The role of the cosmological constant in the notion of the "gravitational entropy"

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IMPORTANT: the "gravitational entropy" I will discuss is NOT:

- ★ The "standard" entropy from Thermodynamics (or Kinetic Theory or Statistical Mechanics) of thermal sources or plasmas in a hydrodynamical regime (short range interactions).

- ★ The Boltzmann Gibbs entropy in Statistical Mechanics of Newtonian self-gravitating systems (isothermal sphere, globular clusters, n-body simulations, etc).

- ★ Tsallis (and related) "non-extensive" entropies.

- ★ Entropy based on Black Hole Thermodynamics and geometric formal analogies (such as "Geometro-Thermodynamics").

However, it can have interesting theoretical connections to all these

So, what entropy I'm talking about?

Two main theoretical proposals of the notion of "gravitational entropy" in the context of GR.

PROPOSAL #1: **Gravitational Entropy constructed from an "effective" energy-momentum tensor associated with the "pure" gravitational field (Weyl tensor).**

Reference: Clifton, Ellis & Tavakol, *Class Quant Grav*, 30, 012301 (2013)

PROPOSAL #2: **Gravitational Entropy as a measure of "accessible gravitational states" defined from scalar curvature fluctuations around an averaged scalar curvature. Information Theory.**

Reference: Hosaya, Buchert & Morita, *Phys Rev Lett*, 92, 141302 (2004)

Surprisingly: both proposals yield:

- **the same conditions for positive entropy production (for Coulomb-like fields)**
- **the Hawking-Beckenstein area formula when applied to a Schwarzschild Black Hole.**

"Gravitational Entropy"

is

the

Entropy

constructed

from a

"geometric fluid"

associated

with the

"free gravitational field"

Free Gravitational Field (Weyl tensor)

→

"Effective" energy-momentum tensor

"square root" of the Bel-Robinson tensor

↓

"Gravitational" fluid variables:

density, pressure, viscosity, heat.

→

"Gravitational" temperature

(Gibbs One-form)

↓

"Gravitational" entropy

& equilibrium states and entropy growth

→

Implications:

Structure Formation

Inflation

Cosmological Constant

What is the "free gravitational" field ?

Under GR "curvature" generates gravity, but "curvature" (Riemann tensor) can be split into "Ricci" and Weyl parts:

$$\mathcal{R}^{ab}_{\ \ cd} = C^{ab}_{\ \ cd} - 2\delta^{[a}_{[c}\mathcal{R}^{b]}_{d]} + \frac{1}{3}\mathcal{R}\delta^{a}_{c]}\delta^{b}_{d]}$$

Riemann **Weyl part** **Ricci part**

Einstein field equations:

$$G^{ab} = \mathcal{R}^{ab} - \frac{1}{2}g^{ab}\mathcal{R} = \frac{8\pi G}{c^4}T^{ab}$$

Curvature **Sources**

$$T^{ab} = \rho u^a u^b + p h^{ab} + \Pi^{ab} + 2q^{(a}u^{b)}$$

Energy Momentum tensor.

Einstein field equations in vacuum (source free): $T^{ab} = 0$

Reduce to: $\mathcal{R}^{ab} = 0$ **in this case: Riemann = Weyl**

Therefore: Free Gravitational field = curvature from the Weyl tensor

BASIC IDEA: construct an "effective" energy-momentum tensor based on the Weyl tensor

Properties of the Weyl tensor:

Covariant decomposition: “electric” and “magnetic” parts

Maxwell tensor	$E_a = F_{ab}u^b, \quad H_a = \frac{1}{2}\eta_{abc}F^{bc},$
Weyl tensor	$E_{ab} = C_{abcd}u^c u^d, \quad H_{ab} = \frac{1}{2}\eta_{acd}C^{cd}_{be}u^e,$

Invariant classification of gravitational fields (solutions of Einstein’s equations) from its algebraic properties:

- **Conformally Flat fields** (zero Weyl tensor): most trivial and homogeneous solutions (FLRW, de Sitter, Minkowski)
- **Coulomb-like fields:** well defined Newtonian limit, do not radiate (black hole & most cosmological inhomogeneous solutions).
- **Wave-like:** gravitational waves and similar radiating solutions
- **The remaining:** Most fields are in between Coulomb-like & wave-like

Homework:

Construct an "effective" energy-momentum tensor from the Weyl tensor

- MUST SATISFY $\mathcal{T}_{ab} = \mathcal{T}_{ba}, \qquad \nabla_b \mathcal{T}^{ab} = 0,$

Energy-Momentum of what ?

of a "geometric fluid" related to the Weyl tensor such that:

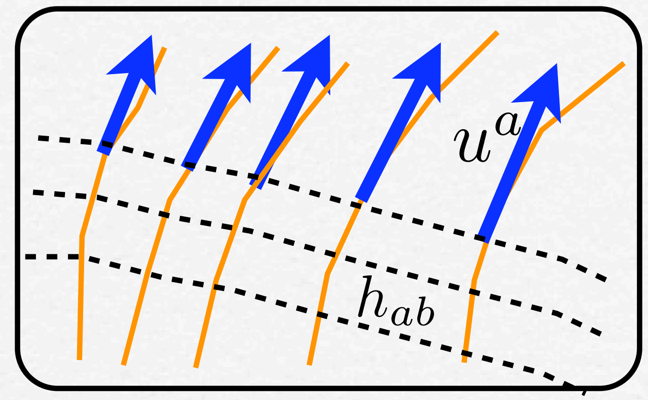
$$C^{abcd} = 0 \quad \Leftrightarrow \quad \mathcal{T}^{ab} = 0$$

Notice: it is an effective fluid: $G^{ab} \neq 8\pi\, \mathcal{T}^{ab} = 0$

How can we construct this geometric fluid ?

