Generalized Wigner function for the analysis of superresolution systems

Kurt Bernardo Wolf, David Mendlovic, and Zeev Zalevsky

The generalized Wigner function is able to represent light distributions that contain spatial and temporal information. The use of such a generalized Wigner distribution function for analysis and understanding of temporally restricted superresolving systems is demonstrated. These systems gain spatial resolution by conversion of the temporal degrees of freedom to spatial degrees of freedom. © 1998 Optical Society of America

OCIS code: 100.6640.

1. Introduction and Motivation

The Wigner function provides a representation of optical wave fields that is intuitively appealing and has specific mathematical properties worthy of application to optical systems whose phase-space description is relevant. In this paper we study the transformations undergone by a signal during the process of time multiplexing for the purpose of superresolution.[1] We use the generalized Wigner function representation in three coordinates: position, momentum (space frequency) and *wavelength*, as proposed in Refs. 2 and 3. Dependence on the wavelength has formerly not been used for analysis because the commonly known Wigner quasi-probability distribution-function formalism[4,5] is borrowed from quantum mechanics. In quantum mechanics the scale between the canonically conjugate observables of position and momentum is fixed to the value $\hbar$ by nature and thus applies strictly to monochromatic paraxial optical signals and wave forms. Note that there is a more generalized version of the Wigner representation that also includes the temporal coordinate.[6] However, for the purposes of this study the three-coordinate representation is sufficient.

The system we analyze in this paper is a Lukosz multiplexer with moving gratings[7] that is capable of sending a one-dimensional object signal through a finite-width aperture by segmenting it into parts separated by small differences in wavelength and reconstituting the signal thereafter. An illustration of such a multiplexer is shown in Fig. 1 and consists of the following modular steps:

(A) The object (input) signal f passes through a moving grating Γ.
(B) The object signal f then passes through a Fourier transformer.
(C) It next passes through the finite-width aperture.
($\bar{\text{B}}$) The signal again passes through a Fourier transformer.
($\bar{\text{A}}$) The countermoving grating $\bar{\Gamma}$ reconstitutes the signal as $\bar{f}$.

In Section 2 we briefly indicate the features we need from a simplified polychromatic paraxial Wigner function and the noncanonical transformation of relative motion. Multiplexing by module (A) occupies Section 3; Section 4 recalls the action of the Fourier transformer in module (B) on the Wigner function and its passage through a narrow slit in module (C). The reverse operations in modules ($\bar{\text{B}}$) and ($\bar{\text{A}}$) reconstitute the wave field into the output signal $\bar{f}$. Section 5 offers some concluding remarks.

2. Polychromatic and Quasi-Monochromatic Wigner Function

In the model of monochromatic paraxial optics, if we are given a signal $f(q, \lambda)$ of wavelength λ, its Wigner function is a bilinear functional of f and also a function of its position q and its conjugate optical momen-

K. B. Wolf is with the Instituto de Investigaciones en Matematicos Aplicadas y en Sistemos, Universidad Nacional Autonoma de Mexico, Apartado Postal 48-3, 62251 Cuernavaca, Morelos, Mexico. D. Mendlovic and Z. Zalevsky are with the Faculty of Engineering, Tel-Aviv University, 69978 Tel-Aviv, Israel.

Received 11 June 1997; revised manuscript received 9 February 1998.

0003-6935/98/204374-06$15.00/0

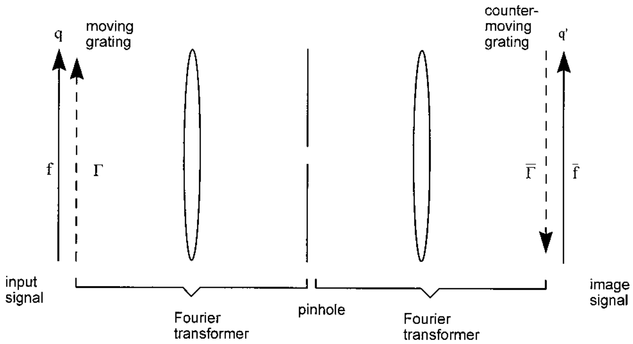

Fig. 1. Lukosz multiplexer.

