Canonical transforms. II. Complex radial transforms

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Continuing the line of development of Paper I [J. Math. Phys. **15**, 1295 (1974)], we enlarge the concept of canonical transformations in quantum mechanics in two directions: first, by allowing the definition of a canonical transformation to be made through the preservation of an so(2,1) algebra, rather than the usual Heisenberg algebra, and providing the bridge between the classical and quantum mechanical descriptions, and, second, through the complexification of the transformation group. In this paper we study specifically the transformations which can be interpreted as the radial part of n-dimensional complex linear transformations in Paper I. We show that we can build Hilbert spaces of analytic functions with a scalar product defined through integration over half the complex plane of a variable which has the meaning of a complex radius. A unitary mapping to the ordinary Hilbert space $\mathcal{L}^2_{r^{n-1}}(0,\infty)$ is provided with a kernel involving a Bessel function. Special cases of this are shown to be the Barut–Girardello, one-dimensional Bargmann and Hankel transforms. The transform kernels provide a series of representations of a subsemigroup of $SL(2,\mathbb{C})$ and allow the construction of coherent states for the harmonic oscillator with an extra centrifugal force. We present a hyperdifferential operator realization of these transforms which yields new Baker–Campbell–Hausdorff and special function relations.

1. INTRODUCTION

In the article which started this series (Ref. 1, henceforth refered to as I), we described complex linear transformations between the quantum-mechanical operators of position $\hat{\mathbf{x}}$ and momentum $\hat{\mathbf{p}}$, and a new pair of quantities given by

$$\hat{\eta} = a\hat{\mathbf{x}} + b\hat{\mathbf{p}}, \qquad \hat{\boldsymbol{\zeta}} = c\hat{\mathbf{x}} + d\hat{\mathbf{p}}, \tag{1.1a}$$

with the unimodularity condition

$$ad - bc = 1, \tag{1.1b}$$

which ensures that (1.1a) is a canonical transformation in the sense that

$$[\hat{\mathbf{x}}, \hat{\mathbf{p}}] = i\mathbb{1} \Leftrightarrow [\hat{\eta}, \hat{\boldsymbol{\zeta}}] = i\mathbb{1}. \tag{1.1c}$$

The motivation for such a program was the observation that particular complex transformations (1.1) have been fruitful: Bargmann[2,3] considered (1.1a) with

$$a = 2^{-1/2} = d, \quad b = -i2^{-1/2} = c \tag{1.2}$$

and the ensuing formalism has been applied to the coherent-state description of quantum optics.[4] Equations (1.1) for a,b,c,d real have provided unitary representations[5,6] of $SL(2,\mathbb{R})$ and, when continued into some regions of the complex plane of the parameters, have been used to relate and evaluate matrix elements of n-body systems subject to Gaussian-potential interactions relevant for the nuclear cluster model.[7]

In I we showed that: (i) The three examples given above are particular cases of a *canonical transform* (1.1) for $a,b,c,d \in \mathbb{C}$, the complex field, between the Hilbert space $\mathcal{H} \equiv \mathcal{L}^2(\mathbb{R})$ of square-integrable functions over the real line $\mathbb{R}$ and spaces $\mathcal{F}_{(a,b,c,d)}$ isomorphic to the Bargmann space of entire analytic functions in $\mathbb{C}$ with the well-known scalar product and decrease conditions.[2] (ii) A unitary transformation between $\mathcal{H}$ and $\mathcal{F}$ could be implemented [for $\mathrm{Im}(a/b) \geq 0$ and b real when $a=0$] which contained the Bargmann transform for (1.2) and the Moshinsky–Quesne transform[5] for $a,b,c,d \in \mathbb{R}$. (iii) The transform kernels provided a representation of a subsemigroup of $SL(2,\mathbb{C})$ for $a,b,c,d \in \mathbb{C}$ subject to certain conditions.[8] (iv) A realization of these transforms through hyperdifferential operators was given, defined at least on spaces of entire functions. The defining conditions for $\mathcal{F}_{(a,b,c,d)}$ were to find a scalar product where $\hat{\eta}$ and $\hat{\zeta}$ had the hermiticity properties derived from (1.1a) and the self-adjointness of $\hat{x}$ and $\hat{p}$ and were represented in the Schrödinger realization η and $-i\,\partial/\partial\eta$ on functions of $\eta \in \mathbb{C}$. The results were seen as a step towards exploiting the fact that quantum mechanics, being a richer structure than classical mechanics, and making use of the complex field in an essential way, should be amenable to a wider class of canonical transformation—defined through (1.1c)—than have been generally considered,[9] introducing scalar products more general than the usual Dirac integral over $\mathbb{R}$.

Among the extensions foreseen in I were to consider *n-dimensional* transformations (1.1) where $\hat{\mathbf{x}} \equiv (\hat{x}_j)$, $j=1,\ldots,n$ etc. were n-vectors, but a,b,c,d remained (complex) multiples of the unit matrix. Equation (1.1c) now takes the familiar form $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$, etc. The "angular" properties, as given by the angular momentum operators in any of the subspaces, remain invariant under (1.1a) since the unimodularity condition (1.1b) insures that

$$L_{jk} \equiv \hat{x}_j\hat{p}_k - \hat{x}_k\hat{p}_j = \hat{\eta}_j\hat{\zeta}_k - \hat{\eta}_k\hat{\zeta}_j. \tag{1.3}$$

The "radial" part of (1.1) is displayed through the three equations

$$\eta^2 = a^2\hat{\mathbf{x}}^2 + 2ab\hat{\mathbf{x}}\cdot\hat{\mathbf{p}} + b^2\hat{\mathbf{p}}^2 - inab, \tag{1.4a}$$

$$\hat{\eta}\cdot\hat{\boldsymbol{\zeta}} = ac\hat{\mathbf{x}}^2 + (ad+bc)\hat{\mathbf{x}}\cdot\hat{\mathbf{p}} + bd\hat{\mathbf{p}}^2 - inbc, \tag{1.4b}$$

$$\boldsymbol{\zeta}^2 = c^2\hat{\mathbf{x}}^2 + 2cd\hat{\mathbf{x}}\cdot\hat{\mathbf{p}} + d^2\hat{\mathbf{p}} - incd. \tag{1.4c}$$

Seen classically, the canonical transformation can be described setting $\mathbf{x}^2 = r^2$, $\mathbf{x}\cdot\mathbf{p} = rp_r$, where the Poisson bracket $\{r, p_r\} = 1$, so that r and p_r are canonically conjugate quantities and $\mathbf{p}^2 = p_r^2 + p_\theta^2/r^2$, p_θ being the (constant) angular momentum. Correspondingly $\eta^2 = \rho^2$, $\boldsymbol{\eta}\cdot\boldsymbol{\zeta} = \rho p_\rho$, $\boldsymbol{\zeta}^2 = p_\rho^2 + p_\theta^2/\rho$. Equations (1.4) then read

$$\rho = [a^2r^2 + 2abrp_r + b^2(p_r^2 + p_\theta^2/r^2)]^{1/2}, \tag{1.5a}$$

$$p_\rho = [acr^2 + (ad+bc)rp_r + bd(p_r^2 + p_\theta^2/r^2)]/\rho, \tag{1.5b}$$

and the transformation of the pairs $(r,p_r) \to (\rho,p_\rho)$ can be checked to be canonical (i.e., $\{r,p_r\}=1 \Leftrightarrow \{\rho,p_\rho\}=1$). As the variable r takes values in $\mathbb{R}^+$ (the half-axis $[0,\infty)$), $\rho^2 \in \mathbb{C}$, and ρ will take values on half this region, which we can choose as

$$\mathbb{C}^+ \equiv \{\rho \in \mathbb{C} \mid \arg(\rho) \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)\}.$$

The transformation (1.5) is not particularly simple-looking, yet its quantum mechanical version will be seen to be implementable. This suggests that the definition of a quantum mechanical canonical transformation be made not in terms of the conservation of the Heisenberg algebra[9] as in (1.1c), which loses its meaning since the "quantization" of (1.5) is not well defined. The alternative, as suggested in this paper, is its definition in terms of the conservation of a higher algebra, in this case $so(2,1)$, which can be built out of the basic classical quantities.

The Schrödinger representation[10] of the operators $\hat{\mathbf{x}}^2$, $\hat{\mathbf{x}}\cdot\hat{\mathbf{p}}$, and $\hat{\mathbf{p}}^2$ is

$$\hat{\mathbf{x}}f(r) = r^2 f(r), \tag{1.6a}$$

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{p}}f(r) = -ir\frac{d}{dr}f(r), \tag{1.6b}$$

$$\hat{\mathbf{p}}^2 f(r) = -\left(\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} + \frac{\lambda}{r^2}\right)f(r), \quad \lambda \in \mathbb{R}, \tag{1.6c}$$

on the (at least twice-differentiable) elements of the Hilbert space $\mathcal{H}_n^+ \equiv \mathcal{L}^2_{r^{n-1}}(\mathbb{R}^+)$ of functions $f(r)$ on the positive half-axis with the scalar product

$$(f,g)_n = \int_0^\infty r^{n-1}\,dr\,f(r)^* g(r), \tag{1.7}$$

(the star indicates complex conjugation). The operators (1.6) are Hermitian between these elements and their domain can be enlarged through the usual adjunction procedure to self-adjoint operators in $\mathcal{H}_n^+$. The constant λ in (1.6c) comes from the spectrum of $L^2 = \frac{1}{2}\sum L_{ij}L_{ij}$ when acting on the $so(n)$-irreducible components of the functions, and has the values

$$\lambda = -l(l+n-2), \quad l = 0,1,2,\cdots. \tag{1.8}$$

The statements concerning the hermiticity of $\hat{\mathbf{p}}^2$ continue to be valid, however, for arbitrary $\lambda \in \mathbb{R}$.

