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SECOND INTERNATIONAL WORKSHOP ON SUPERINTEGRABLE SYSTEMS IN CLASSICAL AND QUANTUM MECHANICS
Theory

Noncommuting Limits of Oscillator Wave Functions*

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Abstract—Quantum harmonic oscillators with spring constants $k > 0$ plus constant forces f exhibit rescaled and displaced Hermite–Gaussian wave functions, and discrete, lower bound spectra. We examine their limits when $(k, f) \to (0, 0)$ along two different paths. When $f \to 0$ and then $k \to 0$, the contraction is standard: the system becomes free with a double continuous, positive spectrum, and the wave functions limit to plane waves of definite parity. On the other hand, when $k \to 0$ first, the contraction path passes through the free-fall system, with a continuous, nondegenerate, unbounded spectrum and displaced Airy wave functions, while parity is lost. The subsequent $f \to 0$ limit of the nonstandard path shows the dc hysteresis phenomenon of noncommuting contractions: the lost parity reappears as an infinitely oscillating superposition of the two limiting solutions that are related by the symmetry.

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1. INTRODUCTION: OSCILLATORS AND FORCES

The one-dimensional harmonic oscillator is a mechanical system whose restitution force $-kx$ is proportional to the spring constant $k > 0$ and opposed to the separation x between a mass point and the oscillator center. When this system is subjected to a constant external force $-f$ ($f > 0$, such as a gravitational field in the direction of the negative x axis), it is characterized by the Hamiltonian operator

$$H^{(k,f)}(\hat{x}, \hat{p}) := \frac{1}{2}\hat{p}^2 + \frac{1}{2}k\hat{x}^2 + f\hat{x} \qquad (1)$$

for unit mass, where $\hat{x}$ and $\hat{p}$ denote the Schrödinger operators of position and momentum, with units chosen so that $[\hat{x}, \hat{p}] = i1$ [1]. The pair of parameters (f, k) thus provide a plane to study the contraction limits to the free quantum particle at the point $(0, 0)$ by taking various paths.

The purpose of this paper is to analyze a case where there are two inequivalent paths to reach the free-particle limit, as shown in Fig. 1. There is the *standard* path, where first f is turned off so the oscillator $H^{(k,0)}$ is centered on the origin, followed by the limit $k \to 0$ to the free $H^{(0,0)}$. And there is the *nonstandard* path, which first turns off k, so that the system $H^{(0,f)}$ is that of free fall, and then lets the force f vanish [2]. The problem posed by this noncommutation of deformation and contraction (for N-dimensional systems), called "dc hysteresis," was followed through the symmetry algebras of the Hamiltonians on the paths of Fig. 1. Here, we examine the case of $N = 1$-dimensional quadratic systems, where the symmetry group of (1) is parity under reflections across the oscillator center, which exchanges the two turning points of the harmonic motion for every energy; this continues being a symmetry in the free limit. In the free-fall system, however, this symmetry under reflection is lost; there is only one point of return. In the subsequent free limit, the symmetry cannot be fully recovered; but—is it lost? Here, we examine the phenomenon of dc hysteresis in the spectra and the wave functions of the oscillator, free-fall, and free systems, with the case of $N = 1$ dimension showing this phenomenon clearly through the asymptotic properties of special functions.

In Section 2, we formalize the proposed limits classically, and in Section 3, we write the Hermite functions of the displaced harmonic oscillator in a form suitable for the proposed limits [3]. These are performed in Section 4 to the free particle, where Hermite functions limit to trigonometric functions, and in Section 5 to the free-fall system, where the limit is to Airy functions. Section 6 examines the

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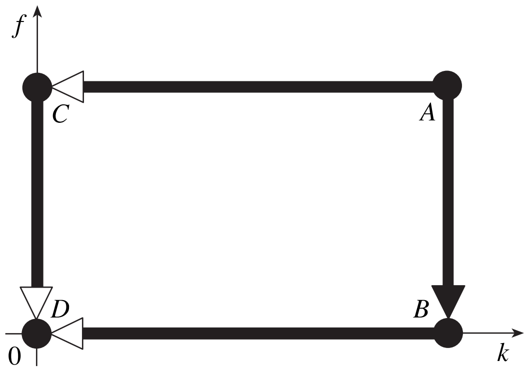