tum $p = n \sin \theta \approx n\theta$ (where n is the refractive index and θ the angle between the ray and the optical axis; henceforth we consider $n = 1$), defined by[4,5]

$$W(f|q, p, \lambda) = \frac{1}{\lambda} \int_{-\infty}^{\infty} \mathrm{d}x f\left(q - \frac{x}{2}, \lambda\right)^* \times \exp(-2\pi i x p/\lambda) f\left(q + \frac{x}{2}, \lambda\right). \quad (1)$$

The polychromatic paraxial model[2] allows for $0 \neq \lambda \subset \mathcal{R}$ and results in the same formula. Since the wavelength shifts $\Delta\lambda$ needed for multiplexing are small, the general formalism is dispensable, and we can allow λ to be simply the third dimension orthogonal to the $q - p$ phase-space plane. The change of scale resulting from $\Delta\lambda$ is assumed negligible.

The Wigner function has built into it the important property of covariance under inhomogeneous linear transformations, i.e., under translations of position and momentum and under general linear maps of phase-space produced by free propagation, thin lenses, and compositions thereof, such as rotations corresponding to fractional Fourier transforms.[8] Under these translations, geometric and wave optics remain in one-to-one correspondence.[9]

Changes in wavelength resulting from relative cross motion between object and screen are not included in the theory, and here we introduce them "by hand" as paraxial, classical Doppler transformations depending on the small parameter $\gamma = v/c \ll 1$, where v is the relative velocity and c the Newtonian velocity of light. Then, for motion across the optical axis we have

$$p \to p + \gamma, \qquad \lambda \to \lambda(1 + \gamma p),$$
$$W(f|q, p, \lambda) \to W[f|q, p + \gamma, \lambda(1 + p\gamma)], \quad (2)$$

where we disregard terms of the order γ^2. An illustrative justification for Eq. (2) can be seen by our saying that the phase of a propagating plane wave in a free space includes $\exp(-ik_x x + 2\pi i \nu_0 t)$ where $k_x = (2\pi/\lambda)p$ and $\nu_0 = c/\lambda$. The movement of $x \to x + vt$ yields a term of $-2\pi i p v t/\lambda + 2\pi i c t/\lambda$. We wish this term to be $2\pi i c t/\lambda'$, and thus $\lambda' \approx \lambda(1 + \gamma p)$. As shown in Fig. 2, a monochromatic signal represented by a Wigner function in (q, p, λ) thus shifts in the direction p and slants the phase plane in wavelength.

3. Wigner Function of a Grated Signal

Assume an input signal $f(q, \lambda)$ is transmitted through a grating Γ of period L, whose Fourier series expansion is

$$\Gamma(q) = \frac{1}{\sqrt{L}} \sum_m \Gamma_m \exp(2\pi i m q/L),$$
$$\Gamma_m = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} \mathrm{d}q \Gamma(q) \exp(-2\pi i m q/L). \quad (3)$$

Then the Wigner function of the grated signal is

$$\begin{aligned} W(\Gamma f|q, p, \lambda) &= \frac{1}{\lambda} \int_R \mathrm{d}x \left\{ f\left(q - \frac{x}{2}, \lambda\right) \frac{1}{\sqrt{L}} \sum_m \Gamma_m \times \exp[2\pi i m(q - x/2)/L] \right\}^* \\ &\quad \times \exp(-2\pi i x p/\lambda) \left\{ f\left(q + \frac{x}{2}, \lambda\right) \times \frac{1}{\sqrt{L}} \sum_{m'} \Gamma_{m'} \exp[2\pi i m'(q + x/2)/L] \right\} \\ &= \frac{1}{\lambda L} \sum_{m,m'} \Gamma_m{}^* \Gamma_{m'} \int_R \mathrm{d}x f\left(q - \frac{x}{2}, \lambda\right)^* \times \exp\left\{2\pi i \left[\frac{m' - m}{L} q - \left(\frac{p}{\lambda} - \frac{m' + m}{2L}\right) x\right]\right\} f\left(q + \frac{x}{2}, \lambda\right) \\ &= \sum_n \left\{ \frac{1}{L} \sum_m \Gamma_m{}^* \Gamma_{n-m} \times \exp[2\pi i (n - 2m) q/L] \right\} \times W\left(f|q, p - \frac{n\lambda}{2L}, \lambda\right), \\ &= \sum_n W_n{}^\Gamma(q) W\left(f|q, p - \frac{n\lambda}{2L}, \lambda\right), \end{aligned} \quad (4)$$

