Finite Kerr medium: Macroscopic quantum superposition states and Wigner functions on the sphere

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We propose a spin model which exhibits the main properties of the Kerr medium. The description of the model uses a recently generalized quasiprobability distribution (Wigner function) for the SU(2) group. This function is naturally defined on the sphere, which plays the role of phase space for the spin system. Our model leads to macroscopic quantum superposition states on the sphere. [S1050-2947(99)06908-5]

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I. INTRODUCTION

To stress the intriguing implications of the quantum superposition of states which are macroscopically different, Schrödinger [1] proposed quantum states with the following structure:

$$\alpha|\text{living cat}\rangle+\beta|\text{dead cat}\rangle, \tag{1}$$

where the two components are superposed with amplitudes α and β. These have become known as Schrödinger cat states. It is certainly difficult to obtain and characterize states (1) in Schrödinger's original setting, so Yurke and Stoler [2] proposed instead to take a single mode of a radiation field, and superpose its coherent states with macroscopically different parameters. Similar ideas were formulated earlier by Bialynicka-Birula [3], and we also point out the odd and even coherent states discussed by Dodonov, Malkin, and Man'ko in Ref. [4].

It was shown in Ref. [2] that a superposition of coherent states with macroscopically different phases can be generated in the course of the evolution of a single-field mode in a Kerr medium with the Hamiltonian [5]

$$H_{\text{Kerr}}=\omega n+\chi n^2, \tag{2}$$

where $\hbar=1$. Here the operators a, $a^\dagger$, and $n=a^\dagger a$ describe a single-field mode of frequency ω, and χ is the Kerr constant. While the photon number distribution is clearly unchanged by Hamiltonian (2), it still leads to a nontrivial phase evolution.

In this paper, we propose a spin model which exhibits the main properties of the Kerr medium. The model is formulated in terms of spin operators S_i, $i=1$, 2, and 3, acting on the $(2l+1)$-dimensional Hilbert space of a particle of spin l. The natural phase space is a sphere of radius l; the spin-coherent states are thus represented by spots on the sphere. Here our model leads to Schrödinger cat states on the sphere, i.e., superpositions of several spots located ''far'' from each other. The description of the model and its dynamics will be made using a recently generalized Wigner quasiprobability distribution on the SU(2) group [6]. In the limit $l\to\infty$, when the su(2) algebra of spin operators contracts to the Heisenberg-Weyl algebra of boson operators, the sphere opens to the phase plane and the model coincides with the quantum harmonic oscillator or, through a renormalization, with the common Kerr medium.

After a brief review of the Kerr dynamics in Sec. II we formulate our model in Sec. III. Section IV describes the Wigner function on the SU(2) group; we discuss the evolution of the spin-coherent states into cat states in Sec. V. Possible physical applications are briefly discussed in the concluding section (VI).

II. KERR MEDIUM

The Kerr Hamiltonian (2) characterizes certain general properties of quantum dynamics. It can be used to generate both quadrature [7] and amplitude squeezing [8] for short times, when the initial wave packet spreads in phase and revolves around the origin in the phase plane [9]. When the phase spread exceeds 2π, the front of the wave packet interfaces with its tail. This self-interference is a quantum feature and has no classical counterpart [10]. At some time instants, the self-interference leads to standing waves which correspond to Schrödinger cat states. A similar phenomenon also appears in other examples of quantum nonlinear evolution, such as the Jaynes-Cummings model [11], where it is called *fractional revivals*; the Dicke model [12]; Rydberg wave packets [13]; and particles in one-dimensional anharmonic potentials [14,15].

It is convenient to ignore the fast rotations of the phase plane generated by the linear term ωn in Eq. (2), and to introduce the time scale $\tau=\chi t$. The nonlinear part n^2 of Hamiltonian (2) has an integer spectrum, and which is a specific feature of the Kerr medium. It is distinct from the Jaynes-Cummings and Dicke models, where the effective-field Hamiltonian (for special initial conditions [16]) is written as $H\sim\sqrt{n+1/2}$ [17]. The latter will also lead to phase spread and self-interference, but the standing waves are not so well pronounced as for the Kerr medium. Due to the integer spectrum of Hamiltonian (2), it is easy to prove that at times $\chi t=\pi K/M$ (where K and M are mutually prime integers) the wave function is a superposition of M copies of the initial state, placed equidistant along the circle $\overline{n}=\text{const}$ [18] (see also Ref. [19]). When $M=1$, the initial state is reproduced up to a global phase factor.