It is the purpose of this article to describe a family of Hilbert spaces $\mathcal{F}^+_{nl\,(a,b,c,d)}$ (the indices a,b,c,d will be suppressed) for which a Schrödinger representation parallel to (1.6) can be implemented for the new variables in (1.4), namely

$$\hat{\eta}^2 \bar{f}(\rho) = \rho^2 \bar{f}(\rho), \tag{1.9a}$$

$$\hat{\eta}\cdot\hat{\boldsymbol{\zeta}}\bar{f}(\rho) = -i\rho\frac{d}{d\rho}\bar{f}(\rho), \tag{1.9b}$$

$$\hat{\boldsymbol{\zeta}}^2 \bar{f}(\rho) = -\left(\frac{d^2}{d\rho^2} + \frac{n-1}{\rho}\frac{d}{d\rho} + \frac{\lambda}{\rho^2}\right)\bar{f}(\rho) \tag{1.9c}$$

on functions of the complex variable ρ restricted to the region $\mathbb{C}^+$ [Eq. (1.6)]. In order that the total derivative with respect to a complex variable be well defined, the functions $\bar{f}$ will be *analytic* functions of ρ and $\partial\bar{f}(\rho)/\partial\rho^* = 0$. The measure for the defining scalar product in $\mathcal{F}^+_{nl}$,

$$(\bar{f},\bar{g})_{nl} = \int_{\mathbb{C}^+} d\mu_{nl}(\rho)\bar{f}(\rho)^*\bar{g}(\rho) \tag{1.10a}$$

is of the form

$$d\mu_{nl}(\rho) = \nu_{nl}(\rho,\rho^*)\,d\,\mathrm{Re}\rho\,d\,\mathrm{Im}\rho, \tag{1.10b}$$

where the weight function $\nu_{nl}(\rho,\rho^*)$ will be found from the hermiticity properties of (1.9)–(1.1) and the hermiticity of $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$. This will be performed in Sec. 2 and the characteristics of the Hilbert space $\mathcal{F}^+_{nl}$ ascertained. In Sec. 3 we will find the unitary transformation between $\mathcal{H}_n^+$ and $\mathcal{F}^+_{nl}$ as given by

$$\bar{f}(\rho) = \int_0^\infty r^{n-1}\,dr\,A_{nl}(\rho,r)f(r), \tag{1.11a}$$

$$f(r) = \int_{\mathbb{C}^+} d\mu_{nl}(\rho)A_{nl}(\rho,r)^*\bar{f}(\rho), \tag{1.11b}$$

through the transform kernel $A_{nl}(\rho,r)$ function of n,l and a,b,c,d. This *complex radial* transform will relate to the complex linear transform of I as the Hankel transform relates to the n-dimensional Fourier transform and, as will be shown, contains the Barut–Girardello transform[11] for the value (1.2) of the parameters[12] and the radial transform of Moshinsky, Seligman, and Wolf in Ref. 13 for a,b,c,d real. In Sec. 4 it is shown that this last transform is indeed regained when a,b,c,d become real and that the scalar product (1.10) collapses to the line integral (1.7). The one-dimensional Bargmann space[2] is also regained when $n=1$ as the direct sum of $\mathcal{F}^+_{10}$ and $\mathcal{F}^+_{11}$. We consider the interest of the complex radial transform to go beyond that of the mere description of the radial part of a known transform: As we will be mapping the radial wavefunctions of potentials of the harmonic oscillator + centrifugal potential ($\sim 1/r^2$) kind on functions of the type ρ^{2N+l}, coherent states for these systems can be defined. This is shown in Sec. 5. In Sec. 6 we make the composition of transforms and shown that the transform kernels provide a representation of a subsemigroup of $SL(2,\mathbb{C})$ in (1.1). Some conclusions of the role of complex canonical transformations in quantum mechanics are presented in Sec. 7. In two appendices we give a hyperdifferential operator realization of the transform (1.8) obtaining a new representation of the associated Laguerre functions and its direct relation to the n-dimensional complex linear transform.

2. THE SPACE $\mathcal{F}^+_{nl}$

We will construct a space $\mathcal{F}^+_{nl\,(a,b,c,d)}$ of functions $\bar{f},\bar{g}$ over $\rho \in \mathbb{C}^+$ endowed with a scalar product of the type (1.10) such that the operators $\hat{\eta}^2$, $\hat{\eta}\cdot\hat{\boldsymbol{\zeta}}$, and $\hat{\boldsymbol{\zeta}}^2$ have the Hermitian conjugation property obtained from inverting (1.4),

$$\begin{aligned}(\hat{\mathbf{x}}^2\bar{f},\bar{g})_{nl} &= ([d^2\hat{\eta}^2 - 2b\hat{\eta}\cdot\hat{\boldsymbol{\zeta}} + b^2\hat{\boldsymbol{\zeta}}^2 + indb]\bar{f},\bar{g})_{nl}\\ &= (\bar{f},\hat{\mathbf{x}}^2\bar{g})_{nl} = (\bar{f},[d^2\hat{\eta}^2 - 2b\hat{\eta}\cdot\hat{\boldsymbol{\zeta}} + b^2\hat{\boldsymbol{\zeta}}^2 + indb]\bar{g})_{nl}\end{aligned} \tag{2.1}$$

and similar companion equations for $(\hat{\mathbf{x}}\cdot\hat{\mathbf{p}})^\dagger = \hat{\mathbf{p}}\cdot\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}^2$ in the Schrödinger representation (1.9). Equation (2.1) and its companions can be turned into differential equations on the weight function $\nu_{nl}(\rho,\rho^*)$ in (1.10) through integration by parts, using, for $\rho = |\rho|\exp(i\theta)$, $d\,\mathrm{Re}\rho\,d\,\mathrm{Im}\rho = |\rho|\,d|\rho|\,d\theta$, and $d/d\rho = \frac{1}{2}\exp(-i\theta)[\partial/\partial|\rho| + (i\rho)^{-1}\partial/\partial\theta]$ so that $\partial\rho^*/\partial\rho = 0$ and, for analytic functions $A(\rho)$, $B(\rho)$,

$$\int_0^\infty |\rho|\, d|\rho| \int_\alpha^\beta d\theta A(\rho)\, \frac{d}{d\rho} B(\rho) = -\int_0^\infty |\rho|\, d|\rho| \int_\alpha^\beta d\theta \left(\frac{d}{d\rho} A(\rho)\right) B(\rho) + \tfrac{1}{2}|\rho| \int_\alpha^\beta d\theta \exp(-i\theta) AB \Big|_{|\rho|\to\infty} - \tfrac{1}{2} i \exp(-i\theta) \int_0^\infty d|\rho| AB(|_{\theta=\beta} - |_{\theta=\alpha}). \quad (2.2)$$

By assuming the boundary integral terms vanish (the restrictions from this condition will be made explicit below), Eq. (2.1) yields the differential equation

$$b^{*2}\left[-\frac{\partial^2}{\partial\rho^{*2}} + \left(2i\frac{d^*}{b^*}\rho^* + \frac{n-1}{\rho^*}\right)\frac{\partial}{\partial\rho^*} + \frac{d^{*2}}{b^{*2}}\rho^{*2} + i\frac{d^*}{b^*}(2-n) - \frac{\lambda+n-1}{\rho^*}\right]\nu_{nl}(\rho,\rho^*) = b^2\left[-\frac{\partial^2}{\partial\rho^2} + \left(-2i\frac{d}{b}\rho + \frac{n-1}{\rho}\right)\frac{\partial}{\partial\rho} + \frac{d^2}{b^2}\rho^2 - i\frac{d}{b}(2-n) - \frac{\lambda+n-1}{\rho^2}\right]\nu_{nl}(\rho,\rho^*), \quad (2.3)$$

and similar ones (i.e., replacing $b\to a$, $d\to c$, etc) for the companions, with vanishing conditions for the boundary terms of $\rho\nu\bar{f}^*\bar{g}$, $\nu\bar{f}^*(\partial_\rho\bar{g})$, $(\partial_\rho\nu)\bar{f}^*\bar{g}$, and $\rho^{-1}\nu\bar{f}^*\bar{g}$ and similar ones replacing ρ and ρ^*. Notice that whereas in I we had two simultaneous first-order differential equations, here we have three second-order ones. Based on I, however, we can make the ansatz that

$$\nu_{nl}(\rho,\rho^*) = \exp\left(\frac{u}{2v}\rho^2\right)\exp\left(\frac{u^*}{2v}\rho^{*2}\right)\mu_{nl}(\rho\rho^*), \quad (2.4)$$

where, as in I, we define

$$u \equiv a^*d - b^*c, \quad (2.5a)$$

$$v \equiv 2\,\mathrm{Im}(b^*a). \quad (2.5b)$$

We obtain the result that the three equations (2.3) yield a single differential equation for μ_{nl} which shows that $\mu_{nl}(\rho\rho^*) = (\rho\rho^*)^{n/2}\beta_{n/2+l-1}(\rho\rho^*/v)$, where β is a solution of Bessel's modified equation: I or K functions. The boundary integral over the semicircle at infinity appearing in the integration by parts of (2.3) will vanish for functions of less or equal growth than $\exp(\frac{1}{2}\rho^2/v)$ if we choose the MacDonald (or modified Hankel) function K. We find, with a specific choice of normalization, justified in Sec. 4 that

$$\nu_{nl}(\rho,\rho^*) = (2/\pi v)\exp[(1/2v)(u\rho^2 + u^*\rho^{*2})] \times (\rho\rho^*)^{n/2} K_{n/2+l-1}(\rho\rho^*/v). \quad (2.6)$$

If we let $u = \omega\exp(i\varphi)$ be the polar representation of u, the behavior of (2.6) at the interval end points is

$$\nu_{nl}(\rho,\rho^*) \underset{|\rho|\to\infty}{\approx} (\tfrac{1}{2}\pi v)^{1/2}|\rho|^{n-1} \times \exp[-(1/v)|\rho|^2(1-\omega\cos\{\varphi+2\theta\})] \quad (2.7a)$$

and

$$\nu_{nl}(\rho,\rho^*) \underset{|\rho|\to 0}{\approx} 2(2v)^{n/2+l-1}\Gamma(\tfrac{1}{2}n+l-1)|\rho|^{2(1-l)}, \quad l > -\tfrac{1}{2}n+1, \quad (2.7b)$$

$$\nu_{nl}(\rho,\rho^*) \underset{|\rho|\to 0}{\approx} -2(\pi v)^{-1}|\rho|^{2(1-l)}\ln(|\rho|^2/v), \quad l = -\tfrac{1}{2}n+1. \quad (2.7c)$$