Fig. 1. Contraction paths in the $k{-}f$ plane of harmonic oscillators with spring constant k and an external force f, whose Hamiltonian is given in Eq. (1), at the point A. The half-plane $k > 0$ belongs to the generic conjugation class of shifted Hamiltonians. The *standard* path $A \to B \to D$ passes through centered oscillators $(k, f = 0)$ at B and then undergoes contraction (indicated by the empty arrowhead) to the free particle $(k = 0, f = 0)$ at D, which belongs to a conjugation class by itself. The *nonstandard* path $A \to C \to D$ contracts the Hamiltonian first to the conjugation class of free-fall systems $(k = 0, f)$ at C and then contracts once again to the free particle at D. The first path conserves parity; the second one does not. The difference between the two paths is dc *hysteresis*.

remaining contraction of Airy to trigonometric functions. Comparing the results of the two contraction paths, we comment on dc hysteresis in Section 7.

2. DISPLACED OSCILLATORS AND CONTRACTIONS

The classical parameters that characterize the motion under the oscillator Hamiltonian (1) are the constant values of

$$\text{the energy,} \quad E^{(k,f)} := H^{(k,f)}(x,p), \tag{2}$$

$$\text{equilibrium point,} \quad \dot{p} = 0 \Rightarrow x^{\text{eq}} := -f/k, \tag{3}$$

$$\text{turning points,} \quad p = 0 \Rightarrow x_{\pm}^{\text{tur}} := x^{\text{eq}} \pm \frac{1}{k}\sqrt{f^2 + 2kE^{(k,f)}}, \tag{4}$$

$$\text{minimal energy,} \quad E_0^{(k,f)} := H^{(k,f)}(x^{\text{eq}}, p = 0) = -f^2/(2k) \leq E^{(k,f)}. \tag{5}$$

When the external force f that acts on the oscillator vanishes, the classical observables of the system (2)–(5) exhibit regular limits to the centered equilibrium point $x^{\text{eq}} = 0$ and to the symmetric turning points at $|x_{\pm}^{\text{tur}}| = \sqrt{2E^{(k,0)}/k}$; the minimal classical energy is $E_0^{(k,0)} = 0$. In the limit $k \to 0$, the symmetric turning points escape to infinity, while the minimal energy is zero throughout the process. This contraction limit can be followed also for the classical trajectories in configuration space $x(t)$, where trigonometric functions limit to linear functions; and it can be seen also in phase space $(x(t), p(t))$, where centered ellipses limit to two parallel straight lines, their slope and intercept depending on the initial conditions.

On the other hand, the limit $k \to 0$ of the observables of the oscillator system (2)–(5), when the external force *persists*, $f \neq 0$, is more delicate. We first appeal to the elementary observation that the system (1) is only a *shifted* oscillator, i.e., that, if we translate the coordinate x to the equilibrium point (3), $x^{\text{eq}} = -f/k$, the system becomes

$$H^{(k,f)}(\hat{x}, \hat{p}) := \frac{1}{2}\hat{p}^2 + \frac{1}{2}k\left(\hat{x} + \frac{f}{k}\right)^2 - \frac{f^2}{2k}, \tag{6}$$

i.e., an ordinary oscillator centered on $x^{\text{eq}} = -f/k$, whose turning points are

$$x_{\pm}^{\text{tur}} := x^{\text{eq}} \pm \frac{1}{k}\sqrt{f^2 + 2kE^{(k,f)}} \underset{k\to 0}{\sim} \begin{cases} E/f - kE^2/(2f^3), \\ -E/f - 2f/k \end{cases} \tag{7}$$

and whose energy is shifted down according to (5)