where we have replaced the summation index with $n = m + m'$. We see thus that the effect of a grating is to produce multiple copies of the original Wigner function spaced apart in momentum p by $\lambda/2L$. In the last expression of Eq. (4) the coefficient $W_n{}^\Gamma(q)$ represents the intensity of each copy and is related directly to the shape of the grating.

To give an example, consider a full-contrast cosinusoidal grating

$$\Gamma^c(q) = \frac{1}{2}\left(1 + \cos \frac{2\pi q}{L}\right) \quad (5)$$

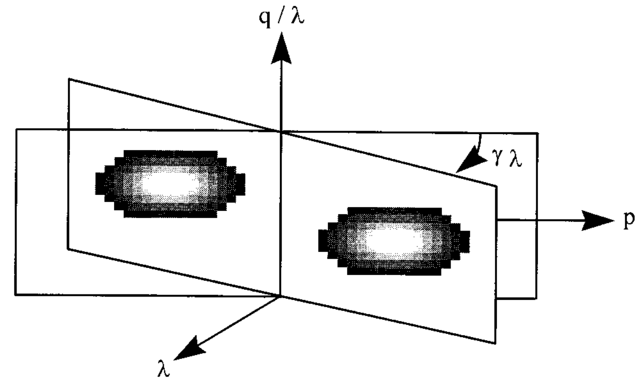

Fig. 2. Transformation of the Wigner function under relative motion between the signal and the grating by $\gamma = v/c$ (the value of $\gamma \ll 1$ is grossly exaggerated). It constitutes a shift in p by γ and a projection in λ by $\Delta\lambda = \lambda p \gamma$. A single-level curve is plotted.

whose Fourier coefficients [Eqs. (3)] are

$$\Gamma_{-1}{}^{c} = \frac{1}{4}\sqrt{L}, \qquad \Gamma_{0}{}^{c} = \frac{1}{2}\sqrt{L}, \qquad \Gamma_{1}{}^{c} = \frac{1}{4}\sqrt{L}, \tag{6}$$

with all others zero. Then the term in brackets of Eq. (4) includes five cross terms for $n = -2, -1, 0, 1, 2$, and the grating will give five replicas of the Wigner function:

$$\begin{aligned} W(\Gamma^{c} f|q, p, \lambda) = {} & \frac{1}{16} W\left(f|q, p - \frac{\lambda}{L}, \lambda\right) \\ & + \frac{1}{4} \cos\frac{2\pi q}{L} W\left(f|q, p - \frac{\lambda}{2L}, \lambda\right) \\ & + \left(\frac{1}{4} + \frac{1}{8}\cos\frac{4\pi q}{L}\right) W(f|q, p, \lambda) \\ & + \frac{1}{4} \cos\frac{2\pi q}{L} W\left(f|q, p + \frac{\lambda}{2L}, \lambda\right) \\ & + \frac{1}{16} W\left(f|q, p + \frac{\lambda}{L}, \lambda\right). \end{aligned} \tag{7}$$

Note that the two extreme terms are true replicas, the middle term ($n = 0$) is positive but presents oscillations in q of period $L/2$, while the $n = \pm 1$ terms oscillate in q with the period L (see Fig. 3).

Next consider a moving grating $\Gamma(q - vt)$, whose Fourier coefficients [Eqs. (3)] will be $\Gamma_m(t) = \Gamma_m \times \exp(-2\pi i m v t/L)$. The Wigner function of the moving grating [Eq. (4)] will follow Eq. (2) for all replicas, and the coefficient in brackets in Eq. (4) will have a further time dependence of

$$\begin{aligned} W_n{}^{\Gamma}(q, t) = {} & \frac{1}{L} \sum_m \Gamma_m{}^{*} \Gamma_{n-m} \\ & \times \exp[2\pi i(n - 2m)(q - vt)/L]. \end{aligned} \tag{8}$$

For the example of Eq. (7), the oscillation can be pictured in Fig. 3, where the middle three replicas mirror the movement of the grating, the central one with double the speed.