Take a spacetime manifold $(\mathcal{M}, g)$

- **satisfies Einstein's equations** $G^{ab} = 8\pi T^{ab}$
- **Admits a 4-velocity field** u^a



Choice of observers comoving with u^a

Spatial metric $h_{ab} = g_{ab} + u_a u_b$

The source of $(\mathcal{M}, g)$ is always expressible as $T^{ab} = \rho u^a u^b + p h^{ab} + \Pi^{ab} + 2q^{(a}u^{b)}$

where we identify:

$\rho = u_a u_b T^{ab}$,	$p = \frac{1}{3} h_{ab} T^{ab}$,	$\Pi^{ab} = T^{\langle a,b \rangle}$,	$q_a = -u^b h_a^c T_{bc}$
Density	**Pressure**	**Viscosity**	**Heat Flux**

WE OBTAIN (to be explained how): $\mathcal{T}^{ab} = \rho_{\rm gr}\, u^a\, u^b + p_{\rm gr}\, h^{ab} + \Pi^{ab}_{\rm gr} + 2\, q^{(a}_{\rm gr}\, u^{b)}$

where we identify:

$\rho_{\rm gr} = u_a\, u_b\, \mathcal{T}^{ab}$,	$p_{\rm gr} = \frac{1}{3} h_{ab}\, \mathcal{T}^{ab}$,	$\Pi^{ab}_{\rm gr} = \mathcal{T}^{\langle ab \rangle}$,	$q^a_{\rm gr} = -u_b\, h^a_c\, \mathcal{T}^{bc}$,
Gravitational Density	**Gravitational Pressure**	**Gravitational Viscosity**	**Gravitational Heat Flux**

FIRST PROBLEM: Weyl tensor is not divergence-less

$$\nabla_d C^{abcd} \neq 0 \quad \text{in general}$$

SOLUTION: Use the ONLY divergence-less tensor constructed from the Weyl tensor: Bel-Robinson tensor

$$T_{abcd} = \frac{1}{4}\left[C_{eabf}\,C^{ef}_{\ \ cd} + C^{*}_{eabf}\,C^{*ef}_{\ \ cd}\right], \qquad C^{*}_{abcd} = \frac{1}{2}\eta_{abc}\,C^{ef}_{\ \ cd},$$

SECOND PROBLEM: The Bel-Robinson tensor is 4th order, but we need a 2nd order tensor

SOLUTION: Obtain its "square root" by the following irreducible algebraic expansion

$$T_{abcd} = \mathcal{T}_{(ab}\mathcal{T}_{cd)} - \frac{1}{2}\mathcal{T}_{e(a}\mathcal{T}_{b}^{\ e}g_{cd)} - \frac{1}{4}\mathcal{T}_{e}^{\ e}\mathcal{T}_{(ab}g_{cd)} + \frac{1}{24}\left[\mathcal{T}_{ef}\mathcal{T}^{ef} + \frac{(\mathcal{T}_{e}^{\ e})^2}{2}\right]g_{(ab}g_{cd)}$$

Notice though that:

$$\forall\ \{T_{ab},\ g_{ab}\}\ \exists\ \text{unique well defined}\ \ T_{abcd}$$

$$\text{but not for all}\ \ \{T_{abcd},\ g_{ab}\}\ \exists\ \text{unique well defined}\ \ T_{ab}$$

However, **for two large classes of space-times it is possible to construct a self consistent tensor** T_{ab} **with the desired properties: symmetric and covariantly conserved** $\nabla_b T^{ab} = T^{ab}{}_{;b} = 0$

- **Coulomb-like fields:** **Petrov type D, with consistent Newtonian analogs**
- **Gravitational Wave-like fields:** **Petrov type N, with consistent radiative analogs**

- **Remaining fields** **are complicated combinations of Coulomb-like and Wave-like fields**

How do we obtain an entropy from an energy-momentum tensor? **EXAMPLE:** **Off equilibrium thermal sources:**

Energy-Momentum tensor:

$$T^{ab} = \rho u^a u^b + p h^{ab} + \Pi^{ab} + 2q^{(a}u^{b)}$$

Energy Density ($\rho u^a u^b$) **Pressure** ($p h^{ab}$) **Viscosity** (Π^{ab}) **Heat flux** ($2q^{(a}u^{b)}$)

State variables by projection:

$$\rho = u_a u_b T^{ab}, \quad p = \tfrac{1}{3} h_{ab} T^{ab}, \quad \Pi^{ab} = T^{\langle a,b \rangle}, \quad q_a = -u^b h^c_a T_{bc}$$

Off-equilibrium Gibbs equation:

$$T\dot{s} = (\rho V)^{\cdot} + p\dot{V} = -V\left[u_a \nabla_b T^{ab} + \nabla_a q^a + \dot{u}_a q^a + \sigma_{ab}\Pi^{ab}\right]$$

This term is zero in equilibrium: $-V\left[u_a \nabla_b T^{ab} + \nabla_a q^a + \dot{u}_a q^a + \sigma_{ab}\Pi^{ab}\right]$

where:

s (specific entropy), V (local volume), T (temperature),

$\dot{u}_a$ (4-acceleration), σ_{ab} (shear),

Now, we proceed "by analogy":

Given the "effective" energy-momentum tensor $\mathcal{T}^{ab}$

project with respect to the same u^a (|| "time"), h^{ab} ($\perp$ "space")

- **"State variables" of the free gravitational field:**

$$\rho_{\rm gr} = u_a u_b \mathcal{T}^{ab}, \quad p_{\rm gr} = \tfrac{1}{3} h_{ab}\mathcal{T}^{ab}, \quad \Pi^{ab}_{\rm gr} = \mathcal{T}^{\langle ab\rangle}, \quad q^a_{\rm gr} = -u^b h^{ab}\mathcal{T}_{bc},$$

- **"Effective" Gibbs equation for the free gravitational field:**

$$T_{\rm gr}\dot{s}_{\rm gr} = (\rho_{\rm gr}V)^{\cdot} + p_{\rm gr}\dot{V} = -V\left[u_a\nabla_b\mathcal{T}^{ab} + \nabla_a q^a_{\rm gr} + \dot{u}_a q^a_{\rm gr} + \sigma_{ab}\Pi^{ab}_{\rm gr}\right]$$

- **"Gravitational temperature" by correspondence with semi-classical definitions**

$$T_{\rm gr} = \frac{c^4}{\pi G}|u_{a;b}\ell^a k^b| = \frac{c^4}{2\pi G}|\dot{u}_a z^a + \mathcal{H} + \sigma_{ab}z^a z^b|,$$

$\ell^a = (u^a - z^a)/\sqrt{2},\ k^a = (u^a + z^a)/\sqrt{2}$ **the 2 real vectors of the NP null tetrad**

$\mathcal{H} \equiv \frac{\Theta}{3} = \frac{1}{3}u^a{}_{;a}$ **Hubble expansion rate**

Results for Coulomb-like fields:

Effective energy momentum tensor:

$$\frac{8\pi G}{c^4}T^{ab} = \left(\frac{2W}{3}\right)^{1/2}\left[(x^a x^b + y^a y^b) - 2(z^a z^b - u^a u^b)\right],$$

where $\{x^a, y^a, z^a, u^a\}$ orthonormal tetrad

and $W \equiv T_{abcd}u^a u^b u^c u^d = \frac{1}{4}E_{ab}E^{ab}, \quad H_{ab} = 0,$