$$E_0^{(k,f)} = E_0^{(k,0)} - f^2/(2k). \tag{8}$$

Letting $k \to 0$ in the above expressions means that x^{eq}, x_{-}^{tur}, $E_0^{(0,f)} \to -\infty$, and that the argument x of the limit wave functions should be finite around the first turning point $x_{+}^{\text{tur}} = E/f$. The final step in this nonstandard path, the limit $f \to 0$, may now appear indeterminate. Classically, additive constants to the Hamiltonian do not alter the equations of motion. In classical phase space $(x(t), p(t))$, displacing ellipses will limit to a parabola, and this to *one* straight line, whose slope and intercept depend again on the initial conditions.

3. OSCILLATOR ENERGIES AND WAVE FUNCTIONS

Quantum mechanics is a theory with more structure than classical mechanics; it specifies when the energy spectrum of a system is quantized or when it is continuous, and the wave functions involve specific special functions of mathematical physics whose pointwise limits should match. Here, we perform the standard contraction from the harmonic oscillator to the free particle.

The Schrödinger equation for the shifted harmonic oscillator (6) is

$$H^{(k,f)}\psi_n^{(k,f)}(x) \tag{9}$$

$$= \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}kx^2 + fx\right)\psi_n^{(k,f)}(x)$$
$$= E_n^{(k,f)}\psi_n^{(k,f)}(x),$$

its spectrum is

$$E_n^{(k,f)} = \sqrt{k}\left(n + \frac{1}{2}\right) - \frac{f^2}{2k}, \quad n \in \{0, 1, 2, \ldots\}, \tag{10}$$

and its square-integrable solutions are well known [1], namely

$$\psi_n^{(k,f)}(x) = \frac{k^{1/8}}{\pi^{1/4}}\frac{\exp[-\frac{1}{2}\sqrt{k}(x + f/k)^2]}{\sqrt{2^n n!}} \times H_n(k^{1/4}(x + f/k)) \tag{11}$$

or

$$\psi_n^{(k,f)}(x) = \frac{k^{1/8}}{\pi^{1/4}\sqrt{n!}}U\left(-n - \frac{1}{2}, (4k)^{1/4}(x + f/k)\right). \tag{12}$$

These solutions have definite parity under reflections across the oscillator equilibrium point of the set $\{\psi_n^{(k,f)}(x)\}_{n=0}^{\infty}$, $(-1)^n$; it follows that the ground-state level is even, and that parity alternates as we go up the energy spectrum.

In (12), we have written the harmonic oscillator wave functions in terms of the parabolic cylinder functions $U(a, z) = D_{-a-1/2}(z)$ {[4], Eq. (19.3.1)} because they are most appropriate for the limiting procedure. They are defined in terms of ${}_1F_1$ hypergeometric (Hermite) polynomials as

$$U(-n - 1/2, z) = \frac{\cos(\pi n/2)}{\sqrt{\pi}} \cdot 2^{n/2} \times \Gamma\left(\frac{1}{2}n + \frac{1}{2}\right)e^{-z^2/4}{}_1F_1\left(-\frac{1}{2}n; \frac{1}{2}; \frac{1}{2}z^2\right) + \frac{\sin(\pi n/2)}{\sqrt{\pi}} \cdot 2^{n/2+1/2}\Gamma\left(\frac{1}{2}n + 1\right) \times ze^{-z^2/4}{}_1F_1\left(-\frac{1}{2}n + \frac{1}{2}; \frac{3}{2}; \frac{1}{2}z^2\right), \tag{13}$$

where, owing to the trigonometric coefficients, only the first or the second summand appears when n is even or odd, and with a sign $(-1)^{n/2}$ or $(-1)^{(n-1)/2}$, respectively. The limit $f \to 0$ of the wave functions (12) to those of a centered oscillator is perfectly regular; the argument of the parabolic cylinder function in (13) simply becomes $z := (4k)^{1/4}(x + f/k) \to (4k)^{1/4}x$.