As λ in (1.8) is invariant under the replacement $l \leftrightarrow -l - n + 2$, only $l \geq -\frac{1}{2}n+1$ need be considered. Correspondingly, we have the property $K_\mu(z) = K_{-\mu}(z)$. The remaining boundary integrals over the imaginary axis will be made to vanish and the finiteness of $(\bar{f},\bar{g})_{nl}$ itself determined by restricting the space of functions. Consider

$$\bar{\phi}'_m(\rho) = c_m \exp[-(u/2v)\rho^2]\rho^m \quad (2.8a)$$

for $m \in \mathbb{R}$ and c_m a normalization constant. In performing the scalar product $(\bar{\phi}'_m, \bar{\phi}'_{m'})_{nl}$ we can separate the integration of $\rho \in \mathbb{C}^+$ into a radial and angular part, the latter being $\int d\theta \exp[i(m'-m)\theta]$ over $[-\frac{1}{2}\pi, \frac{1}{2}\pi)$. This is zero if $m - m'$ is an even nonzero integer, and π if $m = m'$. In the last case, the remaining integral can be evaluated[14] and $\bar{\phi}'_m(\rho)$ normalized through (2.8a) setting

$$c_m = \vartheta_m[\tfrac{1}{2}(2v)^{n/2+m}\Gamma(\tfrac{1}{2}(n+l+m))\Gamma(\tfrac{1}{2}(m-l+2))]^{-1/2}, \quad (2.8b)$$

where ϑ_m is an arbitrary phase and the arguments of the function reflect the fact that the integration is valid and the result finite for $m > l-2$ and $m > -n-l$. The latter is a consequence of the former for $l \geq -\frac{1}{2}n+1$. In checking the vanishing conditions for the boundary terms mentioned below Eq. (2.3), we come to the conclusion that these hold if $m - m'$ is an even integer. If we now write $m = l + \alpha + 2N$ with $N = 0, 1, 2, \cdots$ and $\alpha \in (-2, 0]$ we can see that asking $\bar{\phi}'_m(\rho)$ to be in the invariant common domain of the three operators (1.9) forces $\alpha = 0$. Hence an orthonormal basis for the space $\mathcal{J}^+_{nl}$ object of our construction is, with a specific choice of phase,

$$\bar{\phi}_N(\rho) = (-1)^N[\tfrac{1}{2}(2v)^{n/2}N!\,\Gamma(N+\tfrac{1}{2}n+l)]^{-1/2} \times \exp[-(u/2v)\rho^2][(2v)^{-1/2}\rho]^{2N+l}, \quad N = 0, 1, 2, \cdots. \quad (2.9)$$

Now, the basis (2.9) is *complete* in the Hilbert space $\mathcal{J}^+_{nl}$ of functions $\bar{f}$ of the type $\bar{f}(\rho) = \exp[-(u/2v)\rho^2]\rho^l$ times an entire function in $\rho^2/2v$ of growth (1.1) [or of growth $(2, 1/2v)$ in ρ] completed with respect to the norm induced by (1.10) with the weight function (2.6). The proof is the standard one[15] which proves that convergence in the norm implies pointwise convergence for these functions. Indeed, for

$$\bar{f}(\rho) = \exp[-(u/2v)\rho^2]\rho^l \sum_{N=0}^\infty f_N\rho^{2N} = \sum_{N=0}^\infty \alpha_N f_N \bar{\phi}_N(\rho), \quad (2.10a)$$

$$\alpha_N = (-1)^N(2v)^{N+(n/2+l)/2}[\tfrac{1}{2}N!\,\Gamma(N+\tfrac{1}{2}n+l)]^{1/2}, \quad (2.10b)$$

we have

$$\|\bar{f}\|^2_{nl} \equiv (\bar{f},\bar{f})_{nl} = \sum_{N=0}^\infty \alpha_N^2|f_N|^2. \quad (2.10c)$$

Using the Schwartz inequality, we obtain

$$|\bar{f}(\rho)|^2 = |\exp[-(u/2v)\rho^2]\rho^l|^2 \left|\sum_{N=0}^\infty f_n\rho^{2N}\right|^2 \leq \left|\sum f_N\alpha_N\right|^2 |\exp[-(u/2v)\rho^2]\rho^l|^2 \times \left|\sum \alpha_{N'}^{-1}\rho^{2N'}\right|^2 \leq \sum |f_N|^2\alpha_N^2 |\exp[-(u/2v)\rho^2]\rho^l|^2 \sum \alpha_{N'}^{-2}|\rho|^{2N'}$$

$$= \|\bar{f}\|^2_{nl} |\exp[-(u/2v)\rho^2]|^2 |\rho|^{2-n} \times v^{-1} I_{n/2+l-1}(|\rho|^2/v) = \|\bar{f}\|^2_{nl} K_{nl}(\rho,\rho) \qquad (2.11)$$

[where the function $K_{nl}(\rho,\rho')$ will be defined below], and hence any Cauchy sequence of functions coverging in the norm to a function in $\mathcal{J}^+_{nl}$ implies the uniform convergence of the functions themselves on any compact set in $\mathbb{C}^+$. The reproducing kernel in the integral (1.10)—(2.6) is thus

$$K_{nl}(\rho,\rho') \equiv \sum_{N=0}^{\infty} \bar{\phi}_N(\rho)\bar{\phi}_N(\rho')^* = v^{-1}(\rho\rho'^*)^{1-n/2} \exp[-(1/2v)(u\rho^2 + u^*\rho'^{*2})] \times I_{n/2+l-1}(\rho\rho'^*/v), \qquad (2.12)$$

and appears in the last number of (2.11).

Before closing this section, we will find an algebra of raising, lowering, and weight operators for the basis functions (2.9). Easiest to build, the raising operator is

$$\mathcal{R}\bar{\phi}_N(\rho) \equiv [-(1/2v)\rho^2]\bar{\phi}_N(\rho) = [(N+1)(N+\tfrac{1}{2}n+l)]^{1/2}\bar{\phi}_{N+1}(\rho). \qquad (2.13a)$$

Its Hermitian conjugate under the scalar product (1.10) is the lowering operator

$$\mathcal{L}\bar{\phi}_N(\rho) \equiv -\left[\tfrac{1}{2}v\frac{d^2}{d\rho^2} + \left(u\rho + \tfrac{1}{2}v\frac{n-1}{\rho}\right)\frac{d}{d\rho} + \left(\frac{u^2}{2v}\rho^2 + \tfrac{1}{2}nu + \tfrac{1}{2}v\frac{\lambda}{\rho^2}\right)\right]\bar{\phi}_N(\rho) = [N(N+\tfrac{1}{2}n+l-1)]^{1/2}\bar{\phi}_{N-1}(\rho). \qquad (2.13b)$$

The weight operator

$$\mathcal{N}\bar{\phi}_N(\rho) \equiv \left(\rho\frac{d}{d\rho} + \frac{u}{v}\rho^2 + \tfrac{1}{2}n\right)\bar{\phi}_N(\rho) = (2N+\tfrac{1}{2}n+l)\bar{\phi}_N(\rho) \qquad (2.13c)$$

completes the set of generators of an $so(2,1)$ algebra with commutation relations

$$[\mathcal{N},\mathcal{R}] = 2\mathcal{R}, \quad [\mathcal{N},\mathcal{L}] = -2\mathcal{L}, \quad [\mathcal{R},\mathcal{L}] = -\mathcal{N}. \qquad (2.14)$$

3. THE TRANSFORM BETWEEN $\mathcal{H}^+_n$ AND $\mathcal{J}^+_{nl}$

The transform kernel $A_{nl}(\rho,r)$ in (1.11) can be calculated if we ask for the conditions (1,4), (1.6), and (1.9) to hold, namely, that if $\bar{f}(\rho)$ is the transform of $f(r)$, then $\rho^2\bar{f}(\rho)$ be the transform of

$$\left[a^2r^2 + 2iabr\partial_r - b^2\left(\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{\lambda}{r^2}\right) + niab\right]f(r).$$

Similar conditions stem from $-i\rho\partial_\rho$ and $-\{\partial^2_\rho + [(n-1)/\rho]\partial_\rho + \lambda/\rho^2\}$. By partial integration in (1.11) these can be turned into three second-order differential equations for $A_{nl}(\rho,r)$. From I we make the ansatz that $A_{nl}(\rho,r)$ have the form

$$\exp[(i/2b)(ar^2 + d\rho^2)]B_{nl}(\rho r)$$

whereupon the three differential equations for $A_{nl}(\rho,r)$ yield a single one for $B_{nl}(\rho r)$ as $(\rho r)^{1-n/2}$ times a solution of Bessel's equation. If $f(r)$ belongs to the space $\mathcal{H}^+_n$ with scalar product (1.7), for (1.11a) to be integrable we must require $\operatorname{Im}(a/b) \geq 0$ (i.e., $v \geq 0$) for the exponent and the Bessel function as solution for $B_{nl}(\rho r)$. With a specific choice for phase and normalization to be justified below and in Sec. 4, we write

$$A_{nl}(\rho,r) = b^{-1}\vartheta_{n,l}\exp[(i/2b)(ar^2 + d\rho^2)](\rho r)^{1-n/2} \times J_{n/2+l-1}(\rho r/b) \qquad (3.1a)$$

with

$$\vartheta_{n,l} = \exp[-i\tfrac{1}{2}\pi(\tfrac{1}{2}n+l)]. \qquad (3.1b)$$

The calculation of the explicit form of the orthonormal basis transform to (2.9) can be simplified if we look for the eigenfunctions of the weight operator (2.13c) which through (1.4) becomes

$$\mathcal{N}\phi_N(r) = v^{-1}\left[|a|^2r^2 - \tfrac{1}{2}i\operatorname{Re}(ab^*)r\frac{d}{dr} - |b|^2\left(\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} + \frac{\lambda}{r^2}\right) - \tfrac{1}{4}in\operatorname{Re}(ab^*)\right]\phi_N(r) = (2N+\tfrac{1}{2}n+l)\phi_N(r), \qquad (3.2)$$

plus normalization under (1.7) and a phase to satisfy Eq. (3.4) below. The result is, if we denote the phase of b by $\exp[i\arg b]$ with $\arg b \in [-\pi,\pi)$,

$$\phi_N(r) = \vartheta_N\{2N![\operatorname{Im}(a/b)]^{n/2+l}/\Gamma(N+\tfrac{1}{2}n+l)\}^{1/2} \times \exp[-\tfrac{1}{2}i(a^*/b^*)r^2]r^l L_N^{(n/2+l-1)}[r^2\operatorname{Im}(a/b)], \qquad (3.3a)$$

with

$$\vartheta_N = \exp[i(2N+\tfrac{1}{2}n+l)(\arg b+\tfrac{1}{2}\pi)]. \qquad (3.3b)$$