Effective state variables:

$$\frac{8\pi}{c^4}\rho_{gr} = \left(\frac{2W}{3}\right)^{1/2}, \quad p_{gr} = q^a_{gr} = 0, \quad \frac{8\pi}{c^4}\Pi^{ab}_{gr} = \left(\frac{2W}{3}\right)^{1/2}(x^a x^b + y^a y^b - z^a z^b + u^a u^b),$$

Effective Gibbs equation:

$$T_{gr}\dot{s}_{gr} = (\rho_{gr}V_p)^{\cdot} = -V_p\sigma_{ab}\left[\Pi^{ab}_{gr} + \frac{4\pi G}{c^4}\frac{\rho+p}{\sqrt{2W/3}}E^{ab}\right]$$

Heat flux analogue

Look at the gravitational temperature

$$T_{\rm gr} = \frac{1}{2\pi}\left[\mathcal{H} + \dot{u}_a\, n^a + \sigma_{ab}\, n^a\, n^b\right]$$

$$\mathcal{H} = \frac{\Theta}{3} = h^b_a\,\nabla_b\, u^b, \quad \dot{u}_a = u^b \nabla_b u^a, \quad \sigma_{ab} = \nabla_{(a}u_{b)} - \dot{u}_{(a}u_{b)} - \mathcal{H}\, h_{ab},$$

Hubble expansion — **4-acceleration** — **Shear tensor**

$$\mathcal{H}^2 = \frac{8\pi}{3}\rho - \frac{{}^3\mathcal{R}}{6} - 2\sigma_{ab}\sigma^{ab} + \frac{8\pi}{3}\Lambda$$

Hamiltonian constraint (generalized Friedman equation).

Non-rotating Coulomb-like spacetime.

$$ds^2 = -N^2\, dt^2 + g_{ij} dx^i\, dx^j,$$

→

$$u^a = \frac{1}{N}\,\delta^a_t \quad \exists \quad \ell \propto \sqrt{\det(g_{ij})}$$

$$\dot{u}_a \propto h^b_a \nabla_b (\ln N) \sim \frac{N_{,i}}{N} n_a,$$

$$\sigma_{ab} \propto \partial_t \left[g_{ij,k} g^{kl} n_l - \ell\, g_{ij}\right]\, h^i_a\, h^j_b,$$

→

$$\mathcal{H}^2 \sim \frac{8\pi}{3}\rho + \text{spatial gradients of } g_{ij} + \frac{8\pi}{3}\Lambda$$

Look AGAIN at the gravitational temperature

$$T_{\rm gr} = \frac{1}{2\pi}\left[\mathcal{H} + \dot{u}_a\, n^a + \sigma_{ab}\, n^a\, n^b\right]$$

$$\mathcal{H}^2 \sim \tfrac{8\pi}{3}\rho + \text{spatial gradients of } g_{ij} + \tfrac{8\pi}{3}\Lambda$$

$$\{\dot{u}_a,\, \sigma_{ab}\} \sim \text{radial gradients of the metric,}$$

$$\rho \sim \ell^{-3} \qquad \text{(if we assume CDM)}$$

Asymptotically $\ell \to \infty$

$$\mathcal{H} \to \sqrt{\tfrac{8\pi}{3}\,\Lambda}, \quad \rho \to 0, \quad \{\dot{u}_a,\, \sigma_{ab}\} \to 0$$

Which implies: $T_{\rm gr} \to \sqrt{\frac{2}{3\pi}\,\Lambda}$

Test Gravitational Entropy on solutions of Einstein's equations:

Lemaitre-Tolman-Bondi (LTB) models with a dust source $T^{ab} = \rho u^a u^b$

$$ds^2 = -dt^2 + a^2\left[\frac{\Gamma^2\, dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],$$
$$a = a(t,r), \quad \Gamma \equiv 1 + ra'/a, \quad K = K(r),$$

$$\rho = \frac{\rho_0}{a^3\Gamma}, \quad \mathcal{H} = \frac{\dot{a}}{a} + \frac{\dot{\Gamma}}{3\Gamma}, \quad \sigma^a_b = -\frac{\dot{\Gamma}}{3\Gamma} \times \mathrm{diag}[0,-2,1,1],$$

$$E^a_b = W^2 \times \mathrm{diag}[0,-2,1,1]$$ **("Electric" Weyl tensor)**

$$W^2 = \frac{4\pi G}{3c^4}\left|\mathbf{D}(\rho)\right|, \quad \mathbf{D}(\rho) \equiv \rho - \langle\rho\rangle_q,$$ **Density fluctuation**

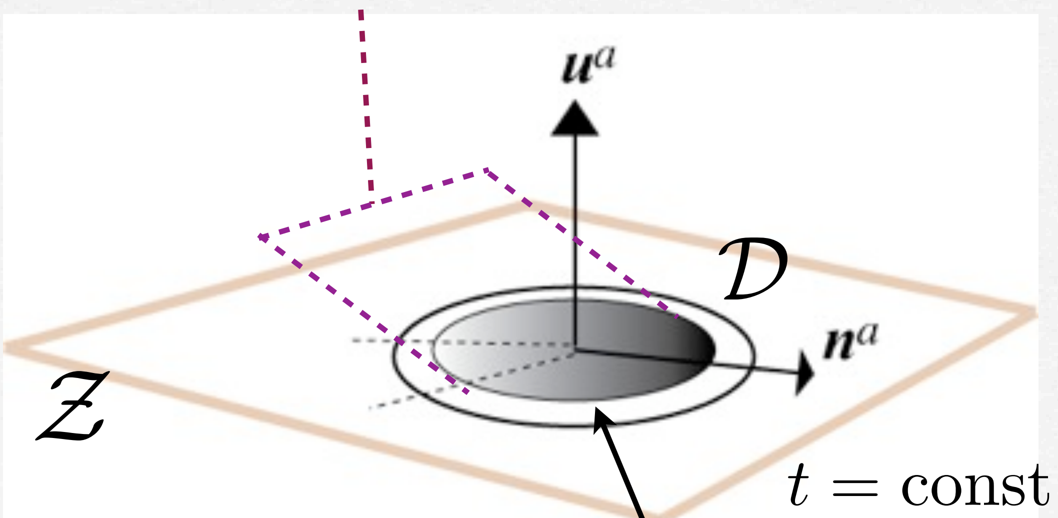
$$\text{where} \quad \langle\rho\rangle_q \equiv \frac{\int \rho\mathcal{F}dV_p}{\int \mathcal{F}dV_p}, \quad \mathcal{F} \equiv \sqrt{1-Kr^2},$$ **Weighed average (the "q-average")**

$$T_{\mathrm{gr}} = \frac{2c^4}{\pi G}\left|\langle\mathcal{H}\rangle_q(1+3\delta^{\mathcal{H}})\right|, \quad \delta^{\mathcal{H}} \equiv \frac{\mathbf{D}(\mathcal{H})}{\langle\mathcal{H}\rangle_q}$$ **"Velocity" fluctuation**