To describe a state Ψ of a quantum system, it is clearest to draw the picture of the corresponding Wigner quasiprobability distribution $W(\Psi|p,q)$, where p and q are classical coordinates of phase space [20]. Generalized coherent states have Gaussian Wigner functions that are positive [21]. Schrödinger cat states cannot be produced from a single Gaussian by linear dynamics. Quantum nonlinear dynamics introduces oscillations into the Wigner functions on phase space [9], with regions where the Wigner function takes negative values. For a two-component cat state, these oscillations are localized near the midpoint between the centers of the Gaussians (i.e., the ''smile of the cat'' [22]).

III. ANALOG OF THE KERR HAMILTONIAN FOR SPIN SYSTEMS

It is well known that the Heisenberg-Weyl algebra can be obtained by contraction from the su(2) algebra. For this purpose we consider the su(2) commutation relations

$$[S_3,S_\pm]=\pm S_\pm, \quad [S_+,S_-]=2S_3, \quad S_\pm=S_1\pm iS_2,$$

and, in a definite unitary irreducible representation l, we build the operators

$$b=\frac{S_-}{\sqrt{2l}}, \quad b^\dagger=\frac{S_+}{\sqrt{2l}}, \quad N=S_3+l. \tag{3}$$

The su(2) algebra in this representation is thus written as

$$[N,b^\dagger]=b^\dagger, \quad [N,b]=b, \quad [b,b^\dagger]=1-\frac{1}{l}N. \tag{4}$$

In the limit $l\rightarrow\infty$ these operators have the commutation relations of the oscillator algebra, with b and $b^\dagger$ being the usual boson operators.

Let us keep l finite, however, and consider the oscillator-like Hamiltonian

$$H=\frac{\omega}{2}(bb^\dagger+b^\dagger b). \tag{5}$$

In terms of the spin-l operators [Eq. (3)], this is written

$$H=\frac{\omega}{4l}(S_+S_-+S_-S_+)=\frac{\omega}{2l}(S^2-S_3^2)=\omega\left(N+\frac{1}{2}-\frac{1}{2l}N^2\right), \tag{6}$$

where ω is a constant. The analogy of Eq. (6) with the usual Kerr Hamiltonian (2) is evident, since the operator of the spin excitation number N has a non-negative integer spectrum $k=0,1,2,\ldots$. The difference between the photon number operator n and N is that the spectrum of the latter is bounded from above: $k\leqslant 2l$.

Using the commutation relation $f(N)S_+=S_+f(N+1)$ we obtain $S_+(t)=e^{itH}S_+e^{-itH}$, and from here the solution of the Heisenberg equations of motion is

$$S_+(t)=e^{i\omega t(2l+1)/2l}e^{-i\omega(t/l)N}S_+=[S_-(t)]^\dagger, \quad S_3(t)=\text{const},$$

in complete analogy with the Kerr medium [7,10]. We can write this solution in the forms

$$\begin{aligned} S_1(t)&=e^{i\omega t/2l}[\cos(\Omega t)S_1-\sin(\Omega t)S_2],\\ S_2(t)&=e^{i\omega t/2l}[\sin(\Omega t)S_1+\cos(\Omega t)S_2], \end{aligned} \tag{7}$$

where $\Omega=\omega(1-N/l)$ is an angular frequency which depends on the excitation number operator N. The time evolution of $\vec{S}(t)$ is a rotation around the z axis, but the precession frequency depends on the latitude on the sphere.

In the limit $l\rightarrow\infty$, Hamiltonian (6) becomes the usual quantum harmonic-oscillator Hamiltonian, i.e., the nonlinearity disappears. We can also modify Eq. (6) to obtain the optical Kerr Hamiltonian in this limit, replacing $\omega/2l$ by a free parameter χ. Indeed,

$$H=\omega(N+\tfrac{1}{2})-\chi N^2 \tag{8}$$

is the common Kerr Hamiltonian (2) as $l\rightarrow\infty$. Note, that in molecular physics [23], this model corresponds to a diatomic molecule approximated by a Morse potential. It was recently shown [24] that the effective Hamiltonian of type (8) describes the interaction of the collective atomic system with the off-resonant quantum radiation field in a dispersive cavity.

