Connection between two Wigner functions for spin systems

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(Received 15 July 1999; published 10 February 2000)

In 1981, Agarwal proposed a Wigner quasiprobability distribution function on the group SU(2) that serves to analyze two-particle spin states on a sphere. Recent work by our group has included the definition of an apparently distinct Wigner function on generic Lie groups whose natural range has the dimension of the group and serves for all square-integrable representations; for the SO(3) case this entails a three-dimensional ''meta-phase'' space. Both have the fundamental properties covariance and completeness. Here we show how the former is obtained as a restriction of the latter.

PACS number(s): 03.65.Bz, 42.50.Fx

I. CONSTRUCTION OF AGARWAL

In Ref. [1], Agarwal proposed the construction of a Wigner quasiprobability distribution function for SO(3) irreducible representation (irrep) classified states, which has been much used to describe two interacting systems of fixed spin S [2]. First one considers the irreducible tensor [*polarization* operator, cf. [3], Eq. (2.4(6))]

$$\hat{T}^{(S)}_{L,M}=\sqrt{\frac{2L+1}{2S+1}}\sum_{m,m'=-S}^{S}|S,m'\rangle C^{S,m'}_{S,m;L,M}\langle S,m|, \qquad (1)$$

where $C^{S,m'}_{S,m;L,M}$ is the Clebsch-Gordan coefficient that couples two representations of spin S to a total spin $0\leqslant L\leqslant 2S$ and projection M. [We note that operators only of *integer* spin L participate, representing SO(3) rather than SU(2).] Then, for a density (Hilbert-Schmidt operator) ρ, the corresponding *Wigner-Agarwal type*-Ω function [4] is defined on the sphere $(\theta,\phi)\in S_2$ through

$$W^{A}_{\Omega}(\rho|\theta,\phi)=\sum_{L=0}^{2S}\sum_{M=-L}^{L}\Omega_{L,M}Y_{L,M}(\theta,\phi)^{*}\mathrm{Tr}(\hat{T}^{(S)}_{L,M}\rho), \qquad (2)$$

where $Y_{L,M}(\theta,\phi)=Y_{L,-M}(\theta,\phi)^{*}$ is the spherical harmonic, Tr is the operator trace, and $\Omega=\{\Omega_{L,M}\}$ are complex numbers.

The simplest analogue of the Wigner function with desirable properties is obtained for $\Omega_{L,M}=1$, while other sets Ω yield other quasi- or distribution functions. [The operator (2) acting on ρ is self-adjoint for $\Omega_{L,M}=\Omega^{*}_{L,-M}$.] In particular, when $\Omega_{L,M}$ is independent of M, the Wigner function is manifestly covariant under SO(3), because similarity transformations of ρ by a rotation will devolve a geometric transformation of the sphere coordinates $(\theta,\ \phi)$. Also (for $\Omega=1$), using Clebsch-Gordan identities, one can easily prove that the overlap (or completeness) relation holds:

$$\int_{s_2}\sin\theta\,d\theta d\phi\,W^{A}_{I}(\rho|\theta,\phi)W^{A}_{I}(\tau|\theta,\phi)=\mathrm{Tr}(\rho\tau). \qquad (3)$$

This relation is important because it allows the formulation of a measurement theory.

II. WIGNER FUNCTION ON A GENERAL LIE GROUP

A rather general construction of a covariant Wigner function has been made on unimodular, exponential-type Lie groups [5], in particular on SU(2) [6], and can be generalized to nonunimodular (such as the affine) groups [7]. Basically, one builds an operator that is the integral over the group manifold of the operator $\exp[\vec{y}\cdot(\vec{x}-\vec{J})]$, where $\vec{J}$ is the vector of its generators, $\vec{y}$ are its polar coordinates, and $\vec{x}$ is a vector whose components are the *meta-phase-space* coordinates over a real space with the dimension of the group manifold. Here we write the SO(3) Wigner function as

$$W^{B}_{\omega}(\rho|\vec{x})=\int\omega(y)dy\int_{s_2}d\frac{\vec{y}}{y}e^{i\vec{y}\cdot\vec{x}}\mathrm{Tr}(e^{-i\vec{y}\cdot\vec{J}}\rho), \qquad (4)$$

where $\omega(y)$ is a weight function of the length $y=|\vec{y}|$, one of the polar coordinates of the group; this ''radius'' labels conjugation classes. Instead of $(0,\ 2\pi)$, the ''radius'' y must have the range $(-\pi,\ \pi)$ to ensure the hermiticity of the Wigner function, $W^{B}_{\omega}(\rho^{\dagger}|\vec{x})=W^{B}_{\omega}(\rho|\vec{x})^{*}$. In the SO(3) manifold of polar coordinates, we recall that diametrically opposite points are identified; hence y actually ranges over a circle. In polar coordinates, the left- and right-invariant measure over the group is $\omega(y)=\frac{1}{2}\sin^2\frac{1}{2}y$.