Interpretation of the “fluctuations”

$$\mathcal{D}(\rho) = \rho(t,\, x^i) - \langle\rho\rangle_{\mathcal{D}}$$

Use $\langle\rho\rangle_{\mathcal{D}} = \frac{\int_{\mathcal{D}} \rho \, d\mathcal{V}}{\int_{\mathcal{D}} d\mathcal{V}}$ average density on domains $\mathcal{D} \subset \mathcal{Z}$



& compare with local density ρ in points $\mathrm{x}^a \in \mathcal{D}$

Conditions for non-negative entropy production: $\dot{s}_{\rm gr} \geq 0$

$$\dot{s}_{\rm gr} \geq 0 \quad \Leftrightarrow \quad \mathbf{D}(\rho)\mathbf{D}(\mathcal{H}) \leq 0$$

Intuitively, it makes sense

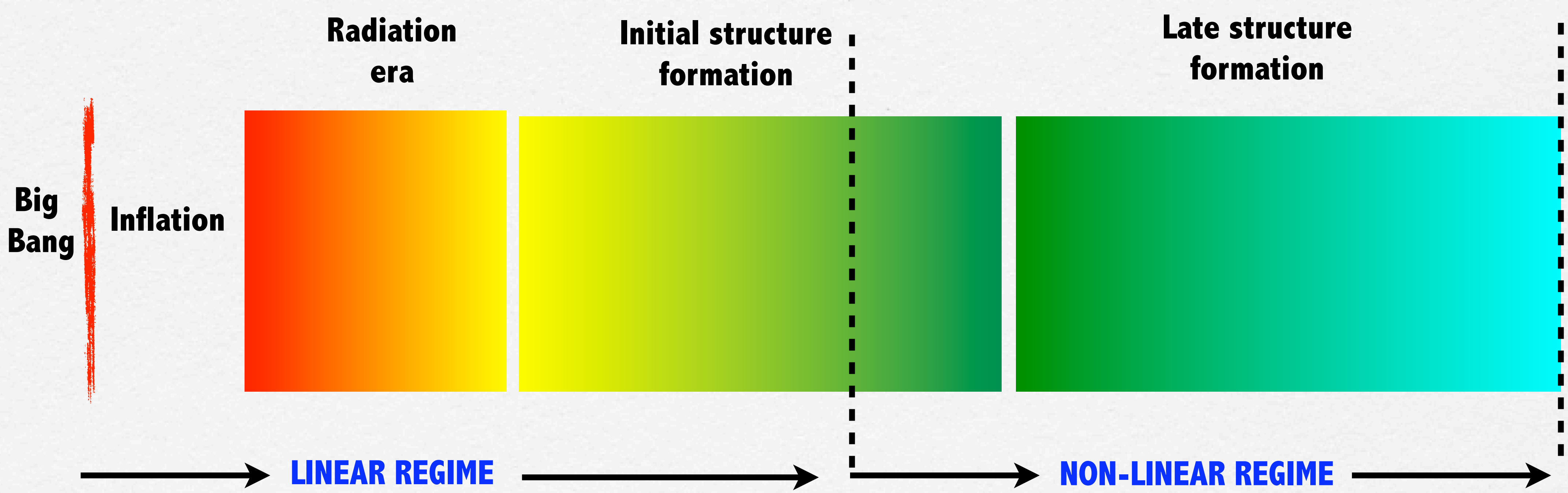
Density increases (collapse) then velocity decreases: $\mathbf{D}(\rho) > 0 \quad \text{then} \quad \mathbf{D}(\mathcal{H}) < 0$

Density decreases (exapansion) then velocity decreases: $\mathbf{D}(\rho) < 0 \quad \text{then} \quad \mathbf{D}(\mathcal{H}) > 0$

But this must be verified, as fluctuations are evaluated along time slices orthogonal to the 4-velocity and thus, they are proportional to spatial gradients:

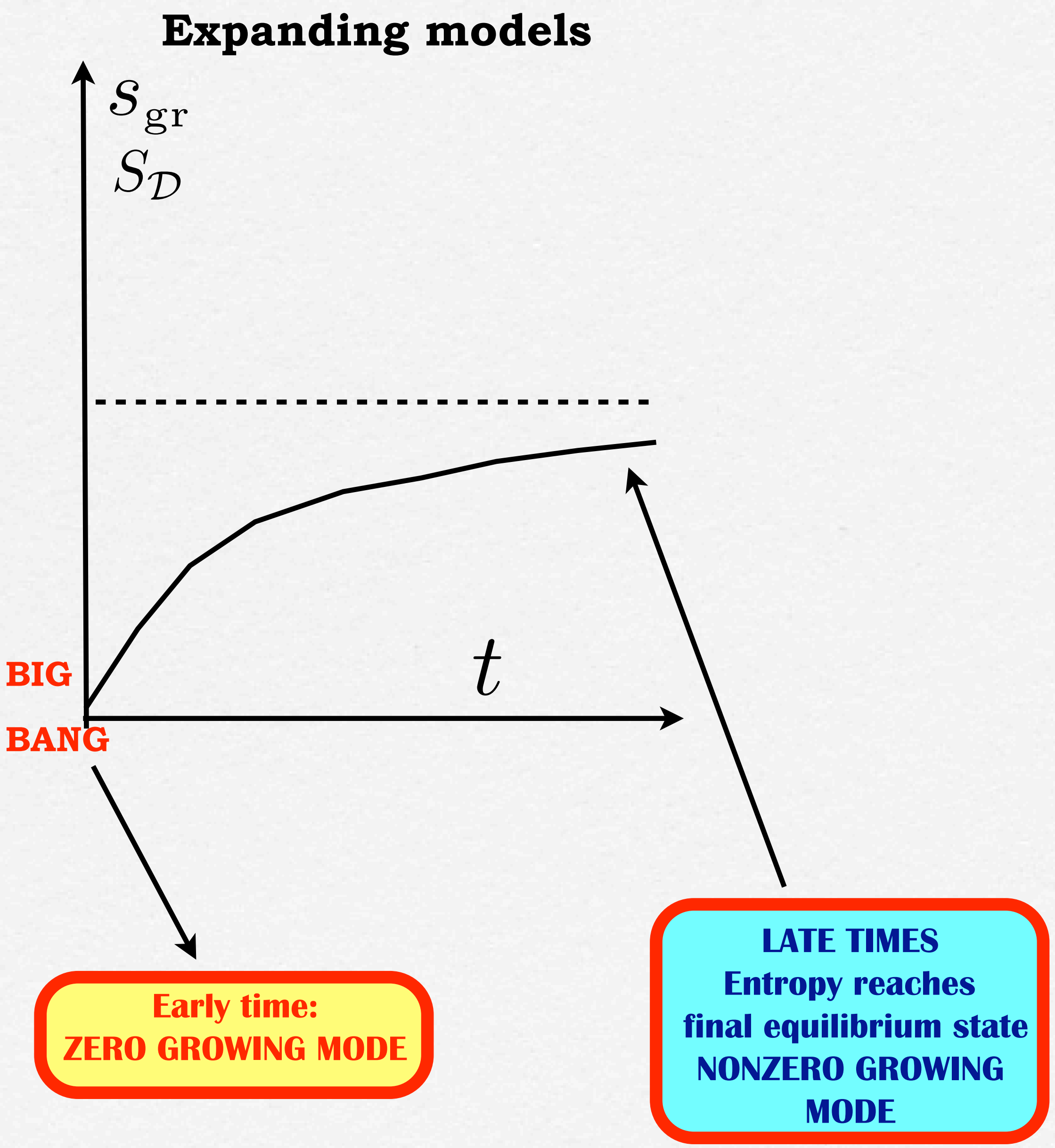
$$\mathbf{D}(\rho) = \frac{\langle\rho\rangle_q'}{3\langle\rho\rangle_q\Gamma} = \frac{1}{a^3\Gamma}\int \rho' a^3\Gamma d\bar{r}, \qquad \mathbf{D}(\mathcal{H}) = \frac{\langle\mathcal{H}\rangle_q'}{3\langle\mathcal{H}\rangle_q\Gamma} = \frac{1}{a^3\Gamma}\int \mathcal{H}' a^3\Gamma d\bar{r},$$