Optical sensors that integrate over time will not see those terms whose coefficients oscillate. The time rms average of the coefficients in Eq. (8) is

$$\langle W_n{}^{\Gamma}(q, t)\rangle = \frac{1}{L} \sum_m \Gamma_m{}^{*} \Gamma_{n-m} \delta_D(n - 2m) = \frac{1}{L} |\Gamma_{n/2}|^2 \tag{9}$$

for n even, and zero for n odd [$\delta_D(n - 2m) = 1$, if $n = 2m$ and 0 otherwise]. Therefore in Fig. 3 the $n = \pm 1$ replica will vanish, while the central one, $n = 0$, will reduce to its constant term.

The time-averaged Wigner function of Eq. (4) will similarly halve the number of terms to n even, becoming

$$\begin{aligned} \langle W[\Gamma(t) f|q, p, \lambda]\rangle = {} & \frac{1}{L} \sum_k |\Gamma_k|^2 W\left\{f|q, p + \gamma - \frac{k\lambda}{L}, \right. \\ & \left. \lambda\left[1 + \left(p - \frac{k\lambda}{L}\right)\gamma\right]\right\}, \end{aligned} \tag{10}$$

for integers k, where we have also replaced the effect of the relative motion on the phase-space and wavelength coordinates. This is the effect of a moving grating on the Wigner function of a signal. In the actual multiplexer, moreover, we should be aware that λ/L is a small quantity compared with the p extent of the signal, so instead of having clearly separated replicas, the single original Wigner function $W(f|q, p, \lambda)$ will unfold into overlapping copies of itself, separated by the wavelength

$$\Delta\lambda/\lambda = -k \frac{\lambda}{L}\frac{v}{c}, \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{11}$$

as suggested in Fig. 4. The time averaging contained in Eq. (10) refers to the Wigner function of the beam in the middle section of the multiplexer of Fig. 1, just after the moving grating.

4. Fourier Transforming and Slitting

After the beam leaves the moving grating, having unfolded into several superposed copies distinguished by small shifts in color, it is ready to undergo passage through a constricting neck in phase-space. If the neck is a finite-width aperture, it will restrict the horizontal spread of Fig. 4. Since a Fourier transform produces a 90° counterclockwise rotation of the Wigner distribution in the $q - p$ plane,[8] the middle section of Fig. 1 is equivalent to the application of a vertical restriction spread slit $R_w(p)$.

We remind the reader that the passage of a signal $f(q, \lambda)$ through a rectangular slit function of width w,

$$R_w(q) = \begin{cases} 1 & -w/2 < q < w/2 \\ 0 & \text{otherwise} \end{cases}, \tag{12}$$

has the effect of multiplying the signal by this function, $f(q, \lambda) \rightarrow R_w(q) f(q, \lambda)$. From its defining equation (1) it is easy to see that the support of the Wigner function $W(R_w f|q, p, \lambda)$ is then also restricted to