We can now verify that[16]

$$A_{nl}(\rho,r) = \sum_{N=0}^{\infty}\bar{\phi}_N(\rho)\phi_N(r)^*. \qquad (3.4)$$

At this point it is apparent that a second pair of transform orthonormal bases for $\mathcal{H}^+_n$ and $\mathcal{J}^+_{nl}$ is useful, since the limit $v \to 0$ of real transformations of (2.9)—(3.3) is not manifest. As in I, we choose the basis functions $\psi_{Nl}(r)$ for $\mathcal{H}^+_n$ to be the radial part of the solutions of a harmonic oscillator with centrifugal force Hamiltonian in n dimensions given by

$$2I_3\psi_{Nl}(r)Y^M_L(\omega) \equiv \tfrac{1}{2}[\hat{\mathbf{p}} + g\hat{\mathbf{x}}^{-2} + \hat{\mathbf{x}}^2]\psi_{Nl}(r)Y^M_L(\omega) = \frac{1}{2}\left[-\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r^2} + \frac{g+L(L+n-2)}{r^2} + r^2\right] \times \psi_{Nl}(r)Y^M_L(\omega) = [2N+\tfrac{1}{2}n+l]\psi_{Nl}(r)Y^M_L(\omega), \qquad (3.5a)$$

where $Y^M_L(\omega)$ is the n-dimensional normalized spherical harmonic, the collective label M standing for the transformation properties under $SO(n-1) \supset \cdots \supset SO(2)$, while the $SO(n)$ label L enters into the differential operator and relates to l through

$$l(l+n-2) \equiv -\lambda \equiv g + L(L+n-2), \qquad (3.5b)$$

giving two values of l for each g and L, in general. The solution of the radial equation is

$$\psi_{Nl}(r) = [2N!/\Gamma(N+\tfrac{1}{2}n+l)]^{1/2}\exp(-r^2/2)r^l L_N^{(n/2+l-1)}(r^2), \qquad (3.6)$$

whose corresponding raising and lowering operators can be obtained from (3.5a) and

$$I_1 \equiv \tfrac{1}{4}[\hat{\mathbf{p}}^2 + g\hat{\mathbf{x}}^{-2} - \hat{\mathbf{x}}^2], \tag{3.7a}$$

$$I_2 \equiv \tfrac{1}{4}[\hat{\mathbf{x}}\cdot\hat{\mathbf{p}} + \hat{\mathbf{p}}\cdot\hat{\mathbf{x}}], \tag{3.7b}$$

which can be verified to close into an $so(2,1)$ algebra. The transform basis functions can be calculated directly using the transform (1.7a), (3.1), (3.6), yielding[17]

$$\overline{\psi}_{Nl}(\rho) = [2N!/\Gamma(N+\tfrac{1}{2}n+l)]^{1/2}(a+ib)^{-n/2-l}[(a-ib)/(a+ib)]^N \times \exp\left(-\frac{1}{2}\,\frac{d-ic}{a+ib}\,\rho^2\right)\rho^l L_N^{(n/2+l-1)}(\rho^2/[a^2+b^2]). \tag{3.8}$$

In particular, notice that when we have the Bargmann case (1.2), (2.5) gives $u=0$, $v=1$, only the leading term of the Laguerre function remains, and both bases coincide as (3.8) becomes proportional to ρ^{2N+l} and equal to $\overline{\phi}_N(\rho)$. This determined our choice of phase for the latter.

The unitarity of the transform pair (1.7) with the kernel (3.1) between $\mathcal{H}_n^+$ and $\mathcal{J}_{nl}^+$ can be established following the same steps as in Bargmann's original work.[18] That it transforms the orthonormal basis $\{\phi_N(r)\}$ to the orthonormal basis $\{\overline{\phi}_N(\rho)\}$ shows that the mapping is isometric. The completeness of the basis $\{\overline{\phi}_N(\rho)\}$ in $\mathcal{J}_{nl}^+$ was found in (2.11)–(2.12) and, moreover, we can perform directly[19]

$$\int_0^\infty r^{n-1}\,dr\,A_{nl}(\rho,r)A_{nl}(\rho',r)^* = K_{nl}(\rho,\rho'), \tag{3.9}$$

when (1.7a) can be performed, i.e., when the kernel (3.1) is bounded, namely for $\mathrm{Im}(a/b)\geq 0$ ($v\geq 0$) or, when $a=0$, b should be real. As $(\overline{f},\overline{g})_{nl} = (f,g)_0$ for any f,g in $\mathcal{H}_n^+$, the mapping is unitary and the existence conditions are identical with those found in I for the linear complex transforms.

4. LIMITS AND PARTICULAR CASES

Real transformations: We want to show that, as in I, when the transformation parameters a,b,c,d in (1.1) become real, the space $\mathcal{J}_{nl}^+$ with a scalar product (1.10) over $\mathbb{C}^+$ collapses to $\mathcal{H}_n^+$ with a scalar product (1.7) over $\mathbb{R}^+$. The said limit involves first determining the behavior of the weight function in (2.6) as, in (2.5), $v\to 0$ and, since $|u|^2+vw=1$ with $w\equiv 2\,\mathrm{Im}\,c^*d$, for $u=\omega\exp(i\varphi)$, $\omega\to 1$. Recalling that[20] $K_\mu(z)\sim[\pi/2z]^{1/2}e^{-z}$ as $|z|\to\infty$, $\omega=|1-vw|^{1/2}\sim 1-\tfrac{1}{2}vw$, $\mathrm{l.i.m.}\,\epsilon^{-1/2}\times\exp[-z^2/\epsilon]=\pi^{1/2}\delta(z)$ for real positive $\epsilon\to 0$ and the fact that $\nu_{nl}(\rho,\rho^*)$ is under the integral $\int_{\mathbb{C}^+} d\,\mathrm{Re}\rho\, d\,\mathrm{Im}\rho = \int_0^\infty |\rho|\,d|\rho|\int_{-\pi/2}^{\pi/2}d\theta$,

$$\begin{aligned}\underset{v\to 0}{\mathrm{l.i.m.}}\,\nu_{nl}(\rho,\rho^*) &= \underset{v\to 0}{\mathrm{l.i.m.}}\,[2/\pi v]^{1/2}\exp[-(|\rho|^2/v)(1-\cos\{\varphi+2\theta\}) \\ &\quad -\tfrac{1}{2}w|\rho|^2\cos\{\varphi+2\theta\}] \\ &= |\rho|^{n-1}\delta(|\rho|\sin(\tfrac{1}{2}\varphi+\theta))\exp[-\tfrac{1}{2}w|\rho|^2\cos(\varphi+2\theta)] \\ &= |\rho|^{n-1}[\delta(\tfrac{1}{2}\varphi+\theta)+\delta(\tfrac{1}{2}\varphi+\theta-\pi)]\exp(-\tfrac{1}{2}w|\rho|^2).\end{aligned} \tag{4.1}$$

Now, as $\theta\in[-\tfrac{1}{2}\pi,\tfrac{1}{2}\pi)$, only the first δ contributes to pick out the value $\theta=\tfrac{1}{2}\varphi$ in the integral, so that for $r'=|\rho|$,

$$\lim_{v\to 0}\int_{\mathbb{C}^+} d\mu_{nl}(\rho)\overline{f}(\rho)^*\overline{g}(\rho) = \int_{\mathbb{R}^+e^{-i\varphi/2}} r'^{n-1}\,dr'\exp(-wr'^2/2)\times\overline{f}(r')^*\overline{g}(r'), \tag{4.2}$$

and the normalization coefficient for ν_{nl} is thus seen to be the appropriate one and the parameter l has disappeared from the right-hand side of (4.2). Since ur'^2 is real, from the discussion below Eq. (2.9) we can see that the functions $\overline{f},\overline{g}$ must be of growth $(2,1/2v-\omega/2v)\sim(2,\tfrac{1}{4}w)$ in r'. In the limit when the transformation parameters become real, $w\to 0$ and $\varphi\to 0$, the integral in the right-hand side of (4.2) is over $\mathbb{R}^+$ and $\mathcal{J}_{nl}^+$ has become identical with $\mathcal{H}_n^+$. The transform kernel $A_{nl}(\rho,r)$ in (3.1) is uneventful in this limit and now becomes a transform in $\mathcal{H}_n^+$ which coincides with the unitary representations of $SO(2,1)$ in "radial" space.[21]

Transformations where $b\to 0$ can be obtained out of the development above since $b\to 0$ implies $v\to 0$, plus the analysis of the behavior of $A_{nl}(\rho,r)$ in (3.1). It can be shown[22] with due care to the phases involved for $r\geq 0$, $\arg r'\in[-\tfrac{1}{2}\pi,\tfrac{1}{2}\pi)$,

$$\underset{b\to 0}{\mathrm{l.i.m.}}\,A_{nl}(r',r) = r'^{1-n}a^{n/2-1}\delta(r-a^{-1}r')\exp[ic/2a)r'^2]. \tag{4.3}$$

Since $\arg a = -\tfrac{1}{2}\arg u = -\tfrac{1}{2}\varphi = \theta = \arg r'$, (4.2) acts under the line integral over $\mathbb{R}^+\exp(-i\varphi/2)$ with the appropriate phase relation between r and r'. The case $a=1$, $c=iq$, q real >0 was used in Ref. 7 to reproduce the matrix elements of a Gaussian potential. The identity transformation is now obtained by simply setting $a=1$, $c=0$ in (4.2), and $A_{nl}(r',r)$ is seen to become the reproducing kernel under the scalar product (1.7). It is thus seen that our choice of the phase factor (3.1b) is appropriate.