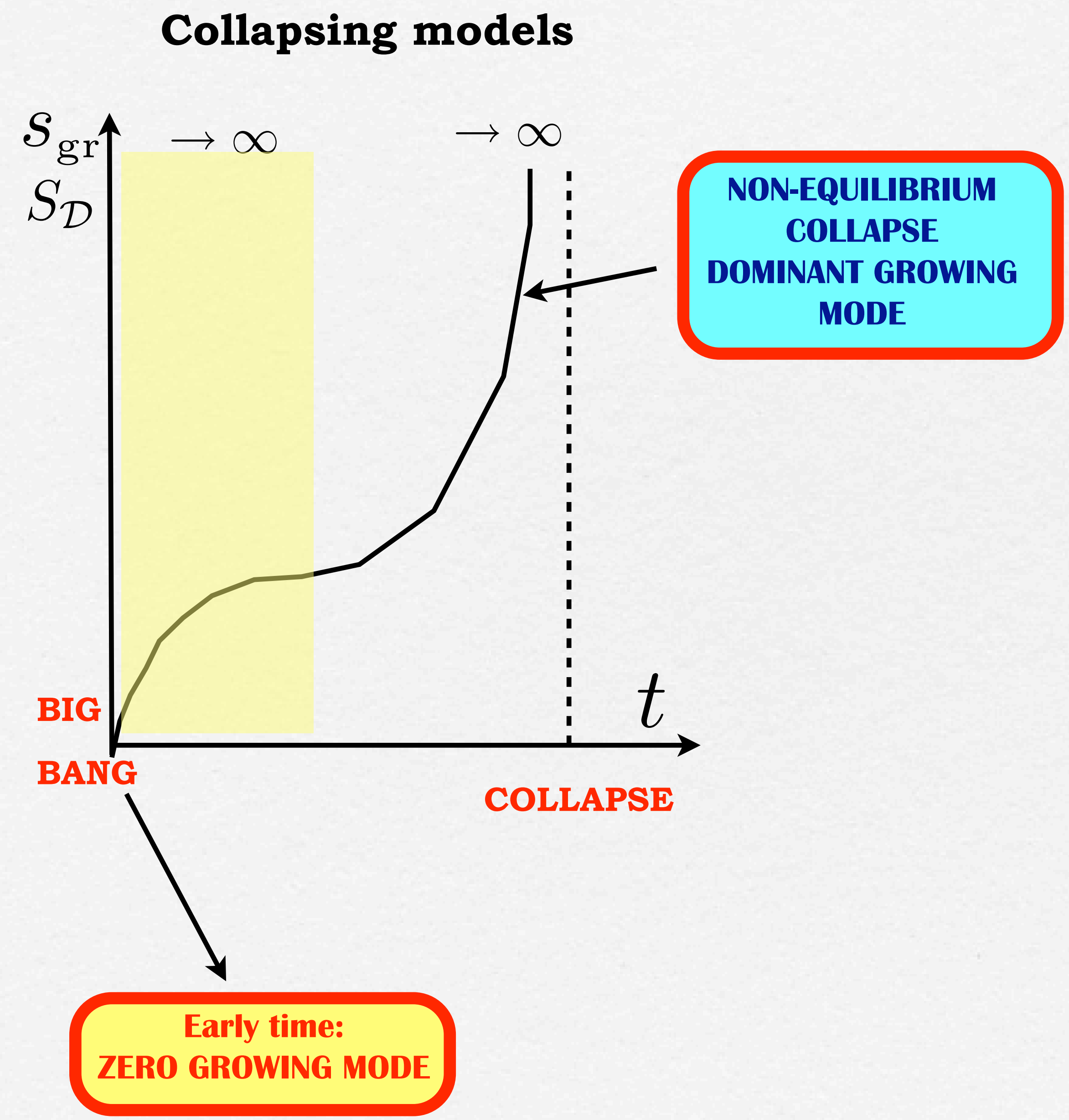
Qualitative time evolution = Gravitational entropy growth



