Viability of $f(R)$ gravity: a quick review

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Outline

SUMMARY

Most of issues that I'll be tackling in this talk are not new. Most of them started since the discovery of the accelerated expansion of the Universe in 1998. Here I'll review one of the most popular alternatives to explain this phenomena and which consist in modifying gravity in the most simple way, without introducing new fields and while respecting most of the basic tenets of Einstein's GR. This alternative is termed $f(R)$ gravity, a particular case being the paradigmatic $f_{GR}(R) = R - 2\Lambda$, (i.e. Einstein's theory + the *infamous* cosmological constant). Suitable modifications of $f_{GR}(R)$ without the Λ term may produce an adequate accelerated expansion with a "dark-energy equation of state" $\omega \approx -1$, but which varies in cosmic time; an interesting possibility that can be tested in a near future. Nevertheless, modifying one of the most successful theories in physics comes with a high price: many of the usual GR predictions might be spoiled (the field equations are different), thus, for every specific proposal $f(R) \neq f_{GR}(R)$ all the gravitational test must be repeated. Here I'll try to summarize until what extent this alternative theories can be viable in several astrophysical and cosmological scenarios.

Introduction

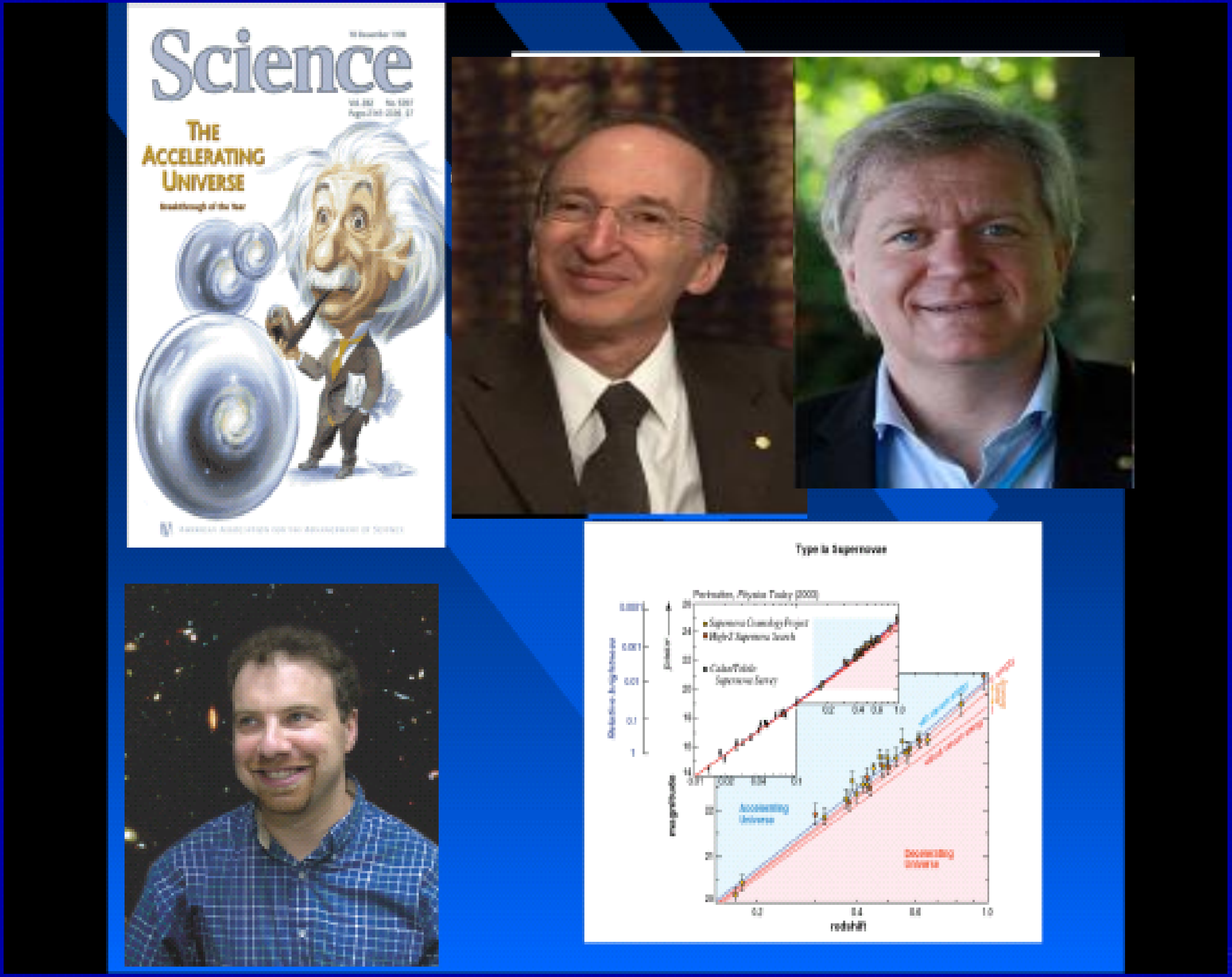
– ΛDM paradigm within GR: the simplest and perhaps most successful cosmological model.