4. STANDARD LIMIT TO THE FREE PARTICLE

For $k \to 0$, the Schrödinger equation of the harmonic oscillator (9) becomes that of the free particle, i.e.,

$$H^{(0,0)}\psi_p^{(0,0)}(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_p^{(0,0)}(x) = \frac{p^2}{2m}\psi_p^{(0,0)}(x), \tag{14}$$

whose energy spectrum $E = p^2/(2m) \geq 0$ is continuous and nonnegative; it is doubly degenerate (except for $E = 0$), and one can use parity (under reflections across the origin) to classify the free-particle wave functions $\phi_p(x) := \psi_p^{(0,0)}(x)$ as

$$\begin{cases} \phi_p^+(x) = \pi^{-1/2}\cos(px/\hbar), & p \geq 0, \\ \phi_p^-(x) = \pi^{-1/2}\sin(px/\hbar), & p > 0. \end{cases} \tag{15}$$

In this section, we show that the limit of the oscillator wave functions $\psi^{(k,0)}(x)$ as $k \to 0$ at *constant energy* E yields the above wave functions $\psi_p^{(0,0)}(x)$. This condition yields the relation

$$E = (n + 1/2)\hbar\omega_n = (n + 1/2)\hbar\sqrt{k_n/m} = p^2/(2m), \tag{16}$$

so that we use the following decreasing sequence of spring constants for $n \to \infty$:

$$k_n = \frac{1}{m\hbar^2}\frac{p^4}{(2n + 1)^2}. \tag{17}$$

By substituting these values into (12), we obtain

$$\psi_n^{(k_n,0)}(x) = \frac{k_n^{1/8}}{\pi^{1/4}\sqrt{n!}}U\left(-n - \frac{1}{2}, z_n\right), \tag{18}$$

where

$$z \equiv z_n = (4mk_n/\hbar^2)^{1/4}x = \frac{px/\hbar}{\sqrt{n + 1/2}}. \tag{19}$$

We now consider the limit of (18) as $n \to \infty$. For this, we use (13) replacing z by z_n. The argument of the ${}_1F_1$ functions in (13) thus becomes

$$\frac{1}{2}z_n^2 = \frac{1}{2n + 1}\left(\frac{px}{\hbar}\right)^2. \tag{20}$$

Foremost, the limit of the Gaussian factor is

$$\exp\left(-\frac{1}{4}z_n^2\right) = \exp(-p^2x^2/(4n + 2)) \to 1.$$

For even or odd parity, $n = 2j$ or $2j + 1$, respectively, the following limits are straightforward to establish for integer $j \to \infty$ ([5], § 8.955):

$$\lim_{j\to\infty} {}_1F_1\left(-j; \tfrac{1}{2}; (px/\hbar)^2/(4j)\right) = \cos(px/\hbar), \tag{21}$$

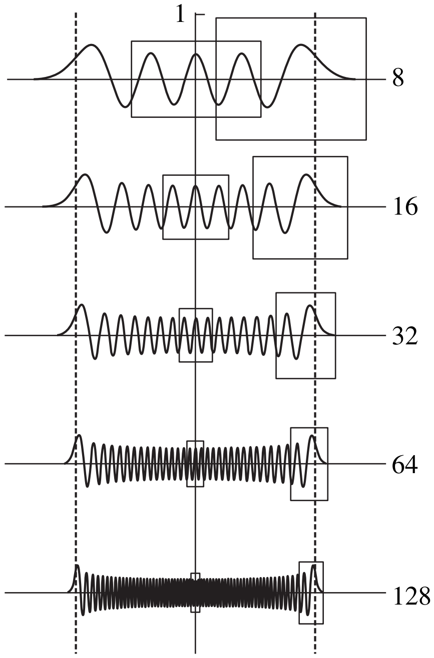

Fig. 2. Contraction of the harmonic oscillator wave functions of increasing energy n to plane waves (24), (25) and to Airy functions (45). The horizontal x axis is scaled so as to keep the classical turning points $x^{\rm tur}_{\pm}$ fixed [see (4)]. The centered boxes indicate the portion of the function that converges to a plane wave p, through scaling in x by $p/2\sqrt{n}$ and normalization by $n^{-1/4}$. The boxes around the turning points indicate the same in the limit of contraction to Airy functions.