Post inflationary conditions: Near homogeneous early Universe

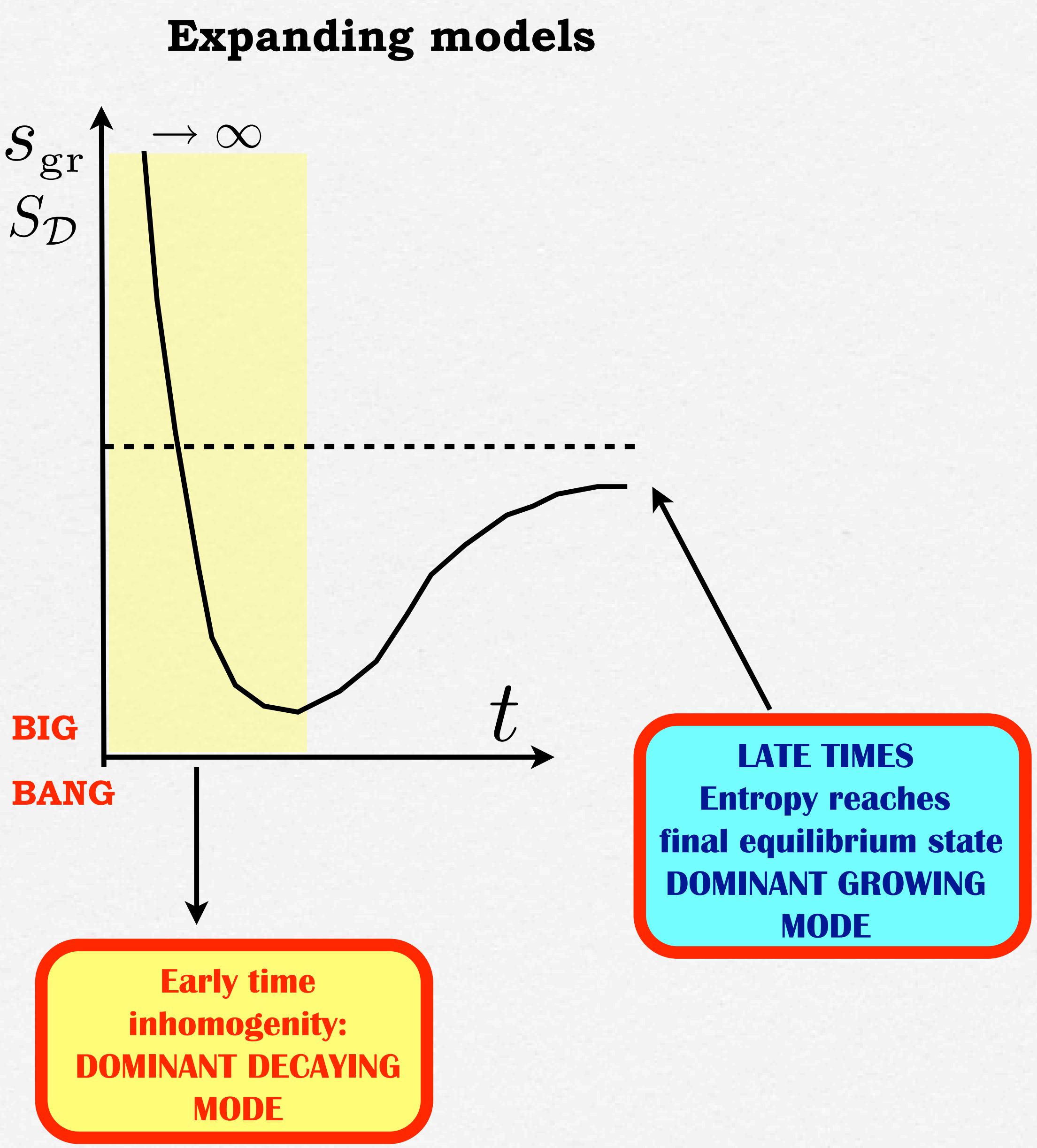
Isotropic BIG BANG: => ~Zero decaying mode

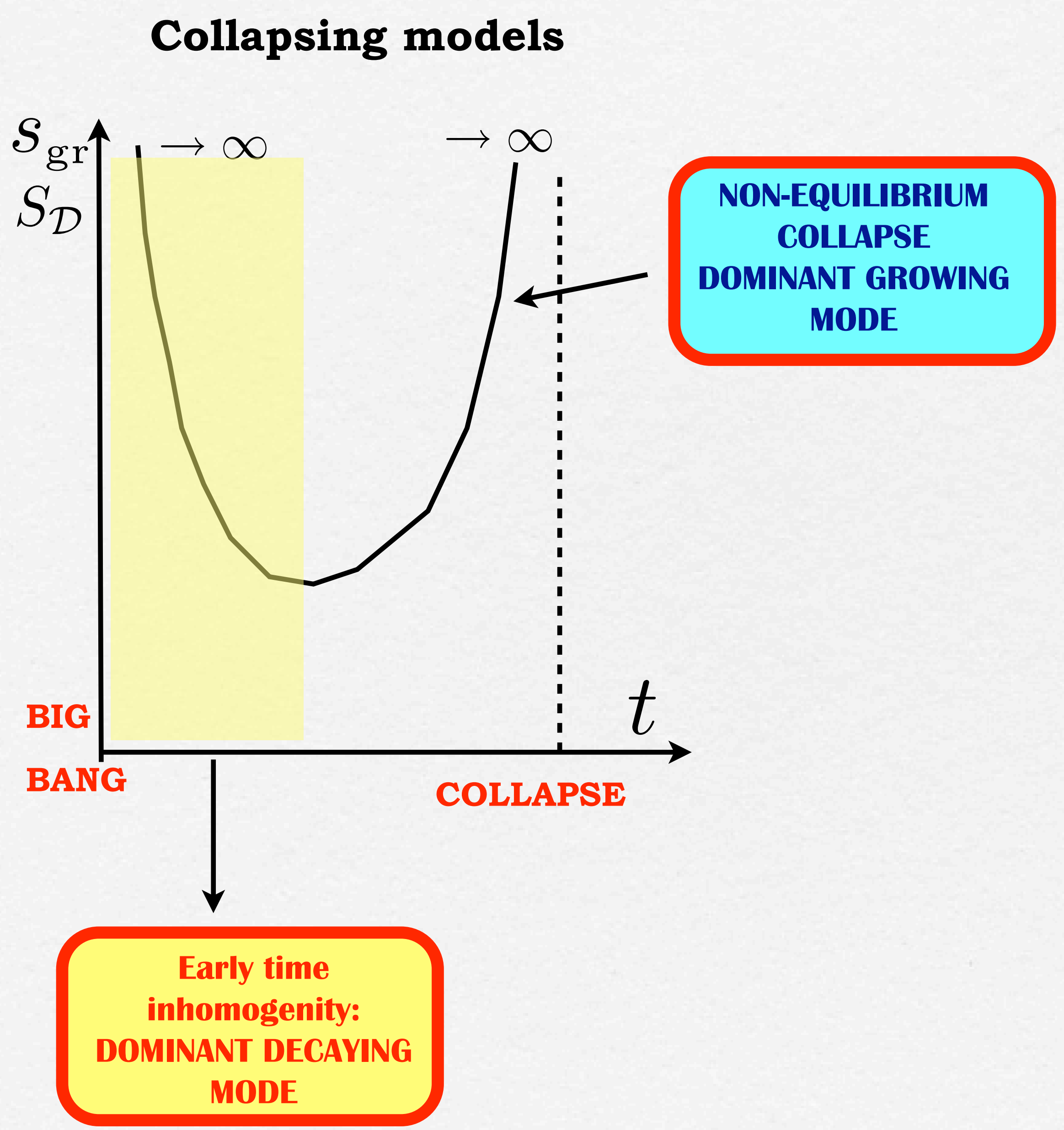




Inhomogeneous early Universe NOT favored.