IV. WIGNER FUNCTION ON SU(2)

In this section we define the Wigner function on the group SU(2) introduced in Refs. [22] and [6]. We do this in a mathematically careful manner to avoid confusion with other different distribution functions on the sphere that have been used for spin systems [25]. We stress that with this definition we shall have full SU(2) covariance, that the generalized overlap formula holds, and that the reconstruction of states and density matrices can be accomplished, just as in quantum mechanics with the usual Wigner function [26].

We denote by $\vec{S}=\{S_i\}_{i=1}^d$ the row vector of spin operators closing into the su(2) algebra. The generic SU(2) group element in *polar parametrization* (indicated by brackets) is given by a 3 vector $\vec{y}$ of length $\xi=|\vec{y}|$ and unit direction $\vec{u}=\vec{y}/\xi=(\sin\theta\sin\phi,\sin\theta\cos\phi,\cos\theta)$,

$$g[\vec{y}]=\exp(-i\vec{S}\cdot\vec{y})=\exp[-i\xi(S_1u_1+S_2u_2+S_3u_3)]. \tag{9}$$

This is a rotation by the angle ξ around the axis $\vec{u}$, with $0\leqslant\xi\leqslant 2\pi, 0\leqslant\theta\leqslant\pi, 0\leqslant\phi<2\pi$; the group manifold is the 3 sphere $\mathcal{S}_3$. The Euler angles are also often used, writing an arbitrary SU(2) group element in the form

$$g(\alpha,\beta,\gamma)=\exp(i\alpha S_3)\exp(i\beta S_1)\exp(i\gamma S_3). \tag{10}$$

The connection between polar and Euler parameters is well known (see, e.g., Ref. [27]).

The three generators $\vec{S}=(S_1,S_2,S_3)$ transform under the action of the SU(2) group as the components of a row vector with the 3×3 orthogonal matrices $\mathbf{R}$ of the *adjoint* representation

$$g[\vec{y}]\vec{S}(g[\vec{y}])^{-1}=\vec{S}\mathbf{R}[\vec{y}]. \tag{11}$$

Applying the group transformation $g=g[\vec{y}]$ to an arbitrary operator $\vec{S}\cdot\vec{x}$ from the su(2) algebra, we have $g(\vec{S}\cdot\vec{x})g^{-1}=\vec{S}\cdot\vec{x}'$, where $\vec{x}'=\mathbf{R}\vec{x}$. Column vectors $\vec{x}$ belong to the three-dimensional real space $\mathbb{R}^3$, conjugate to the vector space of the Lie algebra, and carry the *coadjoint* representation of the group [28], $\vec{x}'=\mathbf{R}\vec{x}$. This space $\vec{x}\in\mathbb{R}^3$ we will call the *metaphase-space* of the classical dynamical system.

Acting with all group transformations $g[\vec{y}]$ on an arbitrary fixed vector $\vec{x}_0$ in the coadjoint representation, we obtain its group *orbit*. This orbit has a symplectic structure and is the common phase space [29,30]. When we choose a definite irreducible representation of spin l, the group acts on the Hilbert space of states $|\Psi\rangle$ of the corresponding quantum system with the scalar product $\langle\Phi|\Psi\rangle=\Sigma_{m=-l}^{l}\phi_m^*\psi_m$. An arbitrary pure spin state $|\Psi\rangle$ is represented by a column vector with components Ψ_m, $m=-l,\ -l+1,\dots,l$. For simplicity we use the same notation for the group element g and its unitary representation image [Eq. (9)].

We now define the *Wigner operator* as an operator-valued function on metaphase space,

$$\mathcal{W}_w(\vec{x})=\int_{\mathrm{SU}(2)}d\vec{y}\,w(\xi)\exp[i(\vec{x}-\vec{S})\cdot\vec{y}]$$
$$=\int_{\mathrm{SU}(2)}d\vec{y}\,w(\xi)\exp(i\vec{x}\cdot\vec{y})g[\vec{y}]. \tag{12}$$

Here $d\vec{y}=dy_1dy_2dy_2$, $\xi=|\vec{y}|$, $w(\xi)$ is a weight function, and the integral is over the group manifold.

In Ref. [6] we considered the Haar invariant measure over the group for integral (12), $w_{\mathrm{Haar}}(\xi)=\frac{1}{2}\xi^{-2}\sin^2\frac{1}{2}\xi$. While this is an obvious choice, it is not the only one; see below. We shall drop the subscript w from the Wigner operator, until more than one weight is used. For a given value of l, the Wigner operator is reduced to the $(2l+1)\times(2l+1)$-dimensional Hermitian matrix $\|W^l_{m,m'}(\vec{x})\|$.

