta, \zeta)\right] p^{J'_n}_{J'_{n-1}}(\phi') \times d^{J'_n}_{J'_{n-1} J_{n-2} J_{n-1}}(\theta'), \quad (4.6)$$

where the primed variables are related to the unprimed one through the transformation (3.7).

5. SOME PROPERTIES OF THE ^Pd-FUNCTIONS

We will not attempt here the explicit evaluation of (4.6). Several properties are apparent, however, from the integral form (4.3)–(4.6):

(i) the group property yields the addition formula (4.4);

(ii) unitarity of the representation gives

$$^Pd^{J_{n+1}}_{J_n J_{n-1} J'_n}(-\zeta) = \overline{^Pd^{J_{n+1}}_{J'_n J_{n-1} J_n}(\zeta)}; \quad (5.1)$$

(iii) invariance of the scalar product (4.3) under the involution $u_n \leftrightarrow u_n^{-1}$ and the unitary of the $U_n D$'s imply

$$^Pd^{\{\alpha, J'_{n-1}, \beta\}}_{J_n J_{n-1} J'_n}(\zeta) = \overline{^Pd^{\{\alpha, J_{n-1}, \beta\}}_{J_n J'_{n-1} J'_n}(\zeta)}; \quad (5.2)$$

(iv) the asymptotic behavior $(\zeta \to \infty)$ is similar to the Regge behavior of the $SO_{n,1}$ d-functiond[4,7,8]: It is exponentialy decreasing in ζ. As the disc (3.7a) streches towards the point $z_n = -1$, $\sin\theta'/\sin\theta \sim e^{-\zeta}$ and

$$^Pd^{\{\alpha, J'_{n-1}, \beta\}}_{J_n J_{n-1} J'_n}(\zeta) \xrightarrow[\zeta \to \infty]{} \delta_{J_{n-1}, J'_{n-1}} \Delta^{\alpha J'_{n-1}\beta}_{J_n J'_n} e^{-[n+i\tau]\zeta}, \quad (5.3)$$

where $\Delta^{\alpha J'_{n-1}\beta}_{J_n J'_n}$ are constants obtainable from (4.6).

6. THE IU_n MATRIX ELEMENTS

We consider now the finite translations generated by z_k and z^k as a multiplier representation on the space of functions f on C_n. The real translation $t_n(\xi) \in ISO_n \subset IU_n$ [taking the place of $r_{n-1,n}(\theta)$ in (2.1b)] is generated by $x_n = \frac{1}{2}(z_n + z^n) = r\cos\theta_{n-1}\cos\phi_n$ and has the action

$$T^{(r)}(t_n(\xi))f(\mathbf{z}) = \exp(i\xi x_n)f(\mathbf{z}), \tag{6.1}$$

which is unitary for real r, but produces no deformation of the C_n manifold. Again, as x_n commutes with the generators of u_{n-1}, the action (6.1) of x_n on C_n can be used to construct the IU_n UIR's through (4.2) and, analogously to (4.3) and (4.6), we find the IU_n ${}^Id$-functions as

$$\begin{aligned} {}^Id^{\{r,J'_{n-1},s\}}_{J_n J_{n-1} J'_n}(\xi) &= \frac{(\dim J_n \dim J'_n)^{1/2}}{\mathrm{vol}U_n} \left(D^{J_n}_{\overline{J'_{n-1}}\,\overline{J_{n-1}}}, T^{(r)}(t_n(\xi))D^{J'_n}_{\overline{J'_{n-1}},J_{n-1}}\right) \\ &= \frac{[\dim J_n \dim J'_n]^{1/2}}{\dim J_{n-1}\dim J'_{n-1}} \frac{(\mathrm{vol}U_{n-1})^2}{\mathrm{vol}U_n\,\mathrm{vol}U_{n-2}} \sum_{J_{n-2}} \dim J_{n-2} \int d\mu_n(\phi,\theta) \\ &\quad\times \overline{p^{J_n}_{J_{n-1}}(\phi)}\;\overline{d^{J_n}_{J'_{n-1}J_{n-2}J_{n-1}}(\theta)} \exp[ir\xi\cos\theta\cos\phi]\,p^{J'_n}_{J'_{n-1}}(\phi)\,d^{J'_n}_{J'_{n-1}J_{n-2}J_{n-1}}(\theta). \end{aligned} \tag{6.2}$$

The iu_n second-order Casimir operator $z_l z^l$ has eigenvalues r^2, and thus r (real) labels the IU_n UIR's corresponding to Chakrabarti's[17] parameter κ. The ${}^Id$-functions (6.2) are independent of the label s. This label enters into the picture when we consider the phase of the translation $\Phi_{n+1}(\phi)$. Its matrix elements follow from (4.1) and will not be considered again. Properties analogous to those presented in the last section follow.

As was the case for the orthogonal groups[7], the $U_{n,1}$ group can be defomed in the Inönü–Wigner sense[19] into the IU_n group when we consider UIR's with $\tau \to \infty$ while keeping $\tau\zeta = r\xi$. The multiplier (3.9) becomes then

$$(\sin\theta'/\sin\theta)^{n+i\tau} \xrightarrow[\tau\to\infty]{} \exp[ir\xi\cos\theta\cos\phi]$$

while, as $\zeta \to 0$, there is no deformation of the group manifold. Comparing (4.6) and (6.2) we see that

$$ {}^Pd^{\{\alpha(\tau),J'_{n-1},\beta(\tau)\}}_{J_n J_{n-1} J'_n}(\zeta) \xrightarrow[\substack{\tau\to\infty\\ \zeta\tau=\xi r}]{} {}^Id^{\{r,J'_{n-1},s\}}_{J_n J_{n-1} J'_n}(\xi),$$

thus, characterizing the value of the last IU_n label when we maintain the eigenvalue of the unitary invariant (3.4b) as zero.

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