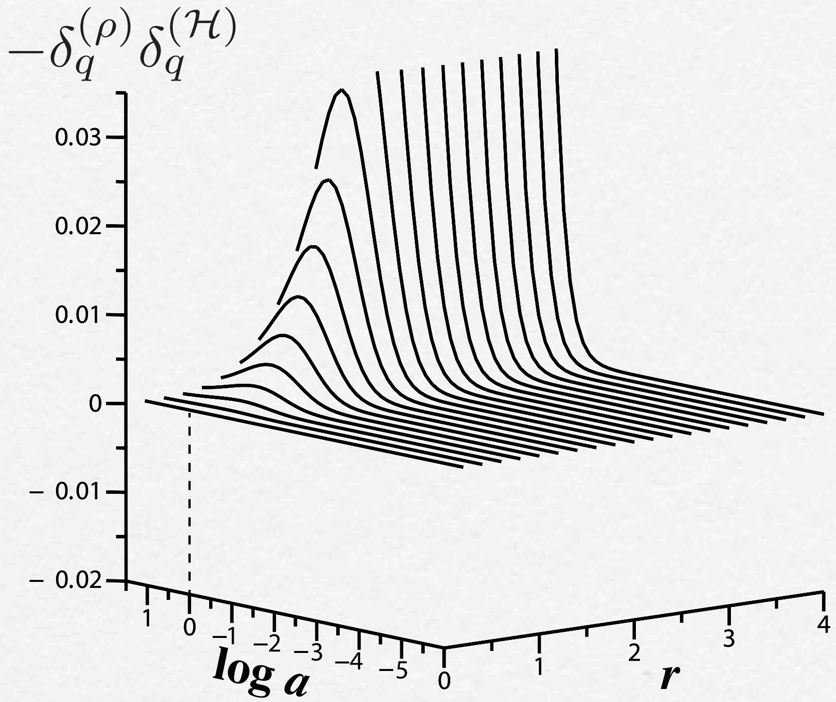
Anisotropic BIG BANG => Large decaying modes





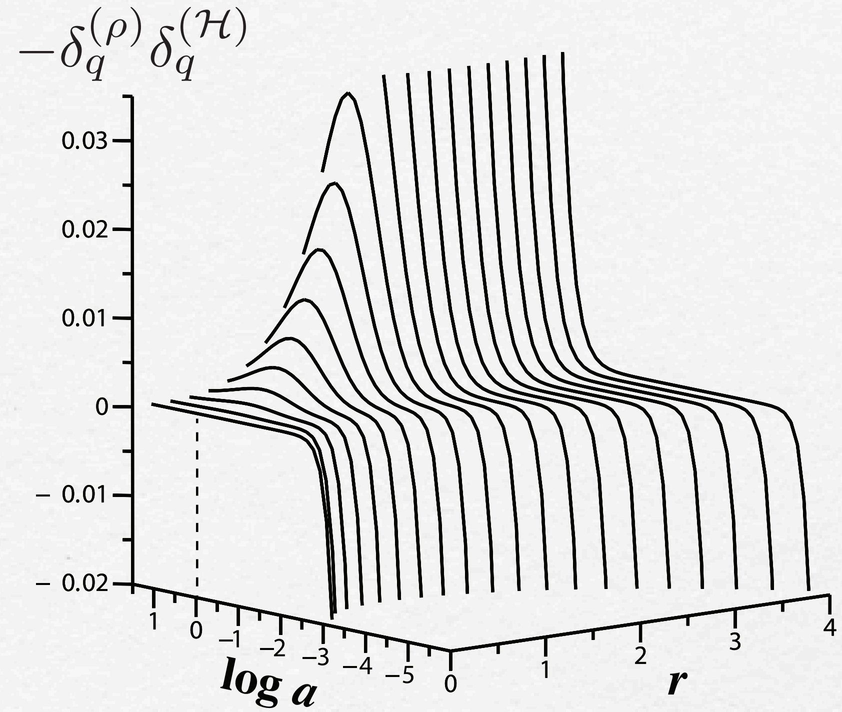
Numerical example

(a)



Cosmic Void with ZERO decaying modes

(b)