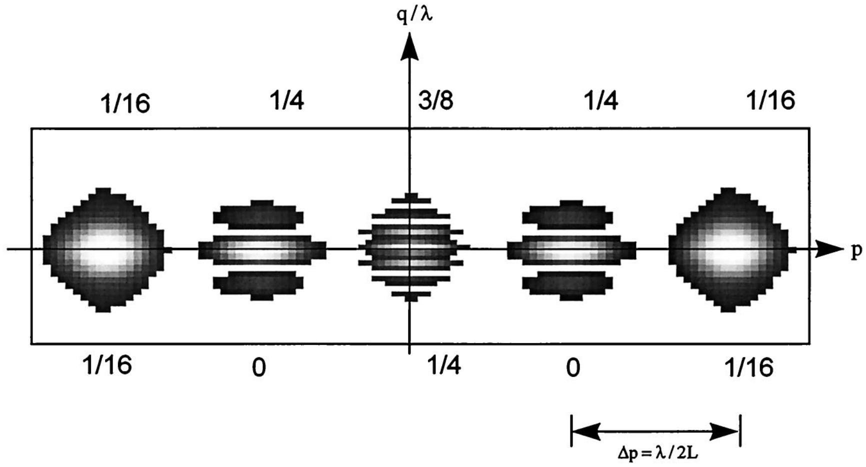

Fig. 3. Five replicas of a Wigner function produced by a cosinusoidal grating $\Gamma^c(q) = 1/2[1 - \cos(2\pi q/L)]$ of period L in q. The replicas stand apart in the angle $p \approx \theta$. Above are the maxima of the coefficients $W_n^{\Gamma}(q)$ of Eq. (4) (they sum to unity). Below are their rms values $\langle W_n^{\Gamma}(q, t)\rangle$ of Eq. (9).

$-w/2 < q < w/2$. In Fig. 5 we show a simple signal (Gaussian), a rectangular-slit function, the slitted signal, and their corresponding Wigner functions.

The multiplexed signal, after Fourier transformation by a lens (90°-counterclockwise rotation of the Wigner chart[8] and passage through the slit), will have the Wigner function shown in Fig. 6. If the width of the slit w is such that it corresponds to the separation in p of the multiplexed copies, as shown in the figure, namely, $\Delta p = w/\lambda$ or $w = \lambda^2/L$, then no part of the Wigner function will be lost. If the finite-width aperture is wider than this quantity, there will be redundancy in the information.

After the neck, the transformation is undone by a further Fourier transform and a grating moving in the opposite direction. When the Lukosz setup is modified by reflection of the beam closely after the finite-width aperture,[1] the same optical elements act in reverse order, the original rotating rosette grating is traversed by the light beam rotating in the opposite direction, and, since the CCD camera has its own integration time, the decoding occurs just as illustrated in the analysis of Section 3.

5. Concluding Remarks

In this paper the generalized Wigner distribution function has been used as an efficient tool for analyzing and understanding superresolution systems. The system under investigation has been a time-multiplexing temporally restricted superresolving setup. Its analysis has revealed the effects of each one of the optical elements in the setup over the generalized Wigner phase-space representation. The simplicity of the mathematical representation indicates the capability of the generalized Wigner distribution function for also analyzing other types of superresolving systems, such as wavelength- or direction-multiplexing setups. Note that a direct analysis would be much more complicated.

Nevertheless, we wish to indicate that the analysis

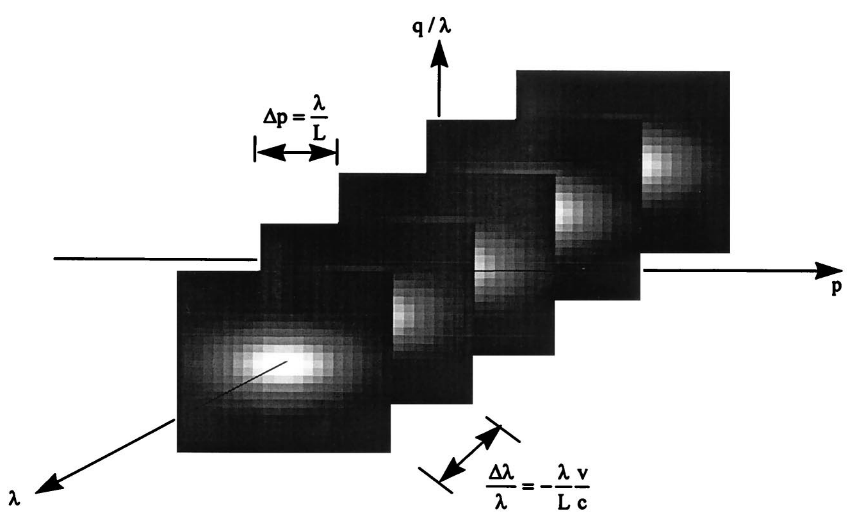

Fig. 4. Multiple copies of an originally monochromatic Wigner function separated by wavelength.