– Alternative Theories of Gravity: try to "replace" Dark (matter-energy) components. This is just one among several possibilities (e.g. inhomogeneous models within GR). More complicated, but it's a worth exploring possibility (I skip the heuristic and philosophyical arguments about the "problems" of Λ. But if you want a thorough and recent review on the subject see: E. Bianchi & C .Rovelli arXiv:1002.3966 and J. Martin: arXiv:1205.3365.)

– $f(R)$ metric theories of gravity: a possible explanation for the accelerated expansion of the Universe as opposed to the Cosmological Constant. (As far as we know, DM must be considered, otherwise it seems impossible to recover the rest of cosmological observations.). These alternative theories of gravity (like others) allows for an "EOS of (geometric) dark energy" that varies in cosmic time, unlike Λ.

SNIa DATA

$$D_L(z) = cH_0^{-1}(z+1)\int_0^z \frac{dz'}{H(z')} \text{ (for } k=0\text{)}, \; \mu = 5\log(D_L/Mpc) + 25 \; .$$



$f(R)$ METRIC GRAVITY

$$S[g_{ab},\psi] = \int \frac{f(R)}{2\kappa}\sqrt{-g}\, d^4x + S_{\rm matt}[g_{ab},\psi] \;, \tag{1}$$

where R =Ricci scalar, $f(R)$ an *a priori* arbitrary function of R, $\kappa := 8\pi G_0$, and ψ represents the matter fields – ordinary and DM – (here $c = 1$).

Varying the action Eq. (1) with respect to the metric yields

$$f_R R_{ab} - \frac{1}{2} f g_{ab} - (\nabla_a \nabla_b - g_{ab}\Box)\, f_R = \kappa T_{ab} \;, \tag{2}$$

where $f_R := \partial_R f$, $\Box = g^{ab}\nabla_a\nabla_b$, and T_{ab} is the EMT of matter. (ordinary matter, and possibly DM as well)

Exercise: Take ∇^a on both sides of Eq. (2) and prove that the EMT of matter is conserved:

$$\nabla^a T_{ab} = 0 \;. \tag{3}$$

When expanding the derivative ∇ acting on f_R and taking the trace we obtain:

$$\Box R = \frac{1}{3f_{RR}}\left[\kappa T - 3f_{RRR}(\nabla R)^2 + 2f - Rf_R\right] , \tag{4}$$

where $(\nabla R)^2 := g^{ab}(\nabla_a R)(\nabla_b R)$, and $T := T^a{}_a$. Using this equation in the field equation to replace $\Box R$ we find

$$G_{ab} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\Big(Rf_R + f + 2\kappa T\Big) + \kappa T_{ab}\right] . \tag{5}$$

We shall be dealing with equations (4) and (5), and treat the theory as a system of second order coupled PDE for the Ricci scalar R and the metric g_{ab}, respectively.