The Wigner (quasiprobability distribution) function is the matrix element of the Wigner operator between the states $|\Psi\rangle$ and $|\Phi\rangle$ with definite l,

$$W(\Psi,\Phi|\vec{x})=\langle\Psi|\mathcal{W}(\vec{x})|\Phi\rangle=\sum_{m,m'=-l}^{l}\Psi_m^*W^l_{m,m'}(\vec{x})\Phi_{m'}. \tag{13}$$

Most often this is used for $\Psi=\Phi$; then we indicate $W(\Psi,\Psi|\vec{x})$ by $W(\Psi|\vec{x})$, and we suppress l when convenient. When instead of pure states we have a density matrix ρ, the Wigner function is

$$W(\rho|\vec{x})=\mathrm{tr}\{\mathcal{W}(\vec{x})\rho\}=\sum_{m,m'=-l}^{l}W^l_{m,m'}(\vec{x})\rho_{m',m}. \tag{14}$$

The W-matrix elements are

$$W^l_{m,m'}(\vec{x})=\int_0^{2\pi}w(\xi)\xi^2d\xi\int_{\mathcal{S}_2}d\vec{v}\,\exp(i\xi\vec{v}\cdot\vec{x})D^l_{m,m'}(\xi\vec{v}), \tag{15}$$

The case $w=w_{\mathrm{Haar}}(\xi)$ was considered in Ref. [6], where the corresponding W-matrix elements were obtained in terms of the common D-matrix elements $D^l_{m,m'}(\vec{y})=\langle l,m|\exp(-i\vec{y}\cdot\vec{x})|l,m'\rangle$, of angular momentum theory [31], also but independently associated with the name of Wigner (see the Appendix). It was shown in Ref. [6] that the radial projection of the Wigner function

$$W(\chi)=\int_{\mathcal{S}_2}d\vec{u}\,W(\Psi|\chi\vec{u}),\quad |\vec{x}|=\chi$$

has its maximum value over the sphere for radius $\xi\approx\sqrt{l(l+1)}$. The figures in this paper are values of the Wigner functions plotted for this radius. We now list the three most relevant properties of the Wigner function: covariance, overlap formula, and density-matrix reconstruction.

Covariance: From the form of integral (12) and the definition of the adjoint representation (11), it follows that the Wigner operator (12) satisfies the SU(2) covariance:

$$g[\vec{y}]\mathcal{W}(\vec{x})(g[\vec{y}])^{-1}=\mathcal{W}(\mathbf{R}(\vec{y})\vec{x}). \tag{16}$$

This means that the linear quantum dynamics can be completely described by the corresponding transformation in classical metaphase space. Covariance holds for any choice of the function $w(\xi)$ in Eq. (12).

Overlap formula: Consider two states of the same spin l, $|\Psi\rangle$ and $|\Phi\rangle$, with corresponding Wigner functions $W_w(\Psi|\vec{x})$ and $W_{\tilde{w}}(\Phi|\vec{x})$, for two different weight functions $w(\xi)$ and $\tilde{w}(\xi)$. We shall prove that the overlap of the two states can be calculated by integration over the metaphase space of the product of their Wigner functions, i.e.,

$$\frac{2l+1}{16\pi^5}\int_{\mathbb{R}^3}d\vec{x}\,W_w(\Psi|\vec{x})W_{\tilde{w}}(\Phi|\vec{x})=|\langle\Psi|\Phi\rangle|^2, \tag{17}$$

if and only if the two weight functions are conjugate in the following sense:

$$w(\xi)\tilde{w}(\xi)=w_{\mathrm{Haar}}(\xi)=\frac{1}{2\xi^2}\sin^2\frac{\xi}{2}. \tag{18}$$

This is the counterpart of the corresponding well-known property of the common Wigner function on the Heisenberg-Weyl group [20], where the Haar weight function is 1.