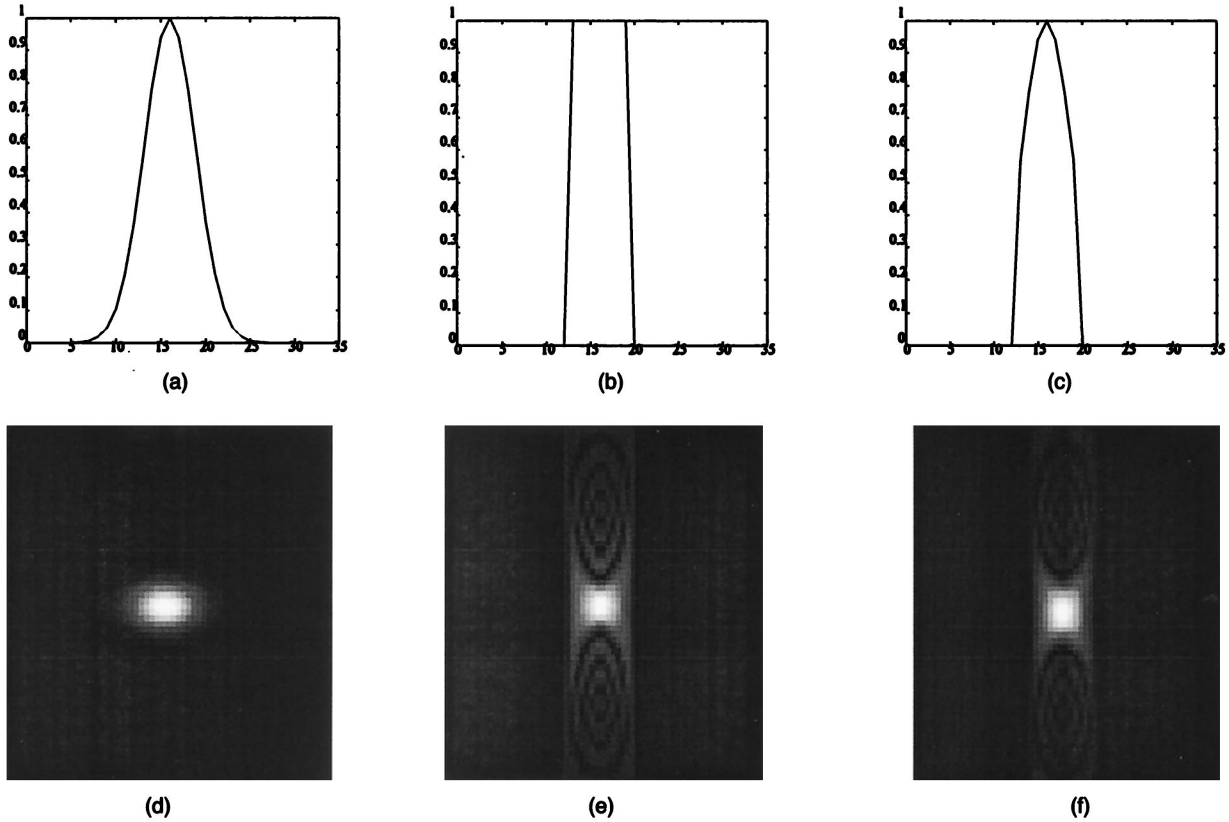

Fig. 5. Slitting of a Gaussian: (a) the Gaussian, (b) the slit, (c) the slitted Gaussian, (d)–(f) their corresponding Wigner functions.