Note that the Wigner function (4) is a function of $\vec{x}\in\mathbb{R}^3$, in contradistinction to the Wigner-Agarwal function (2), which is a function of the sphere $(\theta,\phi)\in\mathcal{S}_2$ only, and that all spins (irreducible representations) of SO(3) are included. It is easy to see also that, for any weight ω, the general Wigner function (4) is covariant. The purpose of this Brief Report is to show in what sense both Wigner functions are equivalent, and in what do they differ.

III. EQUIVALENCE BETWEEN THE TWO WIGNER FUNCTIONS

First we recall two generating functions that are valid for numbers and for SO(3) operators in their $(2S+1)^2$-dimensional reducible representation [3]; the latter is reduced into the irreducibles $L=0,1,...,2S$. They are

$$e^{i\vec{y}\cdot\vec{x}}=\sum_{L=0}^{\infty}\sum_{M=-L}^{L}4\pi i^L j_L(xy)Y_{L,M}\left(\frac{\vec{y}}{y}\right)Y_{L,M}\left(\frac{\vec{x}}{x}\right)^*, \quad (5)$$

$$e^{-i\vec{y}\cdot\vec{J}}=\sum_{L=0}^{2S}\sum_{M=-L}^{L}g_L^S(y)Y_{L,M}\left(\frac{\vec{y}}{y}\right)^*\hat{T}_{L,M}^{(S)},$$

$$g_L^S(y)=\frac{2\sqrt{\pi}(-i)^L}{\sqrt{2S+1}}\chi_L^S(y), \quad (6)$$

where $j_L(z)$ is the spherical Bessel function of order L, and $g_L^S(y)$ is written in terms of the generalized characters of the group, which are [see Ref. [3], Sec. 2.4, Eqs. (2.4(8)) and (4.15(38))]

$$\chi_L^S(y)=\sqrt{\frac{(2S+1)(2S-L)!}{(2S+L+1)!}}\sin^L\tfrac{1}{2}y \times\left(\frac{d}{d\cos\frac{1}{2}y}\right)^L\frac{\sin(S+\frac{1}{2})y}{\sin\frac{1}{2}y)}. \quad (7)$$

The generalized characters are functions that include the group characters in the way that the associated Legendre polynomials include the ordinary Legendre ones.

Now, replacement of Eqs. (5) and (6) into Eq. (4) yields the relation between the two Wigner functions in the form

$$W_\omega^B(\rho|\vec{x})=\sum_{L=0}^{2S}\sum_{M=-L}^{L}h_L^S(x)Y_{L,M}\left(\frac{\vec{y}}{y}\right)^*\mathrm{Tr}(\hat{T}_{L,M}^{(S)}\rho) \quad (8)$$

$$=W^A_{h_L^S(x)}\left(\rho\left|\frac{\vec{x}}{x}\right.\right), \quad (9)$$

$$h_L^S(x)=4\pi i^L\int dy\ \omega(y)j_L(xy)\chi_L^S(y), \quad (10)$$

where we have used the integral over the sphere $\vec{y}/y$ of two spherical harmonics to eliminate two of the sums. The form of Eq. (8) is that of a Wigner-Agarwal function (2) of type $\Omega_{L,M}=h_L^S(x)$. We note that the independence of Ω on the projection number M is a necessary and sufficient condition for covariance.