The integral on the left-hand side of Eq. (17) can be understood as a mean value of an operator $\hat{\sigma}$ which acts on the direct product of two Hilbert spaces of the same value of l,

$$\int d\vec{x}\,W_w(\Psi|\hat{x})W_{\tilde{w}}(\Phi|\vec{x})={}_1\langle\Psi|_2\langle\Phi|\hat{\sigma}|\Psi\rangle_1|\Phi\rangle_2, \tag{19}$$

where

$$\hat{\sigma}=\int_{\mathbb{R}^3}d\vec{x}\,\mathcal{W}_w(\Psi|\vec{x})\otimes\mathcal{W}_{\tilde{w}}(\Phi|\vec{x}). \tag{20}$$

We will show now that $\hat{\sigma}$ is proportional to an exchange operator. Integral (20) is found by replacing the definition of the Wigner operator (12) into Eq. (20); this can be integrated over $\vec{x}$ yielding a Dirac δ. Due to relation (18) between the weight functions, we have

$$\hat{\sigma}=(2\pi)^3\int_{\mathrm{SU}(2)} dg[\vec{y}]\exp(i\vec{y}\cdot\vec{S}_1)\otimes\exp(-i\vec{y}\cdot\vec{S}_2), \tag{21}$$

where $dg[\vec{y}]=\frac{1}{2}\sin^2(\xi/2)d\xi\, d\phi\, d\cos\theta$ is the invariant (Haar) measure, and the integral is over the group manifold. Let $\{|m\rangle_1|m'\rangle_2,\ -l\leqslant m,m'\leqslant l\}$ be an orthonormal basis in the tensor product of the two copies of the Hilbert space. Consider the matrix elements of Eq. (21),

$$_1\langle m|_2\langle m'|\hat{\sigma}|n\rangle_1|n'\rangle_2=(2\pi)^3\int_{\mathrm{SU}(2)} dg\, D^l_{mn}(g)\times D^l_{m'n'}(g^{-1}).$$

From unitarity we have $D^l_{m'n'}(g^{-1})=[D^l_{n',m'}(g)]^*$. Because of the well-known property of orthogonality of the D-matrix elements, we find

$$_1\langle m|_2\langle m'|\hat{\sigma}|n\rangle_1|n'\rangle_2=(2\pi)^3\int_{\mathrm{SU}(2)} dg\, D^l_{mn}(g)\times[D^l_{n',m'}(g)]^*=\frac{16\pi^5}{2l+1}\delta_{mn'}\delta_{nm'}.$$

Thus the operator $\hat{\sigma}$ exchanges the states of the first and the second Hilbert spaces,

$$\hat{\sigma}|\Psi\rangle_1|\Phi\rangle_2=\frac{16\pi^5}{2l+1}|\Phi\rangle_1|\Psi\rangle_2. \tag{22}$$

Comparing this result with Eq. (19), we find Eq. (17).

Clearly, if we choose the ''self-conjugate'' weight function

$$w_0(\xi)=\tilde{w}_0(\xi)=\frac{1}{\sqrt{2}\xi}\sin\frac{\xi}{2}, \tag{23}$$

then the overlap formula holds for the same w Wigner function. Another interesting choice of the weight function is $w(\xi)=\tilde{w}_{\mathrm{Haar}}=1$.

Reconstruction of the state: Given the Wigner function $W(\rho|\vec{x})=\mathrm{tr}\{\mathcal{W}_w(\vec{x})\rho\}$, we can reconstruct the density matrix ρ of the corresponding state of the system. Let $\{|f_n\rangle\}_{n=-l}^{l}$ be an orthonormal basis in the $(2l+1)$-dimensional Hilbert space. Denote by $\tilde{W}_{n,m}$ the matrix elements of the $\tilde{w}$-conjugate Wigner operator between the states $|f_n\rangle$ and $|f_m\rangle$,

$$\tilde{W}_{n,m}(\vec{x})=\langle f_n|\mathcal{W}_{\tilde{w}}(\vec{x})|f_m\rangle.$$

The elements of the density matrix in this basis are given by

$$\rho_{n,m}=\frac{2l+1}{16\pi^5}\int_{\mathbb{R}^3} d\vec{x}\,\tilde{W}_{n,m}(\vec{x})W(\rho|\vec{x}). \tag{24}$$

Indeed, for the pure states $\rho=|\Psi\rangle\langle\Psi|$, from Eqs. (20) and (22) it follows that

$$\frac{2l+1}{16\pi^5}\int_{\mathbb{R}^3} d\vec{x}\,\tilde{W}_{n,m}(\vec{x})W(\Psi|\vec{x}) =\frac{2l+1}{16\pi^5}\,{}_1\langle f_n|_2\langle\Psi|\hat{\sigma}|f_m\rangle_1|\Psi\rangle_2 ={}_1\langle f_n|\Psi\rangle_1\,{}_2\langle\Psi|f_m\rangle_2=\rho_{n,m}.$$