Cosmic Void with NONZERO decaying modes

Decaying modes ⇒ entropy decreasing ONLY in early times

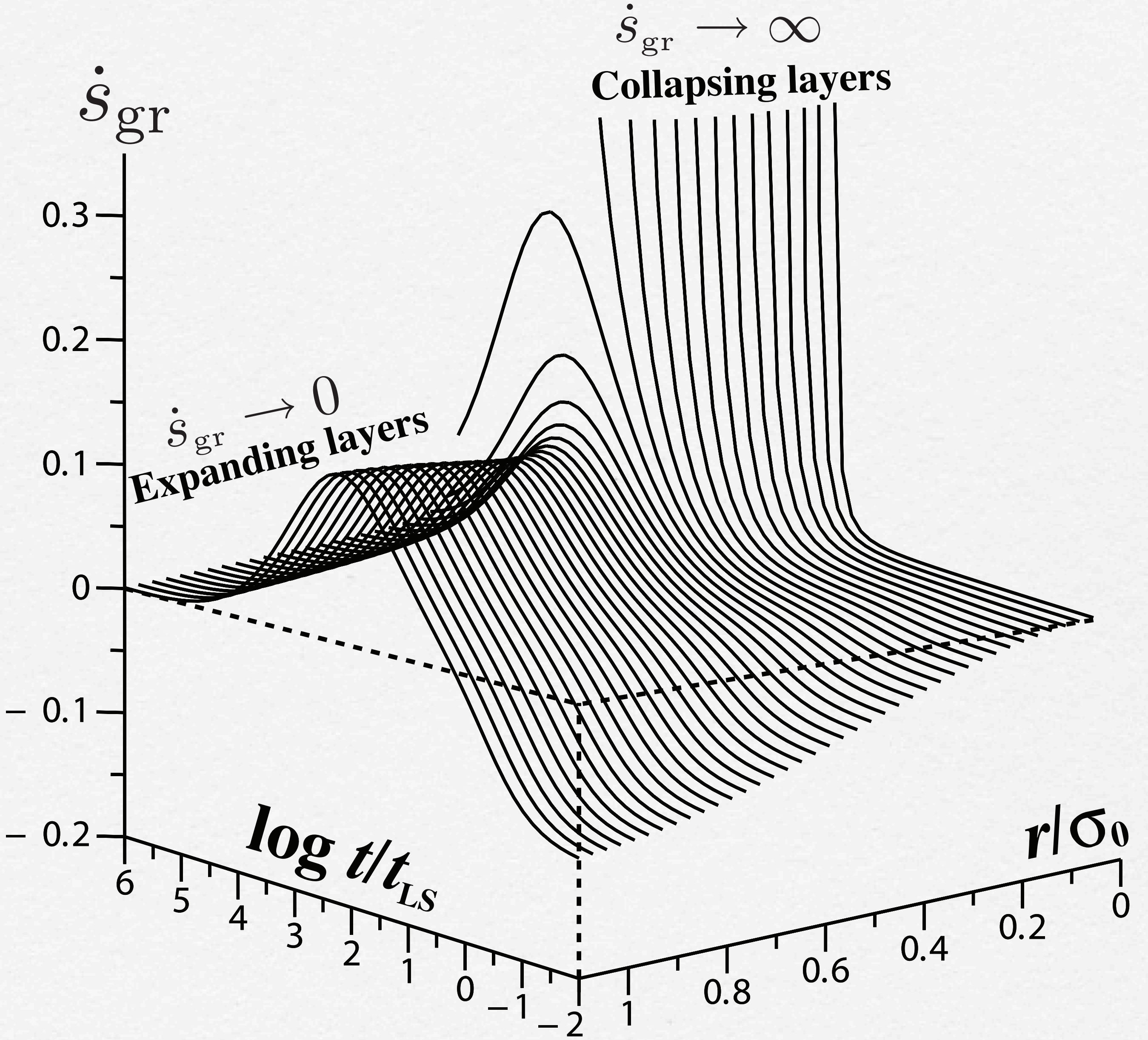
Tuesday, August 5, 2014

Gravitational entropy is positive at early times only if the Big Bang is FULLY isotropic

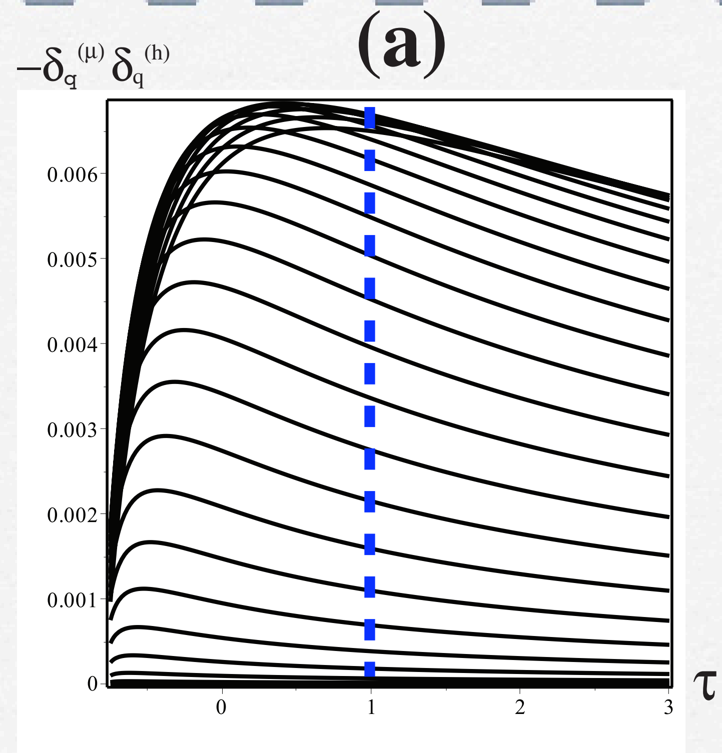
If we postulate that Gravitational Entropy must always increase, then the Universe had to be INITIALLY homogeneous and isotropic

Implications on the need for inflation?

Numerical example: collapse

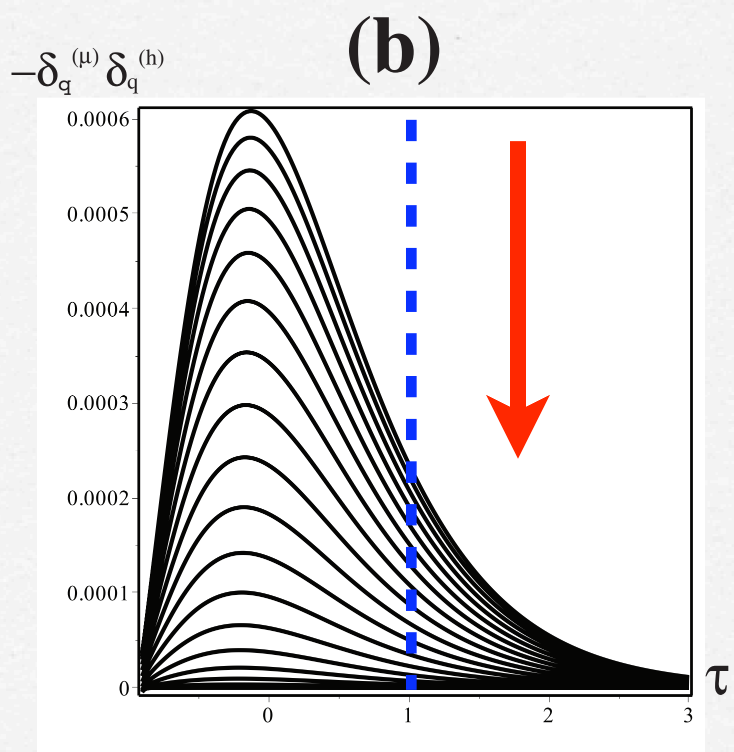


Suppression effect of the Cosmological Constant



Cosmic Void with Lambda = 0

Connection with suppression of the growth of structure
⇒ Observational test of Gravitational Entropy



Cosmic Void with Lambda > 0

Published articles

- Gravitational entropy in generic LTB models, R A Sussman and J Larena, *Classical and Quantum Gravity*, **31**, 075012 (2014)
- Gravitational entropy of cosmic expansion, R. A. Sussman, *Astronomical Notes*,**335**, 592, (2014)
- Suppression of gravitational entropy from a cosmological constant, R. A. and J. Larena, *in preparation*

STILL TO DO:

- consider more general sources and geometries,
- observational predictions,
- test the concept on modified gravity theories.
- explore connections with black hole entropy and non-extensive entropies
- quantum connections?

IN PROGRESS

THANKS FOR YOUR ATTENTION