Exercise: take $f(R) = R - 2\Lambda$ and show that Eqs. (5) and (4) reduce respectively to the GR+Λ theory:

$$G_{ab} + g_{ab}\Lambda = \kappa T_{ab} , \tag{6}$$

$$R = 4\Lambda - \kappa T . \tag{7}$$

GR vs. Alternative Theories of Gravity

The basic axioms of GR are kept in $f(R)$ gravity:

1. The spacetime is a 4-dimensional differential manifold endowed with a Lorentzian metric (M, g_{ab}).
2. Gravitation is described geometrically in terms of the Riemann tensor $R_{abcd} \neq 0$ ($R_{abcd} = 0$ only when the spacetieme is globally flat).
3. The theory should be covariant (diffeomorphism invariant).
4. The equivalence principle holds: test particles move on geodesics of the metric g_{ab}. The laws of physics (those compatible with special relativity) are still valid locally.
5. The only quantity pertaining to spacetime that should appear in the laws of physics is the metric (*minimal coupling*).
6. Assume the usual Levi-Civita connection (no torsion and the theory is metric compatible $\nabla_a g_{ab} = 0$).
7. The field equations should be linear in the second derivatives (quasilinear PDE).

$f(R)$ theories keep all this axioms except "7": only fulfilled when $f(R) = R - 2\Lambda$.

$f(R)$ GRAVITY (BRIEF HISTORICAL REMARKS)

– Non-linear Lagragians in R, R_{ab}, and R_{abcd} date back since the years that followed GR (H. Weyl, 1921; K. Lanczos, 1938; Buchdahl 1970). They were analyzed much later in different contexts. For instance, in cosmology ...

– 1979 (A. Starobinsky), as models for *inflation*.

– 1982 (R. Kerner) as a "cosmological model without singularity". Remark: Several $f(R)$ models considered today are very similar to those considered in this paper.

– 1986 (J.P. Durisseau & R. Kerner) as a "reconstruction of inflationary model".

– As mentioned before, the discovery of the accelerated expansion of the Universe renewed the interest in this kind of models. The first ones proposed within the specific goal of producing an accelerated expansion were: Cappozziello (2002), Cappozziello et al. (2003), Carroll et al. (2004,2005).

– Since 2003 a boom of papers analyzing $f(R)$ gravity in all possible scenarios have appeared in the literature: perhaps more than 1000 papers ! ($\approx 2/week$).

$f(R)$ gravity can mimic Λ

Notice that in vacuum, $R = R_1 = const.$ is a solution of

$$\Box R = \frac{1}{3f_{RR}}\left[\kappa T^{=0} - 3f_{RRR}(\nabla R)^2 + 2f - Rf_R\right] , \quad (8)$$

provided R is a root of $V'(R) = 2f - Rf_R$ (i.e. $2f(R_1) = R_1 f_R(R_1)$), assuming $f_{RR}(R_1) \neq 0$. That is, R_1 is a critical point (e.g. maximum or minimum) of the "potential" $V(R)$. In such an instance, the field equation

$$G_{ab} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\left(Rf_R + f + 2\kappa T^{=0}\right) + \kappa T_{ab}^{=0}\right] . \quad (9)$$

reduces to

$$G_{ab} = -g_{ab}\frac{R_1}{4} \qquad \text{(in vacuum)} . \quad (10)$$

$f(R)$ theory behaves like GR with an *effective cosmological constant* $\Lambda_{eff} = R_1/4$!
The fact that the theory can admit this solution for R allows one to find non-trivial solutions that asymptotically (past,future, or spatial infinity) match a De Sitter solution, which in turn can explain several cosmological observations.

FRW COSMOLOGY

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \;, \qquad (11)$$

where $k = \pm 1, 0$. When obtaining numerical solutions we shall focus only on the "flat" case $k = 0$.