We now show that, if we integrate the general Wigner function $W_\omega^B(\rho\vec{x})$ over $x\geqslant 0$ with measure $x^\nu dx$, $0<\nu<1$, we can find a weight function $\omega(y)$ such that it yields the Wigner-Agarwal function. In Eq. (8), this integral affects only the factor $h_L^S(x)$, yielding

$$\int_0^\infty x^\nu dx h_L^S(x)=4\pi i^L\int_{-\pi}^{\pi}dy\,\omega(y)\chi_L^S(y)\int_0^\infty dx\,x^\nu j_L(xy)$$

$$=\frac{4\pi^2 2^\nu}{\sqrt{2S+1}}\frac{\Gamma(\frac{1}{2}(L+\nu+1))}{\Gamma(\frac{1}{2}(L-\nu+2))}\int_{-\pi}^{\pi}dy\frac{\omega(y)}{y^{\nu+1}}\chi_L^S(y). \quad (11)$$

(See Ref. [8], Eq. 6.561.14.) An orthogonality relation for the generalized characters [which does not appear in Ref. [3], but can be proven from its Eq. 4.15(2)] is

$$\int_{-\pi}^{\pi}dy\ \chi_L^S(y)\chi_{L'}^S(y)=2\pi\delta_{L,L'}\frac{2S+1}{2L+1}. \quad (12)$$

This allows us to extract the function $\omega_S(y)$ such that Eq. (11) is $\int_0^\infty x^\nu dx h_L^S(x)=1$, namely

$$\omega_{S,\nu}(y)=\frac{y^{\nu+1}}{2^\nu(2\pi)^3}\sum_{L'=0}^{2S}\frac{2L'+1}{\sqrt{2S+1}}\frac{\Gamma(\frac{1}{2}(L'-\nu+2))}{\Gamma(\frac{1}{2}(L'-\nu+1))}\chi_{L'}^S(y), \quad (13)$$

which will be independent of L. Then, the relation between the general and the Wigner-Agarwal functions (4) and (2) is

$$\int_0^\infty x^\nu dx\ W^B_{\omega_{S,\nu}}(\rho|\vec{x})=W_1^A\left(\rho\left|\frac{\vec{x}}{x}\right.\right), \quad 0<\nu<1. \quad (14)$$

The conclusion of this development is that the new, group-theoretically motivated Wigner function (4), which is defined over $\mathbb{R}^3$ and is a functional of a density (Hilbert-Schmidt) operator, when restricted to spin S, yields the covariant Wigner-Agarwal function (2), which is defined over the sphere, endowing it with a radius-dependent type $\Omega_L(x)$. As shown in Eq. (10), it is the spherical Hankel transform of order L of the weight function $\omega(y)$ times the generalized characters (7) of the group.

IV. OVERLAP FORMULA AND THE WEIGHT FUNCTION

The freedom we have assumed for the weight function $\omega(y)$ of the conjugation classes of the group (and their harmonic transform function over the irreducible representations) is curtailed when we demand that the general Wigner function satisfy the overlap condition (3). Indeed, when we use Eq. (8) to compute this integral over $\mathbb{R}^3$ in polar coordinates, we find

$$\int_{\mathbb{R}^3}d\vec{x}\ W_\omega^B(\rho|\vec{x})W_\omega^B(\tau|\vec{x})$$
$$=\sum_{L=0}^{2S}\sum_{M=-L}^{L}(-1)^M\mathrm{Tr}(\hat{T}_{L,M}^{(S)}\rho)\mathrm{Tr}(\hat{T}_{L,-M}^{(S)}\tau)$$
$$\times\int_0^\infty x^2dx[h_L^S(x)]^2, \quad (15)$$

where we have used the integration over the sphere of the two spherical harmonics to eliminate two of the sums. Now we can compute the remaining integral over x, replacing the square of Eq. (10) and noting the Parseval relation between norms of two functions related by the Hankel transform,

$$\int_0^\infty x^2dx|h_L^S(x)|^2=\frac{32\pi^4}{2S+1}\int\frac{dy}{y^2}[\omega(y)\chi_L^S(y)]^2. \quad (16)$$

When $\omega(y)$ is such that this integral is a constant, then the right-hand side of Eq. (15) is Tr ($\rho\tau$), and the usual overlap formula (3) holds. This occurs when $\omega(y)=y\sin\frac{1}{2}y$, independently of the assumed bound on the spins S.

While the Wigner-Agarwal function was introduced prompted by physical considerations in quantum optics, the Wigner function (4) was built in Ref. [5] for polychromatic wave optics, but with the view to extend its definition to general Lie groups. Several particular cases were studied [6], among them the SU(2) case that appears in this report. A more comprehensive mathematical treatment in [7] shows that for the affine group, the proposed Wigner function matches the concept of hyperimage in wide-band radar imaging [9]. It is good to see the relation between these apparently disjoint functions.

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