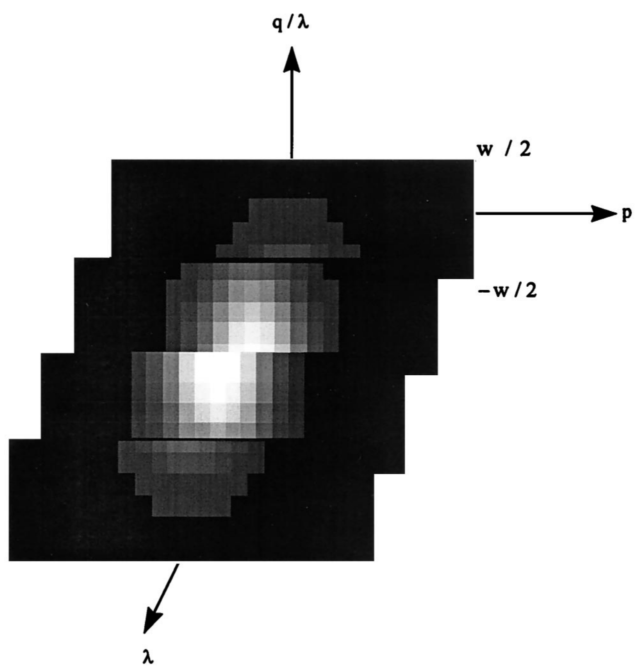

Fig. 6. Multiple copies of the original Wigner function after the plane of the slit. In this way each copy bears a different portion of the signal and is capable of passing the information through the spatial neck of the finite-width aperture.

presented here has not been a rigorous quantative proof for the validity of the time-multiplexing superresolution approach. The analysis has merely investigated the influence of the different parts of the setup over the generalized Wigner distribution by showing an additional application for it. It was more qualitative than quantative. Thus it has demonstrated the qualitative validity of the time multiplexing and showed that, for proper reconstruction, one needs $\omega = \lambda^2/L$.

The advantage of the analysis by means of the generalized Wigner function over the conventional analysis is that the generalized Wigner function presents simultaneously the behavior of the signal in the spatial, spectral, and wavelength coordinates. Thus it presents the theory of conversions of signal's degrees of freedom from one domain to the other more schematically.[10,11]

This paper has aimed to promote the generalized Wigner representation and to indicate additional aspects for its applicability as a mathematical–physical first-class tool of optical investigation.

Acknowledgements are due to the Israeli Ministry of Science and Art under the research program "Diffractive optical elements: fabrication and applications" and to the National Autonomous University of Maxico under project "Optica Matematica." Z. Zalevsky acknowledges the Eshkol fellowship, granted by the Israeli Ministry of Science and Art.

References

1. D. Mendlovic, A. W. Lohmann, N. Konforti, I. Kiryuschev, and Z. Zalevsky, "One-dimensional superresolution optical system for temporally restricted objects," Appl. Opt. **36,** 2353–2359 (1997).
2. K. B. Wolf, "Wigner distribution function for paraxial polychromatic optics," Opt. Commun. **132,** 343–352 (1996).
3. D. Mendlovic and Z. Zalevsky, "Definition and properties of the generalized temporal-spatial Wigner distribution function," Optik (Stuttgart) **107,** 49–61 (1997).
4. E. P. Wigner, "On quantum correction for thermodynamic equilibrium," Phys. Rev. **40,** 749–759 (1932).
5. H. W. Lee, "Theory and application of the quantum phase-space distribution functions," Phys. Rep. **259,** 147–211 (1995).
6. M. J. Bastiaans, "Transport equations for the Wigner distribution function in an inhomogeneous and dispersive medium," Opt. Acta **26,** 1333–1344 (1979).
7. W. Lukosz, "Optical systems with resolving powers exceeding the classical limit," J. Opt. Soc. Am. **56,** 1463–1472 (1966).
8. A. W. Lohmann, "Image rotation, Wigner rotation, and the fractional Fourier transform," J. Opt. Soc. Am. A **10,** 2181–2186, (1993).
9. O. Castanos, E. Lopez-Moreno, and K. B. Wolf, "Canonical transforms for paraxial wave optics," Vol. 250 of Springer-Verlag Lecture notes in Physics (Springer Veilag, Heidellerg, 1986), pp. 159–182.
10. D. Mendlovic and A. W. Lohmann, "Space–bandwidth product adaptation and its application to superresolution: fundamentals" J. Opt. Soc. Am. A **14,** 558–562 (1997).
11. D. Mendlovic, A. W. Lohmann, and Z. Zalevsky, "Space–bandwidth adaptation and its application to superresolution: examples," J. Opt. Soc. Am. A **14,** 563–567 (1997).