The Hankel transform is obtained when, as for the ordinary Fourier transform in (1.1), we set $a=0=d$, $b=1=-c$. The transform kernel becomes[23]

$$A_{nl}^H(r',r) = \vartheta_{nl}(r'r)^{1-n/2}J_{n/2+l-1}(r'r). \tag{4.4}$$

The Barut-Girardello transform[11] was introduced in developing the formalism for coherent states associated with noncompact groups, these being eigenstates of the lowering operator of an $so(2,1)$ algebra in the ("discrete") $D^\pm(\Phi)$ representations ($\Phi=-\tfrac{1}{2},-1,-\tfrac{3}{2},\cdots$). It can be obtained as a particular case of complex radial transforms for the values (1.2) of the parameters. The scalar product in the $\mathcal{J}_{nl}^+$ space has the weight function

$$\nu_{nl}^B(\rho,\rho^*) = 2\pi^{-1}|\rho|^n K_{n/2+l-1}(|\rho|^2). \tag{4.5a}$$

Similarly, the transform kernel becomes

$$A_{nl}^B(\rho,r) = 2^{1/2}(\rho r)^{1-n/2}\exp[-\tfrac{1}{2}(r^2+\rho^2)]J_{n/2+l-1}(2^{1/2}\rho r), \tag{4.5b}$$

and the orthonormal basis

$$\overline{\phi}_N^B(\rho) = (-1)^N[2^{n/2-1}N!\,\Gamma(N+\tfrac{1}{2}n+l)]^{-1/2}(2^{-1/2}\rho)^{2N+l}, \tag{4.5c}$$

with the reproducing kernel

$$K_{nl}^B(\rho,\rho') = (\rho\rho'^*)^{1-n/2}I_{n/2+l-1}(\rho\rho'^*). \tag{4.5d}$$

When $l=0$, this agrees with the scalar product in the Barut–Girardello[24] space $z\equiv\tfrac{1}{2}\rho^2\in\mathbb{C}$ for $D^+(\Phi)$ when the latter is multiplied by a factor of $2^{n/2-2}\Gamma(\tfrac{1}{2}n)$ and we set $\Phi=-\tfrac{1}{4}n$. The results of Ref. 7 are regained when we multiply our weight function by a factor $2^{n/2-2}$ and set $|q|=\tfrac{1}{2}n-1$, integer.[25] It should be noticed that the basis functions (3.6) are bases for an $so(2,1)$ representation given by the eigenvalue of $I^2 = I_3^2 - I_1^2 - I_2^2$ obtained from (3.5)–(3.7) to be $Q \equiv \frac{1}{4}[(\frac{1}{2}n + l - 1)^2 - 1] = \Phi(\Phi + 1)$ i.e., labelled by $\Phi = -\frac{1}{2} \pm \frac{1}{2}(\frac{1}{2}n + l - 1)$. Multivalued "discrete series" representations of the $SO(2,1)$ group are important as can be seen from the fact that for the ordinary one-dimensional harmonic oscillator ($n=1$, $\lambda = 0$) we have the $\Phi = -\frac{1}{4}$ and $-\frac{3}{4}$ representations of $SO(2,1)$.[26]

The one-dimensional "radial" spaces are the cases when $n=1$. As no angular momentum operators exist, in (1.8), $0 = \lambda = -l(l-1)$. There are two solutions for this: $l=0$ and $l=1$, i.e., $\frac{1}{2}n + l - 1 = \mp\frac{1}{2}$, and correspondingly two spaces, $\mathcal{F}_{10}^+$ and $\mathcal{F}_{11}^+$ are transforms of $\mathcal{H}_1^+$. The weight function in both spaces is, recalling $K_{\pm 1/2}(z) = [\pi/2z]^{1/2} e^{-z}$,

$$\nu_1(\rho, \rho^*) = 2(2\pi v)^{-1/2} \exp[(1/2v)(u\rho^2 - 2\rho\rho^* + u^*\rho^{*2})] \equiv \nu^1(\rho, \rho^*), \tag{4.6}$$

which is formally identical to the weight for the complex *linear* transform spaces in I. It has to be recalled, however, that, there,[27] the scalar product involves integration over all of $\mathbb{C}$. We shall explain this below. The two transform kernels are, using the particular expressions for $J_{\pm 1/2}$,

$$A_{10}(\rho, r) = \exp(-i\pi/4)(2/\pi b)^{1/2} \times \exp[(i/2b)(ar^2 + d\rho^2)] \cos(\rho r/b), \tag{4.7a}$$

$$A_{11}(\rho, r) = -i \exp(-i\pi/4)(2/\pi b)^{1/2} \times \exp[(i/2b)(ar^2 + d\rho^2)] \sin(\rho r/b). \tag{4.7b}$$

Hence in $\mathcal{F}_{10}^+$, the transform functions have the property $\bar{f}_0(\rho) = \bar{f}_0(-\rho)$ under inversion of the space, while in $\mathcal{F}_{11}^+$ $\bar{f}_0(\rho) = -\bar{f}_0(-\rho)$, as can be seen from the bases (2.9). Now if for a given function $f(r)$ on $r \in \mathbb{R}^+$ we extend the domain to the whole of $\mathbb{R}$ and write $f(r) = f_+(r) + f_-(r)$, $f_\pm(r) = \frac{1}{2}[f(r) \pm f(-r)]$, expanding f into its odd and even components and further demand that a transform $\bar{f}(\rho)$ have the same parity under inversion of the argument as the original function [this corresponds to having L^2 with the same eigenvalue λ in both spaces, the transformation properties under $O(n)$ now collapsing to C_2], we can write $\bar{f}_0$ as the transform of f_+ and $\bar{f}_1$ as that of f_-. Suppressing arguments,

$$\bar{f} \equiv \bar{f}_0 + \bar{f}_1 = \int_{\mathbb{R}^+} dr\, A_{10} f_+ + \int_{\mathbb{R}^+} dr\, A_{11} f_- = \int_{\mathbb{R}^+} dr\, A^1 f \tag{4.8}$$

with

$$A^1(\rho, r) \equiv \tfrac{1}{2}(A_{10} + A_{11})(\rho, r) = (2\pi b)^{-1/2} \exp(-i\pi/4) \exp[(i/2b)(ar^2 - 2r\rho + d\rho^2)], \tag{4.9}$$

regaining the complex linear transform in I between $\mathcal{H} \equiv \mathcal{L}^2(-\infty, \infty)$ and $\mathcal{F}$ with the scalar product

$$(\bar{f}, \bar{g})^1 = 2(\bar{f}_0, \bar{g}_0)_{10} + 2(\bar{f}_1, \bar{g}_1)_{11} = \int_{\mathbb{C}} d\,\mathrm{Re}\rho\, d\,\mathrm{Im}\rho\, \nu^1(\rho, \rho^*) \bar{f}(\rho)^* \bar{g}(\rho). \tag{4.10}$$

For the values (1.2) of the parameters, this is the Bargmann transform.[2]

Another, quite different, way of obtaining back the complex linear transforms is to follow the procedure of Barut and Girardello[11] of considering functions of $z = \epsilon^{-1/2}$ with $z = \frac{1}{2}\rho^2$ and letting $n \to \infty$ such that ϵn remain a finite number. This effects the contraction of the representations of the $so(2,1)$ algebra in (2.14) in the orthonormal basis (2.13) to that of the Heisenberg algebra. The limiting procedure is a delicate one, and we shall not pursue this point further.

5. COHERENT STATES FOR THE RADIAL HARMONIC OSCILLATOR WITH A CENTRIFUGAL FORCE

The Bargmann transform has proven to be the natural tool for the construction of coherent states for the harmonic oscillator since they map the eigenstates $\psi_N(x)$ of the one-dimensional system on functions of the complex variable $z \in \mathbb{C}$, $\bar{\psi}_N(z) = [(2\pi)^{1/2} N!]^{-1/2} z^N$ (using the normalization of I). The coherent states, defined[28] as $|z) = \sum |N\rangle \bar{\psi}_N(z)$ are eigenstates of the lowering operator $\hat{z} = 2^{-1/2}(\hat{x} + i\hat{p})$ with eigenvalue z. They resolve the identity as $\mathbb{1} = \int |z)\, d\mu^1(z) (z|$ [using the measure $d\mu^1(z)$ of I] and are overcomplete[29] as $(z|z') = K^1(z, z')$, the reproducing kernel in the scalar product with measure $d\mu^1(z)$.