$$\lim_{j\to\infty} {}_1F_1\Big(-j; \tfrac{3}{2}; (px/\hbar)^2/(4j)\Big) = \sin(px/\hbar)/(px/\hbar).$$

The numerical factors in front of the parabolic cylinder function in (12) have the following asymptotic behavior for $n = 2j$ or $2j+1$, when $j \to \infty$:

$$\frac{k_{2j}^{1/8}\cdot 2^j}{\pi^{3/4}} \frac{\Gamma(j+1/2)}{\sqrt{(2j)!}} \underset{j\to\infty}{\sim} \sqrt{\frac{p}{2\pi j}}, \qquad (22)$$

$$z\frac{k_{2j+1}^{1/8}\cdot 2^j}{\pi^{3/4}} \frac{\Gamma(j+3/2)}{\sqrt{(2j+1)!}} \underset{j\to\infty}{\sim} px\sqrt{\frac{p}{\pi(2j+1)}}. \qquad (23)$$

Thus, for every finite j, an oscillator of the sequence $\{k_j\}_{j=0}^{\infty}$ has discrete energy levels $E_n^{(k_j,0)}$, bounded from below by a ground state of even parity with $E_0^{(k_j,0)} = \frac{1}{2}\sqrt{k_j}$, equally spaced by $\sqrt{k_j}$, nondegenerate, and of definite and alternating parity. In the limit $j \to \infty$, where $k_j \to 0$, all states of finite n will collapse to the ground level, but the corresponding sequence of increasingly higher states $n = j \to \infty$ will limit to a free state with energy $E_n^{(k_n,0)} \approx \sqrt{k_n}n \approx F_\nu = \nu$ and the same parity. The argument of the oscillator wave functions (12) is $(4k_n)^{1/4}x = \sqrt{2\nu/n}x = px/\sqrt{n}$, so from (13), (21), and (22), (23) we can write for any $p \geq 0$ the two parity-respecting limits in the form

$$\lim_{j\to\infty} j^{1/4}(-1)^j \psi_{2j}^{(k_{2j},0)}(x) = \frac{1}{\sqrt{\pi}}\cos(px), \qquad (24)$$

$$\lim_{j\to\infty} j^{1/4}(-1)^j \psi_{2j+1}^{(k_{2j+1},0)}(x) = \frac{1}{\sqrt{\pi}}\sin(px). \qquad (25)$$

Because the oscillator wave functions are normalized to unity, the norm of the functions in the sequence of the left-hand side of the previous two equations is $\approx\sqrt{n} \to \infty$. The Kronecker normalization of the bound states thus becomes the Dirac normalization for continuous $p \geq 0$ and the two values of parity. In Fig. 2, we show centered harmonic oscillator wave functions of increasing energy $n \approx \frac{1}{2}p^2$ and the portion of the function that converges pointwise to a plane wave in the interval $-4\pi \leq px \leq 4\pi$. (See the caption of Fig. 2; the limit to Airy functions will be detailed in the next section.)

5. LIMIT $k \to 0$ TO THE FREE-FALL SYSTEM

The classical turning point of a particle with energy E in a potential $V(x)$ is defined by $V(x_E^{\rm tur}) = E$. The Schrödinger equation may there be approximated by an Airy equation, because the first term of the expansion of the energy-shifted potential in the $x^{\rm tur}$-shifted coordinate is

$$V(x - x_E^{\rm tur}) - E \approx \frac{dV(y)}{dy}\Big|_{y=x_E^{\rm tur}} x, \qquad (26)$$

i.e., a linear or free-fall potential due to a constant force. For quadratic potentials, relation (26) is exact.