For mixed states, consider the basis $\{|r_n\rangle\}_{n=-l}^{l}$ where the density matrix is diagonal: $\rho=\Sigma c_n|r_n\rangle\langle r_n|$. Then $W(\rho|\vec{x})=\Sigma c_n W(r_n|\vec{x})$, and we have

$$\frac{2l+1}{16\pi^5}\int_{\mathbb{R}^3} d\vec{x}\,\tilde{W}_{n,m}(\vec{x})W(\rho|\vec{x}) =\frac{2l+1}{16\pi^5}\sum_k c_k\int_{\mathbb{R}^3} d\vec{x}\,\tilde{W}_{n,m}(\vec{x})W(r_k|\vec{x})=\rho_{n,m}.$$

We note that the SU(2) Wigner function provides the SU(2) Q function used by Agarwal, Puri, and Singh in Ref. [24]. Indeed, the overlap of the w Wigner function of a state $|\Psi\rangle$ with the $\tilde{w}$ Wigner function of a coherent states $|\alpha,\beta\rangle$ on the sphere (see below) is $Q(\alpha,\beta)=|\langle\alpha\beta|\Psi\rangle|$, the nonnegative function studied in Ref. [24].

V. EVOLUTION OF COHERENT STATES ON THE SPHERE

The above concepts will be now applied to describe the evolution in a Kerr medium of spin-coherent states.

Spin-coherent states. We now particularize the spin-l model to a system of $2l$ two-level atoms prepared initially in a state symmetric under permutations of atoms. If the interaction Hamiltonian also has this symmetry, then there are $2l+1$ accessible states $|m=k-l\rangle$, $0\leqslant k\leqslant 2l$, called the Dicke states [32]; here $2l-k$ is the number of atoms in the ground state, and k is the number of atoms in the excited state.

The simplest examples of spin-coherent states [33,30], are the states $|0\rangle$ (spin down) and $|2l\rangle$ (spin up), both of which have uncertainties $\Delta S_3=0$ and $\Delta S_1=\Delta S_2=\sqrt{l/2}$. All the other spin-coherent states $|\alpha,\beta\rangle$ can be produced by rotation from the state $|0\rangle$, as

$$g(\alpha,\beta,\gamma)|0\rangle=e^{i\alpha S_3}e^{i\beta S_1}e^{i\gamma S_3}|0\rangle =e^{-il\gamma}e^{i\alpha S_3}e^{i\beta S_1}|0\rangle=e^{-il\gamma}|\alpha,\beta\rangle, \tag{25}$$

where we have used Eq. (10). The rotation angle γ leads to a phase factor which may be disregarded. Mathematically this corresponds to the reduction from SU(2) to the factor space SU(2)/SU(1)$=\mathcal{S}_2$ [30]. The general form of spin-coherent states is thus $|\alpha,\beta\rangle=e^{i\alpha S_3}e^{i\beta S_1}|0\rangle$. The manifold of spin-coherent states is phase space [30], and is a sphere of radius l (called the *Bloch* sphere in quantum optics and the *Poincaré* sphere in polarization optics [35]). The components of spin-coherent states in the eigenbasis of $\mathcal{S}_3$ are the SU(2) d-matrix elements [27]