A similar construction for the radial functions of an n-dimensional harmonic oscillator with centrifugal force can now be made. The angular part of the wavefunctions continues to be the n-dimensional spherical harmonic in the $n-1$ angles of real or complex space as in (3.5a) (see Appendix B). We shall now examine the proper quantum-mechanical solutions of the radial part of the operator (3.5a). These are (3.6) plus the conditions that I_3 be self-adjoint between them, which means that the constant terms in the partial integrations be zero (which imposes conditions on the behavior of the functions at $r=0$) and that ψ_{Nl}, $r^{-1}\psi_{Nl}$, and $(d/dr)\psi_{Nl}$ be square-integrable.[30] From (3.5b) we see that for each n, L, and g, the two solutions

$$l_\pm = (1 - \tfrac{1}{2}n) \pm [(1 - \tfrac{1}{2}n)^2 + L(L + n - 2) + g]^{1/2} \tag{5.1}$$

are real for centrifugal forces which include attractive ones but which are not more attractive than those allowed by the zero of the discriminant for the lowest angular momentum $L = 0$ namely

$$g \geq -(1 - \tfrac{1}{2}n)^2. \tag{5.2}$$

Given this condition is fulfilled, square-integrability of ψ_{Nl} under the scalar product in $\mathcal{H}_n^+$ (since it is assured that the behavior at infinity is adequate), places restrictions on the behavior at the origin: $l > -\frac{1}{2}n$. The same conditions on $r^{-1}\psi_{Nl}$ and $(d/dr)\psi_{Nl}$ narrows the choice to $l > 1 - \frac{1}{2}n$. Hence, only l_+ of the two choices in (5.1) is possible for general g and n satisfying (5.2). Only in the case when the latter two conditions are absent (i.e., $g=0$, $n=1$, $L \equiv 0$, and $l_- = 0$), do we need the two solutions of (5.1). This is convenient since for all cases, except the one-dimensional oscillator with no centrifugal force, the space $\mathcal{F}_{nl_+}^+$ contains all the states of the system for a given angular momentum[26] L. Henceforth denote $l_+ \equiv l(L, n, g)$. Recalling (4.4) define now the kets

$$|\rho)_{nL} \equiv \sum_{N=0}^{\infty} |N\rangle_{nl}\, \bar{\phi}_N^B(\rho)$$

$$=2^{-(n/2+l-1)/2}\rho^l \sum_{N=0}^{\infty} |N\rangle_{nl}[N!\Gamma(N+\tfrac{1}{2}n+l)]^{-1/2}(-1)^N\rho^{2N}, \quad \rho\in\mathbb{C}^+, \tag{5.3}$$

where $|N\rangle_{nl}$ stands for the state (3.6). The ket (5.3) can be considered as a *coherent* state for the system since it is an eigenket of the lowering operator defined, parallel to (2.14), with (3.7) as

$$L\equiv I_1-iI_2=-\tfrac{1}{2}[2^{-1/2}(\hat{\mathbf{x}}+i\hat{\mathbf{p}})]^2+\tfrac{1}{4}g r^{-2} \tag{5.4a}$$

with eigenvalue $-\frac{1}{2}\rho^2$, as the bracketing suggests for $g\to 0$. This can be proven immediately using the $so(2,1)$ raising and lowering operator matrix elements (2.13):

$$L|\rho)_{nL}=\sum_{N=0}^{\infty}[N(N+\tfrac{1}{2}n+l-1)]^{1/2}|N-1\rangle_{nl}\overline{\phi}_N^B$$
$$=\sum_{N'=0}^{\infty}|N'\rangle_{nl}[(N'+1)(N'+\tfrac{1}{2}n+l)]^{1/2}\overline{\phi}^B_{N'+1}$$
$$=\sum_{N'=0}^{\infty}|N'\rangle_{nl}\mathcal{R}\,\overline{\phi}^B_{N'}=-\tfrac{1}{2}\rho^2|\rho)_{nL}. \tag{5.5}$$

The usual coherent-state properties follow,[28] as ${}_{nL}(\rho|\rho')_{nL}=K^B_{nl}(\rho,\rho')$ and $\int_{\mathbb{C}^+}|\rho)_{nL}\,d\mu^B(\rho)_{nL}(\rho|=\mathbb{1}$. It would seem desirable to change the labels $z=\frac{1}{2}\rho^2\in\mathbb{C}$ so as to coincide with the treatment in Ref. 11 with $l=0$ and $n=-4\Phi$. There is the problem, however, that for $l\neq 2\times$ integer, an $f(z)=(\rho|f)$ would not be an entire function of z, but one with a branch cut from 0 to ∞. A completeness statement[29] on the coherent states (5.3) is also wanting. Since a connection exists between the radial differential equations of the harmonic oscillator and Coulomb systems,[13,31] one expects that similar coherent states can be defined for the latter. This will be taken up elsewhere.

6. COMPOSITION OF TRANSFORMS AND REPRESENTATIONS OF $HSL(2,\mathbb{C})$

Two related topics which are virtually identical with their counterparts for complex linear transforms will now be presented in the briefest manner. The first one pertains the possibility of composition of transforms, seen as *active* transformations $\mathcal{A}_1:\mathcal{H}^+=\mathcal{F}_1^+$ and $\mathcal{A}_2:\mathcal{H}^+=\mathcal{F}_2^+$ into one transform $\mathcal{F}_2^+=\mathcal{A}_2\mathcal{A}_1^{-1}\mathcal{F}_1^+\equiv\mathcal{A}_{21}\mathcal{F}_1^+$ between $\mathcal{F}_1^+$ and $\mathcal{F}_2^+$ with the same n,l but differing in the parameters a,b,c,d, as

$$\bar{f}^{(2)}(\rho)=\int_{\mathbb{C}^+}d\mu_1(\rho')A_{(2,1)}(\rho,\rho')\bar{f}^{(1)}(\rho'), \tag{6.1a}$$
$$\bar{f}^{(1)}(\rho')=\int_{\mathbb{C}^+}d\mu_2(\rho)A_{(2,1)}(\rho,\rho')^*\bar{f}^{(2)}(\rho), \tag{6.1b}$$

where $d\mu_1(\rho')$ and $d\mu_2(\rho)$ are the corresponding measures and the transform kernel is

$$A_{(2,1)}(\rho,\rho')=\int_{\mathbb{R}^+}r^{n-1}\,dr\,A_{(2)}(\rho,r)A_{(1)}(\rho',r)^*$$
$$=\Phi(b_2,-b_1^*;b)^n\vartheta_{nl}b^{-1}\exp[(i/2b)(a\rho'^{*2}+d\rho^2)]$$
$$\times J_{n/2+l-1}(\rho\rho'^*/b)$$
$$=A_{(1,2)}(\rho',\rho)^* \tag{6.2a}$$

where

$$M\equiv\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}^{*-1} \tag{6.2b}$$

and

$$\Phi(b',b'';b)=\exp\{-\tfrac{1}{2}i[\arg b'+\arg b''-\arg b-\arg(b'b''/b)]\}$$
$$=\pm 1, \tag{6.2c}$$

when the conditions for existence of $\mathcal{A}_1$ and $\mathcal{A}_2$ are fulfilled [i.e., $\operatorname{Im}(a_1/b_1)\geq 0$ and $\operatorname{Im}(a_2/b_2)\geq 0$, etc].

The second point is that the composition of transforms can also be seen as that of *passive* transformations of the space $\mathcal{H}_n^+$ onto itself through a set of operators (6.2) and such that with each matrix M as defined in (6.2b) we associate a "D function"

$$D^{(0)}_{rr'}(M)=A_{(2,1)}(r,r') \tag{6.3}$$

which satisfies[32]

$$\int_{\mathbb{R}^+}r'^{n-1}\,dr'\,D^{(0);n,l}_{rr'}(M_1)\,D^{(0);n,l}_{r'r''}(M_2)$$
$$=\Phi(b_1,b_2;b_{12})^nD^{(0);n,l}_{rr''}(M_1M_2). \tag{6.4}$$

We have thus a ray representation of that subset of $M\in SL(2,\mathbb{C})$ for which integration is possible. The conditions for the kernels to be bounded (or Hilbert—Schmidt) were examined in I. This forms a subsemigroup of $SL(2,\mathbb{C})$ called $HSL(2,\mathbb{C})$ in Ref. 7 and (6.3) is a representation of $HSL(2,\mathbb{C})$ labeled by the indices n,l. A continuum of such representations can be built as

$$D^{(k);n,l}_{\rho,\rho'}(M)\equiv D^{(0);n,l}_{\rho,\rho'}(M_kMM_k^{*-1})=D^{(k);n,l}_{\rho',\rho}(M^{*-1})^*, \tag{6.5}$$

for $M_k\in HSL(2,\mathbb{C})$, with a composition law which replaces the integration over $\mathbb{R}^+$ with $\int_{\mathbb{C}^+}d\mu_k(\rho)$. From (6.5) we see that for $M\in SL(2,\mathbb{R})\subset HSL(2,\mathbb{C})$, the representation is unitary.

7. CANONICAL TRANSFORMATIONS IN QUANTUM MECHANICS, EXTENDED

In the way of conclusion, the results of I and this paper seem to indicate that the definition of a canonical transformation in quantum mechanics as that which preserves the Heisenberg algebra[9] in (1.1c) can be extended. Equation (1.1c) is the quantum analog of the classical concept of a canonical transformation to that which preserves the Poisson bracket between canonically conjugate variables. The validity of (1.1c) is thus restricted to those transformations where the new operators $\hat{\eta}$ and $\hat{\xi}$ exist and have the same domain and spectrum as the original, usual $\hat{x}$ and $\hat{p}$. Classical mechanics can work with the radial coordinate r and it conjugate momentum p_r and establish that (1.5) is a proper canonical transformation and, being a local theory, avoid specifying what happens at $r=0$. The translation of (1.5) to quantum mechanics appears difficult, as operators "$\hat{\rho}$" and "$\hat{p}_\rho$" are not of the usual kind as they have no self-adjoint extension.[33]

The picture we seem to be arriving at overcomes this limitation on two accounts: First, we make use of operators which *are* properly defined [as the $so(2,1)$ generators (3.5a)—(3.7) or their linear combinations $\hat{\mathbf{x}}^2$, $\frac{1}{2}(\hat{\mathbf{x}}\cdot\hat{\mathbf{p}}+\hat{\mathbf{p}}\cdot\hat{\mathbf{x}})$ and $\hat{\mathbf{p}}^2$ with the extra centrifugal force term added to the angular momentum one] and say that the transformation

$$\begin{pmatrix} I_1' \\ I_2' \\ I_3' \end{pmatrix} = \begin{pmatrix} \frac{1}{2}[a^2-b^2-c^2+d^2] & -ab+cd & \frac{1}{2}[-a^2-b^2+c^2+d^2] \\ -ac+bd & -ac-bd & ac+bd \\ \frac{1}{2}[-a^2+b^2-c^2+d^2] & ab+cd & \frac{1}{2}[a^2+b^2+c^2+d^2] \end{pmatrix} \times \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{7.1}$$

obtained from (1.1a), (1.3) and (3.5)–(3.7) is canonical in this extended context since, as can be verified

$$[I_j, I_k] = i\epsilon_l I_l \Leftrightarrow [I_j', I_k'] = i\epsilon_l I_l' \tag{7.2}$$

with (j,k,l) cyclic permutations of $(1,2,3)$ and $\epsilon_1 = \epsilon_2 = -\epsilon_3 = 1$. The $so(2,1)$ algebra is thus conserved and we can turn the procedure of finding the weight function ν_{nl} and transform kernel A_{nl} to stem from (6.1) and the hermiticity conditions on the $\{I_j'\}$ implied by the $\{I_j\}$ being self-adjoint. Although a Heisenberg algebra is undefined here, ρ itself retains the meaning of an underlying space variable. The classical limit of (6.1) is (1.5).