The limit $k \to 0$ of the shifted oscillator can be analyzed through its energy and turning point, which are

$$E = \frac{p^2}{2m} + fx, \quad x_E^{\rm tur} = \frac{E}{f}, \quad E \in \mathfrak{R}. \qquad (27)$$

We rewrite the Schrödinger equation for this system as

$$(H^{(0,f)} - E)\psi^{(0,f)}(x) \qquad (28)$$
$$= \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + fx - E\right)\psi^{(0,f)}(x)$$
$$= -\left(\frac{\hbar^2 f^2}{2m}\right)^{1/3}\left(\frac{d^2}{dz^2} - z\right)\psi^{(0,f)}(x) = 0$$

in terms of the energy-dependent variable

$$z = z_E = \left(\frac{2mf}{\hbar^2}\right)^{1/3}\left(x - \frac{E}{f}\right) \qquad (29)$$

$$= (2f)^{1/3}(x - x_E^{\text{tur}}).$$

The square-integrable eigenfunctions of (28) are then given in terms of Airy functions ([6], § 7.2.8 and 8.5.3)

$$\psi^{(0,f)}(x,E) = \phi_E(x) = 2^{1/12} f^{-1/6} \text{Ai}(z_E), \quad (30)$$

which are Dirac-normalized to satisfy

$$\int_{-\infty}^{\infty} dx \phi_E^*(x)\phi_{E'}(x) = \delta(E - E'). \quad (31)$$

The eigenfunctions (30) have the same form as the functions of z_E in (28); their dependence on the energy E is through the displacement of their argument to the turning point, i.e., $z_E \propto (x - E/f)$.

As in Section 4, we consider a sequence of oscillators with spring constants $\{k_j\}_{j=1}^{\infty}$ tending monotonically to zero, also indicated $k \to 0$. The classical minimal energy of the shifted oscillators (8) drops without bound, $E_0^{(k_j,f)} = -f^2/(2k_j) \to -\infty$, while the equilibrium point moves without bound to the left, $x_j^{\text{eq}} = -f/k_j \to -\infty$. To maintain a finite energy value E, the integer level number n of the wave function (12) must grow, in each oscillator k_j of the sequence according to (10), with $n = j$, as

$$E_n^{(k_n,f)} = \sqrt{k_n}(n + 1/2) - f^2/(2k_n) \approx E, \quad (32)$$
$$n \in \{0, 1, 2, \ldots\},$$

and, hence,

$$2n + 1 \approx \frac{f^2 + 2k_n E}{k_n^{3/2}} \underset{k_n \to 0}{\sim} \frac{f^2}{k_n^{3/2}} \quad (33)$$
$$\Rightarrow k_n \underset{n\to\infty}{\sim} \left(\frac{f^2}{2n+1}\right)^{2/3}$$

[cf. Eq. (17)]. For that energy E, we should assign a new position coordinate ξ to the oscillator turning point (7) which escapes to the right, $x_+^{\text{tur}} \to \infty$, while we contract the old position coordinate x with the scale

$$x_+^{\text{tur}} - x^{\text{eq}} = \frac{1}{k}\sqrt{f^2 + 2kE} \quad (34)$$
$$\approx k^{-1/4}\sqrt{2n+1} \underset{k\to 0}{\sim} \frac{f}{k} + \frac{E}{f},$$

where it is understood that $k \equiv k_n$ and n are asymptotically related by (33), and similarly for all n-dependent quantities such as x_+^{tur} and x^{eq}. Keeping only the leading power of k, we define

$$\xi := \frac{x - x^{\text{eq}}}{x_+^{\text{tur}} - x^{\text{eq}}} = 1 + \frac{x - x_+^{\text{tur}}}{x_+^{\text{tur}} - x^{\text{eq}}} \quad (35)$$

or

$$\xi := \frac{(4k)^{1/4}(x + f/k)}{2\sqrt{n + 1/2}} \underset{k\to 0}{\sim} 1 \quad (36)$$
$$+ \frac{k}{f}\left(x - \frac{E}{f}\right) =: 1 + y,$$

where y will be the argument of the free-fall wave functions $\phi_E(x)$ in (30).