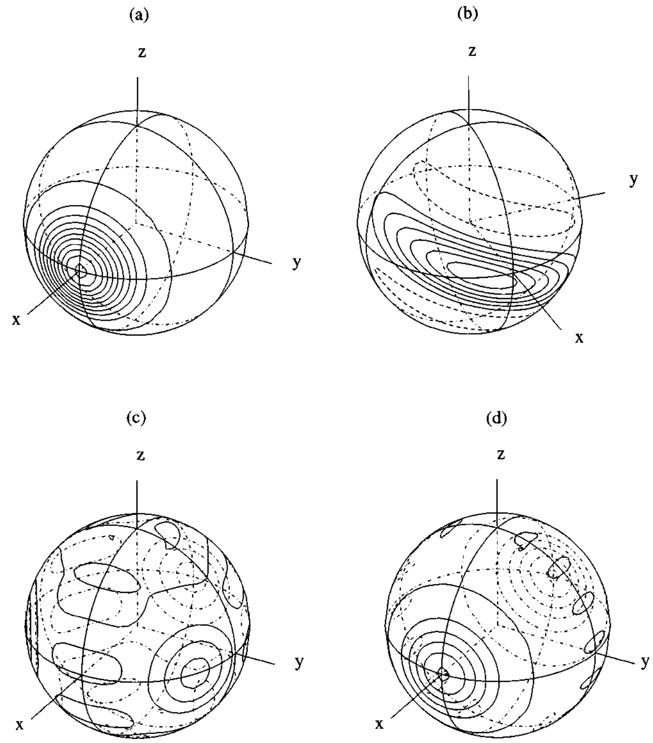

FIG. 1. Time evolution of the Wigner function on the sphere governed by a nonlinear finite Kerr Hamiltonian, for $l=8$. The plots have the following line code: ——, visible grid, positive, and zero-level lines; —.—.—.—, invisible grid and positive-level lines; — — —, visible negative- and zero-level lines; $\cdots$, invisible negative-level lines. The times are (a) $\chi t=0$ (initial coherent state), (b) $\chi t=0.15$, (c) $\chi t=\pi/3$, and (d) $\chi t=\pi/2$.

$$\langle m|\alpha,\beta\rangle = e^{im(\alpha-\pi/2)} d^{l}_{m,0}(\beta) = e^{im(\alpha-\pi/2)} \times\left(\frac{(2l)!}{(l-m)!(l+m)!}\right)^{1/2} \cos^{l+m}\frac{\beta}{2}\sin^{l-m}\frac{\beta}{2}. \tag{26}$$

The Wigner function of the spin-coherent state $|\alpha, \beta\rangle$ is a round, concentrated blob on this sphere, centered at the point (α, β). The radius of the Wigner blob on the sphere is $\sqrt{l/2}$, whereas the radius of the sphere is l. The classical limit for the spin system occurs when the dimension of the representation, $2l+1$, grows without bound and thus the direction of the vector $\vec{S}$ becomes sharper. Hence the classical limit of a spin system is a magnetic needle, i.e., a compass.

The evolution of the initial spin coherent state under the finite Kerr Hamiltonian [Eq. (6)] is shown in Figs. 1(a)–1(f) for $l=8$. The level plots of the Wigner function with $w(\xi) = w_{\rm Haar}(\xi)$ for the value of the radial coordinate $|\vec{x}| = \sqrt{l(l+1)}$ are shown for various subsequent times. Let us analyze these figures.

We start in Fig. 1(a) at $t=0$ with an eigenstate of S_1 of eigenvalue l; this is a coherent state. As time evolves, the points of phase space move only in α (geographical longitude, or phase), keeping β (colatitude) constant. This evolution multiplies eigenstates $|m=k-l\rangle$ by phases

$$\exp\{-it[\omega(k+\tfrac{1}{2})-\chi k^2]\}, \tag{27}$$

and keeps the probability distribution of the states $|m\rangle$. The phase $\omega t k$ leads to the rigid rotation of the sphere around the vertical axis; this will be extracted so that we consider only the nonlinear phase shift $t\chi k^2$. In analogy with the usual Kerr medium, as time evolves we can expect phase spread and squeezing along some direction different from the 3-axis. Indeed, this is shown in Fig. 1(b). Slanted squeezing was reported previously for this Hamiltonian by Kitagawa and Ueda [36]. For longer times, self-interference appears. This is shown in Figs. 1(c) and 1(d), for triple and double resonances, respectively, at times $\chi t=\pi/3$ and $\pi/2$. We now proceed to prove that the latter are true Schrödinger cat states [24] and find the amplitudes of the components in the cat states.

Schrödinger cats on the sphere. The figures suggest that at resonance times $\chi t=\pi K/M$ (with K and M mutually prime; we take $K=1$ for simplicity), the initial coherent state unfolds into a sum of M coherent states placed equidistant around the equator.

Indeed, the nonlinear phase in Eq. (27) is produced by the unitary operator $\exp(i\chi t N^2)$, where N is the level number operator defined in Eq. (3). At times $\chi t=\pi/M$, for M even, the eigenvalues $\exp(i\pi k^2/M)$ of the operator $\exp(i\chi t N^2)$ are periodic in k with period M (invariant under the substitution $k\rightarrow k+M$). For M odd, we consider instead the operator $\exp[i\chi t N(N-1)]$ times an oscillator phase (linear in N). As for the common Kerr medium [19,18], this periodicity in N allows us to use the finite Fourier transform basis of phases $\{M^{-1/2}e^{-2\pi isk/M}\}_{s=0}^{M-1}$. Thus we expand [34],