Second, we have permitted the transformation parameters a,b,c,d to be complex. This is in line with the fact that quantum mechanics allows—indeed needs—the complex field as the domain of definition of its functions. The consequence of the second extension is to require Hilbert spaces of functions which include the usual Dirac[10] and Bargmann[2,3] spaces. The transformation (6.1) is the most general one allowed by (6.2), since the group of linear real automorphisms of the algebra $so(2,1)$ is $O(2,1)$ and its complexification is $SL(2,\mathbb{C})$.

Among the canonical transformations which have been useful in classical mechanics is the one mapping the phase-space coordinates on a conserved quantity—angular momentum or the Hamiltonian—and its conjugate—angle or time. One of the aims of this program[26] is to give an extended quantum mechanical meaning to these mappings.

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APPENDIX A: REALIZATION THROUGH HYPERDIFFERENTIAL OPERATORS

As in I, we introduce a Lie algebra structure for the $SL(2,\mathbb{C})$ set of canonical transforms, disregarding the Hilbert-space structure of the functions involved, as

$$\bar{f}(r) = \int_{\mathbb{R}^+} r'^{n-1}\, dr' A_{nl\,(\tau)}(r,r') f(r') = \exp\left[i\tau H\left(r, \frac{d}{dr}\right)\right] f(r) \tag{A1}$$

where τ labels one-parameter subgroups and asking only the integrals involved to exist. The operator $H(r, d/dr)$ need not be bounded.[34] The differential operator $H(r,d/dr)$ can be found by inspection from

$$H\left(r, \frac{d}{dr}\right) f(r) = -i\int_{\mathbb{R}^+} r'^{n-1}\, dr' \left(\frac{\partial}{\partial\tau} A_{nl\,(\tau)}(r,r')\Big|_{\tau=0}\right) f(r') \tag{A2}$$

and by using the differential equations satisfied by the integration kernel, to pass the partial derivatives to act on f through partial integration, assuming the constant terms to vanish.

In agreement with what we expect from I, we find

$$\exp[ic\tfrac{1}{2}(r^2)] = \exp[ic\tfrac{1}{2}\hat{\mathbf{x}}^2]:$$

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \tag{A3a}$$

$$\exp(ib\tfrac{1}{2}\{\partial_r^2 + [(n-1)/r]\partial_r + \lambda/r^2\}) = \exp(-ib\tfrac{1}{2}\hat{\mathbf{p}}^2):$$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \tag{A3b}$$

$$\exp(i\alpha\tfrac{1}{4}\{\partial_r^2 + [(n-1)/r]\partial_r + \lambda/r^2 + r^2\}) = \exp[-i\alpha\tfrac{1}{4}(\hat{\mathbf{p}}^2 - \hat{\mathbf{x}}^2)]:$$

$$\begin{pmatrix} \cosh\frac{1}{2}\alpha & \sinh\frac{1}{2}\alpha \\ \sinh\frac{1}{2}\alpha & \cosh\frac{1}{2}\alpha \end{pmatrix}, \tag{A3c}$$

$$\exp[-\beta(r\partial_r + \tfrac{1}{2}n)] = \exp[-i\beta\tfrac{1}{4}(\hat{\mathbf{x}}\cdot\hat{\mathbf{p}} + \hat{\mathbf{p}}\cdot\hat{\mathbf{x}})]:$$

$$\begin{pmatrix} e^{\beta/2} & 0 \\ 0 & e^{-\beta/2} \end{pmatrix}, \tag{A3d}$$

$$\exp(i\gamma\tfrac{1}{4}\{\partial_r^2 + [(n-1)/r]\partial_r + \lambda/r^2 - r^2\}) = \exp[-i\gamma\tfrac{1}{4}(\hat{\mathbf{p}}^2 + \hat{\mathbf{x}}^2)]:$$

$$\begin{pmatrix} \cos\frac{1}{2}\gamma & \sin\frac{1}{2}\gamma \\ -\sin\frac{1}{2}\gamma & \cos\frac{1}{2}\gamma \end{pmatrix}. \tag{A3e}$$

The generators of the last three transforms constitute the $so(2,1)$ dynamical algebra for the radial oscillator with centrifugal force. Associating thus products of 2×2 complex matrices to hyperdifferential operators yields Baker–Campbell–Hausdorff relations[35] including ∂_r^2, $(1/r)\partial_r$, $r\partial_r$, r^2, and r^{-2} terms. A particular composition used in I is

$$\begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} = \begin{pmatrix} 1 & -\tanh\theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\cosh\theta & 0 \\ 0 & \cosh\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\tanh\theta & 1 \end{pmatrix} \tag{A4}$$

and involves the use of (A3) for $b = -\tanh\theta = c$, $\beta = -2\ln\cosh\theta$. Rather than write the lengthy resulting relation, we take $\theta = i\frac{1}{4}\pi$. This gives the Bargmann (i.e., Barut–Girardello, for arbitrary l) transform (4.4) as

$$\bar{f}(r) = \exp\left[\tfrac{1}{8}\pi\left(\frac{d^2}{dr^2} + \frac{n-1}{r}\,\frac{\lambda}{dr} + \frac{\lambda}{r^2} + r^2\right) f(r)\right. = 2^{-n/4}\exp\left[\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{n-1}{r}\,\frac{d}{dr} + \frac{\lambda}{r^2}\right)\right] e^{r^2/4} f(2^{-1/2}r). \tag{A5}$$

Writing for f the radial wavefunction (3.6) and for $\bar{f}$ the corresponding (3.8) [i.e., (2.9) for $u=0$, $v=1$] and recalling (1.8), we obtain

$$\exp\left[\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{n-1}{r}\,\frac{d}{dr} - \frac{l(l+n-2)}{r^2}\right)\right](2^{-1/2}r)^l L_N^{(n/2+l-1)}(\tfrac{1}{2}r^2) = \frac{(-1)^N}{N!}(2^{-1/2}r)^{2N+l}. \tag{A6}$$

A special function relation which seems to be new is obtained setting $z=\frac{1}{2}r^2$ and inverting (A6) as

$$\exp\left[-\left(z\frac{d^2}{dz^2}+\tfrac{1}{2}n\frac{d}{dz}-\frac{l(l+n-2)}{4z}\right)\right]z^{N+l/2} = (-1)^N N!\, z^{l/2} L_N^{(n/2+l-1)}(z) \qquad \text{(A7)}$$

and can be verified to hold independently by expanding in series.

APPENDIX B: THE PASSAGE FROM n-DIMENSIONAL TO RADIAL TRANSFORMS

In I, Appendix B, we gave results concerning the extension to n dimensions of the complex linear transforms. For the case when the canonical transform is of the type (1.1), that is, when the transformation submatrices A, B, C, D of $n\times n$, are multiples a, b, c, d of $\mathbb{1}$, these take the form

$$A^n(\eta,\mathbf{x})=\{(2\pi|b|)^{-1/2}\exp[-\tfrac{1}{2}i(\tfrac{1}{2}\pi+\arg b)]\}^n \times\exp[(i/2b)(a\mathbf{x}^2-2\mathbf{x}\cdot\eta+d\eta^2)]. \qquad \text{(B1)}$$

the integration over $\mathbf{x}$-space being over $\mathbb{R}^n$, with measure $d^n\mathbf{x}$, and the scalar product $(\bar{f},\bar{g})^{(n)}$ involving an integration over η-space, over $\mathbb{C}^n$ with measure $\nu^n(\eta,\eta^*)\,d^n\operatorname{Re}\eta\,d^n\operatorname{Im}\eta$,

$$\nu^n(\eta,\eta')=(\tfrac{1}{2}\pi v)^{-n/2}\exp[(1/2v)(u\eta^2-2\eta\cdot\eta^*+u^*\eta^{*2})]. \qquad \text{(B2)}$$

We want to show here how expressions (B1) and (B2) relate to the corresponding radial kernel (3.1) and measure (1.10)–(2.6). Consider first the two-dimensional case ($n=2$). Parametrize $\mathbb{R}^2$ as $x_1=r\sin\theta$, $x_2=r\cos\theta$ with $r\in[0,\infty)$, $\theta\in[0,2\pi)$, and $d^2\mathbf{x}=r\,dr\,d\theta$. Now parametrize $\mathbb{C}^2$ as $\eta_1=\rho\sin\Theta$, $\eta_2=\rho\cos\Theta$ with $\rho\in\mathbb{C}^+$ {i.e., $\arg\rho\in[-\frac{1}{2}\pi,\frac{1}{2}\pi)$}, $\operatorname{Re}\Theta\in[0,2\pi)$, $\operatorname{Im}\Theta\in(-\infty,\infty)$. Noticing that if $y=f(z)$ and $dy=f'(z)\,dz$, then $d\operatorname{Re}y\,d\operatorname{Im}y=|f'(z)|^2\,d\operatorname{Re}z\,d\operatorname{Im}z$, we have that the measure in $\mathbb{C}^2$ is

$$d^2\operatorname{Re}\eta\,d^2\operatorname{Im}\eta=|\rho|^2\,d\operatorname{Re}\rho\,d\operatorname{Im}\rho\,d\operatorname{Re}\Theta\,d\operatorname{Im}\Theta.$$