The coordinate ξ in (35) enters the asymptotic limit of the parabolic cylinder function to the Airy functions reported in [4], Eq. (19.7.3),

$$U\left(-n - \frac{1}{2}, \sqrt{4n+2}\xi\right) \underset{n\to\infty}{\sim} \frac{\Gamma(-n/2)}{2^{n/2}} \quad (37)$$
$$\times \frac{[(6n+3)\theta]^{1/6}}{(\xi^2 - 1)^{1/4}} \text{Ai}([(6n+3)\theta]^{2/3}),$$

where

$$\theta := \tfrac{1}{4}\left[\xi\sqrt{\xi^2 - 1} - \ln\left(\xi + \sqrt{\xi^2 - 1}\right)\right]. \quad (38)$$

With the change of variables (36) and keeping the first two powers of $y = \xi - 1$ in all expressions, we arrive at the following asymptotics for the subexpressions in (38) and those of the factors in (37):

$$\xi\sqrt{\xi^2 - 1} \underset{n\to\infty}{\sim} \sqrt{2}\left(y^{1/2} + \frac{5}{4}y^{3/2}\right), \quad (39)$$

$$\ln(\xi + \sqrt{\xi^2 - 1}) \underset{n\to\infty}{\sim} \sqrt{2}\left(y^{1/2} - \frac{1}{12}y^{3/2}\right), \quad (40)$$

$$\Rightarrow \theta \underset{n\to\infty}{\sim} \frac{1}{3}\sqrt{2}y^{3/2}; \quad (41)$$

$$\frac{2^{-n/2}\Gamma(-n/2)}{\sqrt{n!}} \underset{n\to\infty}{\sim} (2\pi)^{1/4}k^{3/8}f^{-1/2}, \quad (42)$$

$$\frac{[(6n+3)\theta]^{1/6}}{(\xi^2 - 1)^{1/4}} \underset{n\to\infty}{\sim} 2^{-1/6}k^{-1/4}f^{1/3}, \quad (43)$$

$$[(6n+3)\theta]^{2/3} \underset{n\to\infty}{\sim} (2f)^{1/3}\left(x - \frac{E}{f}\right) = z_E. \quad (44)$$

In this way, for $k_n \underset{n\to\infty}{\sim} f^{4/3}/(2n)^{3/2}$, we have proven that

$$\lim_{n\to\infty} k_n^{-1/4}\psi_n^{(k_n,f)}(x - x_E^{\text{tur}}) \quad (45)$$
$$= 2^{1/12}f^{-1/6}\text{Ai}((2f)^{1/3}(x - E/f))$$

[cf. Eqs. (30)]. The inner product of the left-hand sides is $k_n^{-1/2}\delta_{n,n'}$; in the limit $k \to 0$, this becomes the Dirac $\delta(E - E')$.

6. THE FREE FALL TO FREE LIMIT

The limit $f \to 0$ of the free-fall Airy wave functions (30) closes the nonstandard contraction path in Fig. 1. We recall the asymptotic properties of the Airy function ([4], § 10.4) for large positive and large negative arguments,

$$\mathrm{Ai}(z) \underset{z\to\infty}{\sim} \frac{1}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right), \tag{46}$$

$$\mathrm{Ai}(-z) \underset{x\to-\infty}{\sim} \frac{1}{\sqrt{\pi}} (-z)^{-1/4} \sin\left(\frac{2}{3}(-z)^{3/2} - \frac{1}{4}\pi\right). \tag{47}$$