$$e^{i\pi N^2/M} = \frac{e^{i\pi/4}}{\sqrt{M}}\sum_{s=0}^{M-1} f_s e^{-2\pi isN/M},$$

$$f_s = e^{-i\pi s^2/M}, \quad M \text{ even}, \tag{28}$$

$$e^{i\pi N(N-1)/M} = \frac{e^{i\pi/4}}{\sqrt{M}}\sum_{s=0}^{M-1} f_s e^{-2\pi isN/M},$$

$$f_s = e^{-i\pi(s-1/2)^2/M}, \quad M \text{ odd}. \tag{29}$$

The evolution operator at cat times is therefore a sum of rotations $e^{2\pi isN/M}$ by angles $2\pi s/M$, $s=0,1,\ldots,M-1$, acting on the initial coherent state. The wave function is thus a sum of M coherent states with these rotation angles and with the amplitudes f_s i.e., an M-component Schrödinger cat state. As in the case of the Heisenberg-Weyl algebra (the Wigner function on the plane), the Wigner function on the sphere shows interference fringes between different components of the cat state (the ''smile of the cat''); see Figs. 1(c) and 1(d).

VI. CONCLUSIONS

We have used a recently defined Wigner function on SU(2) to understand spin-coherent states and their evolution under a physically realistic Kerr Hamiltonian. This spin system is realized in quantum optics as a collective atomic system [32,12], as a pair of radiation field modes of two different polarizations [35], and also as an atomic interferometer [37,38]. The effective nonlinearity in the case of a collective atomic system arises due to the dynamical Stark shift [39,40] (and it has been proposed to generate the atomic Schrödinger cat states in Ref. [41]) or due to the interaction with radiation field in a dispersive cavity out of resonance with the atomic transition frequency; see Ref. [24], where the equivalent model was described in terms of the atomic Q function (the diagonal matrix element of the atomic density matrix between the spin-coherent states). In an atomic interferometer the nonlinearity may be created by a nonlinear active element such as a Coulomb coupler [42].

In wave optics our model describes the propagation of a finite number of transverse modes in a waveguide [6]. An ideal (Gaussian) waveguide corresponds to the infinite equidistant spectrum and usually is described by a harmonic oscillator. In real waveguides with a finite number of modes, the spectrum is always different from the equidistant one, which leads to the effective nonlinearity. Finally, in molecular physics [23], our model corresponds to a diatomic molecule approximated by a Morse potential.

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APPENDIX

Here we give a convenient formula for the calculation of the Wigner function for a spin system. From the covariance it follows that the Wigner operator at arbitrary point $\vec{x} = \eta(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$ can be obtained by the rotation of the Wigner operator at the North pole, $W(\eta\vec{k})$, $\vec{k} = (0,0,1)$,

$$W(\vec{x}) = e^{-i\phi S_z} e^{-i\theta S_x} W(\eta\vec{k}) e^{i\theta S_x} e^{i\phi S_z}.$$

In turn, it is easy to prove that at the Wigner operator is diagonal at the North pole, $W^l_{m,m'} = \delta_{m,m'} W_m$. The quantities $W_m(\eta)$ are the eigenvalues of the Wigner operator; they depend only on $\eta = |\vec{x}|$ and can be written in the form

$$W_m(\eta) = 2\pi \sum_{k=-l}^{l} \int_{-1}^{1} d\cos\theta |d^l_{mk}(\cos\theta)|^2 F(\eta\cos\theta - k),$$

$$F(y) = \int_0^{2\pi} d\xi\, \xi^2 w(\xi) e^{i\xi y}.$$

where, for the case $w(\xi) = (1/2\xi^2)\sin^2(\xi/2)$,

$$F(y) = \frac{(-1)^{2l}\sin(2\pi y)}{8}\left[\frac{-1}{y+1} + \frac{2}{y} + \frac{-1}{y-1}\right],$$

and for the case $w(\xi) = (1/\xi)\sin(\xi/2)$,

$$F(y) = (-1)^{2l}\left\{\pi e^{i2\pi y}\left(\frac{1}{y+1/2} - \frac{1}{y-1/2}\right) + \frac{(e^{i2\pi y} + (-1)^{2l})}{2i}\left(\frac{1}{(y-1/2)^2} - \frac{1}{(y+1/2)^2}\right)\right\}.$$

Note that k takes integer or half-integer values depending on whether l is an integer or half-integer; this leads to the factor $(-1)^{2l}$ in the last equation.

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