Now, using $\mathbf{x}\cdot\eta=r\rho\cos(\theta-\Theta)$ and the Bessel generating function, we have

$$\begin{aligned}A^2(\eta,\mathbf{x}) &= (2\pi b)^{-1}\exp(-i\tfrac{1}{2}\pi)\exp[(i/2b)(ar^2+d\rho^2)] \times\sum_{m=-\infty}^{\infty}(-i\exp[-i(\theta-\Theta)])^m J_m(\rho r/b)\\ &=\sum_{m=-\infty}^{\infty}A_{2,m}(\rho,r)[(2\pi)^{-1/2}\exp(im\theta)]^* \times[(2\pi)^{-1/2}\exp(im\Theta)],\end{aligned} \qquad \text{(B3)}$$

where $A_{2,m}(\rho,r)$ is given, with correct phase and normalization, by (3.1). This means that if we have a function $f(\mathbf{x})$ of definite eigenvalue m under L_{12} in the form $f_m(r)[(2\pi)^{-1/2}\exp(im\theta)]$ (so that the angular part be normalized), then

$$\begin{aligned}\bar{f}(\eta) &= \int_{\mathbb{R}^2}d^2\mathbf{x}\,A^2(\eta,\mathbf{x})f(\mathbf{x})\\ &=\int_{\mathbb{R}^+}r\,dr\,A_{2,m}(\rho,r)f_m(r)[(2\pi)^{-1/2}\exp(im\Theta)]\\ &=\bar{f}_m(\rho)[(2\pi)^{-1/2}\exp(im\Theta)].\end{aligned} \qquad \text{(B4)}$$

and the dependence of $\bar{f}$ on Θ is the same as that of f on θ (the range of the former being now over a strip in the complex plane), and only a transform of the *radial* part has taken place. The scalar product in the transform space of two such functions can now be calculated using (B2) and $\eta\cdot\eta^*=|\rho|^2\cosh(2\operatorname{Im}\Theta)$, and an integral representation of the Macdonald function[36]

$$\begin{aligned}(\bar{f},\bar{g})^{(2)} &= \int_{\mathbb{C}^2}d^2\operatorname{Re}\eta\,d^2\operatorname{Im}\eta\,\nu^2(\eta,\eta^*)\\ &\quad\times\{\bar{f}_m(\rho)(2\pi)^{-1/2}\exp[im(\operatorname{Re}\Theta+i\operatorname{Im}\Theta)]\}^*\\ &\quad\times\{\bar{g}_m(\rho)(2\pi)^{-1/2}\exp[im(\operatorname{Re}\Theta+i\operatorname{Im}\Theta)]\}\\ &=(2/\pi v)\int_{\mathbb{C}^+}|\rho|^2\,d\operatorname{Re}\rho\,d\operatorname{Im}\rho\\ &\quad\times\exp[(1/2v)(u\rho^2+u^*\rho^{*2})]\bar{f}_m(\rho)^*\bar{g}_m(\rho)\\ &\quad\times\int_{-\infty}^{\infty}d\operatorname{Im}\Theta\exp[-(1/v)|\rho|^2\cosh(2\operatorname{Im}\Theta)]\\ &\quad\times\exp(-2m\operatorname{Im}\Theta)\\ &=\int_{\mathbb{C}^+}d\operatorname{Re}\rho\,d\operatorname{Im}\rho\,\nu_{2,m}(\rho,\rho^*)\bar{f}_m(\rho)^*\bar{g}_m(\rho)\\ &=(\bar{f}_m,\bar{g}_m)_{2,m},\end{aligned} \qquad \text{(B5)}$$

where $\nu_{2m}(\rho,\rho^*)$ is given correctly by (2.6). Indeed, had we used different angle dependence for $\bar{f}$ and $\bar{g}$, a Kronecker δ in their eigenvalue under L_{12} would appear.

The problem for the n-dimensional case can be formulated similarly: Parametrize the real n-space $\mathbb{R}^n$ in the usual hyperspherical coordinates where the jth component reads $x_j=r\sin\theta_{n-1}\cdots\sin\theta_j\cos\theta_{j-1}$ for $1\le j\le n-1$ ($\theta_0\equiv 0$) and $x_n=r\cos\theta_{n-1}$. The ranges are $r\in[0,\infty)$, $\theta_1\in[0,2\pi)$, and $\theta_k\in[0,\pi]$ for $2\le k\le n-1$. Now parametrize the complex n-space $\mathbb{C}^n$ replacing r by ρ and θ_k by Θ_k with $\rho\in\mathbb{C}^+$. $\operatorname{Re}\Theta_k$ having the same ranges as θ_k and[37] $\operatorname{Im}\Theta_k\in(-\infty,\infty)$. The measure in $\mathbb{R}^n$ is $d^n\mathbf{x}=r^{n-1}dr\,d^{n-1}\omega_{n-1}$ with $d^{n-1}\omega_{n-1}=\sin^{n-2}\theta_{n-1}\,d\theta_{n-1}\,d^{n-2}\omega_{n-2}$ and $d\omega_1=d\theta_1$ while, in $\mathbb{C}^n$, $d^n\operatorname{Re}\eta\,d^n\operatorname{Im}\eta$ is found from the former with the weight function given by the absolute square of the weight function in $\mathbb{R}^n$. In order to express the n-dimensional transform kernel (B1) in a suitable way, expand the factor $\exp(-i\mathbf{x}\cdot\eta/b)$ in a series of Bessel times Gegenbauer polynomials,[38] the former in $r\rho/b$ and the latter in

$$\cos\theta_{n-1}\cos\Theta_{n-1}+\sin\theta_{n-1}\sin\Theta_{n-1}[\cos\theta_{n-2}\cos\Theta_{n-2}+\sin\theta_{n-2}\sin\Theta_{n-2}(\cdots)]$$

which can be identified with a degenerate $SO(n)$ $d^{\,l}_{000}$ function[39] and turned into a sum of products of hyperspherical harmonics in $\omega\equiv\{\theta_j\}$ and $\Omega\equiv\{\Theta_j\}$ as

$$\exp(-i\mathbf{x}\cdot\eta/b)=(2\pi)^{n/2}(r\rho/b)^{1-n/2}\sum_{l=0}^{\infty}\exp(-i\pi l/2) \times J_{n/2+l-1}(r\rho/b)\sum_M Y_l^M(\omega)^*Y_l^M(\Omega), \qquad \text{(B6)}$$

where the sum over the collective index M runs over the allowed $SO(n-1)\supset\cdots\supset SO(2)$ irreducible representation labels. Replacement of (B6) in (B1) and comparison with (3.1) gives

$$A^n(\eta,\mathbf{x})=\sum_{l=0}^{\infty}A_{nl}(\rho,r)\sum_M Y_l^M(\omega)^*Y_l^M(\Omega), \qquad \text{(B7)}$$

which is the n-dimensional version of (B3) and which tells us, performing the integrations parallel to (B4) that the angular dependence of $\bar{f}$ is the same as that of f, with only the additional domain of the angles in the complex plane. Finally, in order to show the n-dimensional analog of (B5),

$$(\bar{f},\bar{g})^{(n)}=\int_{\mathbb{C}^n}d^n\operatorname{Re}\eta\,d^n\operatorname{Im}\eta\,\nu^n(\eta,\eta^*) \times[\bar{f}_l(\rho)Y_l^M(\Omega)]^*[\bar{g}_l(\rho)Y_l^M(\Omega)]$$

$$= \int_{\mathbb{C}^+} d\,\mathrm{Re}\rho\, d\,\mathrm{Im}\rho\, v_{nl}(\rho,\rho^*) \bar{f}_l(\rho)^* \bar{g}_l(\rho) = (\bar{f}_l, \bar{g}_l)_{nl}, \tag{B8}$$

we must prove

$$\int d^{n-1}\mathrm{Re}\Omega\, d^{n-1}\mathrm{Im}\Omega\, Y_l^M(\Omega)^* Y_l^M(\Omega) \exp[-(1/v)\eta\cdot\eta^*] = (2\rho\rho^*/\pi v)^{1-n/2} K_{n/2+l-1}(\rho\rho^*/v), \tag{B9}$$

where the integration ranges over the strips in the complex plane of each of the angles as indicated above. The direct proof of Eq. (B9) is difficult. Differential or recursion-relation manipulations run into hopeless multiple integrals or combinatorics. A procedure which has allowed the verification of a fair number of individual cases for low l is that which uses the fact that (B9) is independent of M and shows that the Nth moment of the two sides of Eq. (B9) in $|\rho|^2$ are equal. For this, multiply Eq. (B9) by $(\rho\rho^*)^{2N+n+2l-1}$ and integrate over $\rho \in \mathbb{C}^+$. By using (2.6) and (2.9), the right-hand side has the value

$$\tfrac{1}{2}\pi^{n/2} v^n (2v)^{l+2N} N!\,\Gamma(N+\tfrac{1}{2}n+l)$$

while the left-hand side has become, for $\xi = v^{-1/2}\eta$ the Bargmann integral over $\mathbb{C}^n$ of the absolute square of $(\xi^2)^N \mathcal{Y}_l^l(\xi)$, where

$$\mathcal{Y}_l^l(\xi) = [\Gamma(\tfrac{1}{2}n+l)/2\pi^{n/2}\Gamma(l+1)]^{1/2}(\xi_1 + i\xi_2)^l$$

is the extreme, normalized, solid spherical harmonic. This seems to point out that no true Bargmann-type integral tables exist. The separation of n-dimensional integrals into radial and angular[40] parts can be seen as a step in that direction.

Note added in proof: It has been pointed out by Professor M. Toller that the semigroup $HSL(2,\mathbb{C})$ used here and in Ref. 1 has also been exploited in the harmonic analysis approach to multiperipheral dynamics. See G. Soliani and M. Toller, Nuovo Cimento **15**, 430 (1973) and S. Ferrara, G. Mattioli, G. Rossi, and M. Toller, Nucl. Phys. **B53**, 366 (1974). A particular case of Eq. (A7), for $l=0$, appears in C.M. King, M. Sc. Thesis, Auburn University (1963), unpublished.

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