For energy E, the argument of the Airy function in (30)–(45) is

$$z := (2f)^{1/3}\Big(x - \frac{E}{f}\Big) = -\frac{\sqrt{2}E}{\sqrt{f}}\Big(1 - \frac{f}{E}x\Big), \tag{48}$$

where the turning point is at $x^{\mathrm{tur}} = E/f$. When we consider a sequence of free-fall systems with external forces $f \to 0^+$ (we have assumed throughout that $f > 0$), their turning points escape to the right for positive energy ($E > 0$) or to the left for negative energy ($E < 0$). Hence, for a fixed, finite position x, there exists always a (small) f in the sequence such that $z < 0$ for positive energy and $z > 0$ for negative energy. In other words, for positive energies, the limit $f \to 0$ concerns the region $x < x^{\mathrm{tur}} \to \infty$, where the wave function oscillates on the whole x axis, and (47) applies. For negative energies, the limit concerns $x > x^{\mathrm{tur}} \to -\infty$, where (46) applies and the wave function is reduced to zero; this eliminates the negative-energy states from the spectrum of the free limit. Finally, the boundary case $E = 0$ in the limit $f \to 0$ contains an Airy function of $z = \sqrt{2f}x \to 0$, which indicates that the wave function (30) becomes the constant $\phi_0(x) = 2^{1/12} f^{-1/6} \mathrm{Ai}(0) \to \infty$.

For $E > 0$, we use (47) further expanding the argument of the sine function as

$$\frac{2}{3}(-z)^{3/2} = \frac{(2E)^{3/2}}{3f}\left(1 + \frac{f}{E}x\right)^{3/2} \underset{f\to0}{\sim} \frac{(2E)^{3/2}}{3f} + \sqrt{2E}x. \tag{49}$$

The free-fall wave functions (30) multiply the previous limit expressions by the factor $2^{1/12} f^{-1/6}$, and thus we have

$$\phi_{E>0}(x) \underset{f\to0}{\sim} \frac{1}{\sqrt{2\pi}}((\cos\chi + \sin\chi)\cos(px) + (\cos\chi - \sin\chi)\sin(px)), \tag{50}$$

where the label $p := \sqrt{2E} > 0$ is the same as in (14)–(25), and

$$\chi := (2E)^{3/2}/3f \to \infty. \tag{51}$$

Lastly, $\phi_{E<0}(x) \underset{f\to0}{\sim} 0$ and $\phi_{E=0}(x) \underset{f\to0}{\sim} \pi^{-1/2}$. This is the result of following the nonstandard path.

7. CONCLUSIONS

We now compare the results of the standard and the nonstandard paths of Fig. 1 when contracting the oscillator with a constant force (1) to the free system (14). Along the standard path, the discrete and lower bound spectrum of the oscillator collapsed to the double positive continuous spectrum of the free particle classified by parity, and the Hermite–Gaussian wave functions contracted as expected to the full set of plane-wave solutions given in (24), (25). On the other hand, the nonstandard path through the free-fall system (45) led to the results (50), (51). The phenomenon of dc hysteresis due to the loss of parity can be seen by comparing the last result (50), (51) with the free eigenfunction set (15) for $p \in \mathfrak{R}$.

As we pointed out in the Introduction, along the nonstandard path, the discrete lower bound oscillator energy spectrum first contracts to the real line of the free-fall spectrum, where the Airy wave functions do not have definite parity. When the external force is then made to vanish, the lower half of this spectrum is lost because the corresponding wave functions vanish, while the upper half remains, and expression (50) indeed becomes a solution to the free-particle system. Although the sum of squares of the coefficients of the $\cos(px)$ and $\sin(px)$ summands is $1/\pi$, their ratio becomes indeterminate in the limit $f \to 0$ because their mixing angle χ in (51) grows without bound. The wave functions in the nonstandard limit exhibit an infinite oscillation between the two solutions of distinct parity.

Generally, dc hysteresis is due to the loss of symmetry [2] along a nonstandard contraction path; in one dimension, the symmetry lost is parity. This is now clearly manifest in the limiting spectra and wave functions, of which only "half" is recovered, in the peculiar way in which expression (50) contains the sum of the two original limit solutions, in an infinitely oscillating linear combination.

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