$$H^2 + \frac{k}{a^2} + \frac{1}{f_R}\left[f_{RR}H\dot{R} - \frac{1}{6}(f_R R - f)\right] = \frac{\kappa\rho}{3f_R} \;, \qquad (12)$$

$$\ddot{a}/a = \dot{H} + H^2 = \underbrace{\frac{1}{f_R}\left(f_{RR}H\dot{R} + \frac{f}{6} - \frac{\kappa\rho}{3}\right)}_{R_1/12=\Lambda_{\rm eff}/3 \;\; when \;\; \rho\to 0 \;\; and \;\; if \;\; R\to R_1} \;, \qquad (13)$$

where $H = \dot{a}/a$ is the Hubble expansion. From Eq. (4) we find

$$\ddot{R} = -3H\dot{R} - \frac{1}{3f_{RR}}\left[3f_{RRR}(\dot{R})^2 + \underbrace{2f - f_R R}_{V'(R)} - \kappa(\rho - 3p)\right] \;. \qquad (14)$$

If during the cosmic evolution R reaches an extrema of the potential $V(R)$ (with vanishing $\dot{R}$), say, at present time where the matter contributions ρ, p are "small" compared to $\rho_{\rm crit}$ then $R \approx R_1$ with $V'(R_1) = 0$, and so $\dot{H} + H^2 = \ddot{a}/a \sim R_1/12 = \Lambda_{\rm eff}/3 > 0$ if $\Lambda_{\rm eff} > 0$. Thus $\ddot{a} > 0 \rightarrow$ Accelerated expansion !! This what happens precisely when solving the full equations numerically

Now, the expresion for the Ricci scalar is given by

$$R = 6\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right) \quad . \tag{15}$$

Note that by using Eqs. (12) and (13) in Eq. (15) we obtain an identity $R \equiv R$, which shows the consistency of the equations (c.f. the SSS case) !

T_{ab} of matter is a mixture of three kinds of perfect fluids: baryons, radiation and dark matter, in a epoch where they don't interact with each other except gravitationally. Then for each matter component the EMT conserves separately and $\nabla_a T_i^{ab} = 0$ ($i = 1, 3 \to$ baryons, radiation, DM) leads to

$$\dot{\rho}_i = -3H(\rho_i + p_i) \quad . \tag{16}$$

The total energy-density is $\rho = \sum_i \rho_i = -T^t_{\ t}$ and since $p_{\rm bar,DM} = 0$, and $p_{\rm rad} = \rho_{\rm rad}/3$ then $T = T_{\rm bar} + T_{\rm DM} = -(\rho_{\rm bar} + \rho_{\rm DM})$. Then Eq. (16) integrates

$$\rho = \frac{\rho^0_{\rm bar} + \rho^0_{\rm DM}}{(a/a_0)^3} + \frac{\rho^0_{\rm rad}}{(a/a_0)^4} \quad , \tag{17}$$

where the knotted densities are the densities today. Here $a_0 = a(t_0)$, t_0 is present cosmic time. The differential equations will depend explicitly on $a(t)$ via the matter terms.

Equation of State (EOS) of GDE (1st part)

In the ΛCDM model the equation of state $\omega_\Lambda = p_\Lambda/\rho_\Lambda = -1$. We shall define an EOS for the modified gravity contribution given by (for $f(R) \neq R$)

$$\omega_X = \frac{p_X}{\rho_X} \ , \tag{18}$$

where ρ_X is defined from the modified Friedmann equation, so that it reads $H^2 = \frac{\kappa}{3}\left(\rho + \rho_X\right) = \frac{\kappa \rho_{\rm tot}}{3}$, which leads to

$$\rho_X = \frac{1}{\kappa f_R}\left\{\frac{1}{2}\left(f_R R - f\right) - 3 f_{RR} H \dot{R} + \kappa\rho\left(1 - f_R\right)\right\} \ , \tag{19}$$

In a similar way we define ρ_X, so that the dynamic equation for H reads

$$\dot{H} + H^2 = -\frac{\kappa}{6}\left\{\rho + \rho_X + 3\left(p_{\rm rad} + p_X\right)\right\} = -\frac{\kappa \rho_{\rm tot}}{6}\left\{1 + 3\omega_{\rm tot}\right\} \ , \tag{20}$$

where $\omega_{\rm tot} = p_{\rm tot}/\rho_{\rm tot}$. From this latter, we obtain

$$p_X = -\frac{1}{3\kappa f_R}\left[\frac{1}{2}\left(f_R R + f\right) + 3 f_{RR} H \dot{R} - \kappa\left(\rho - 3 p_{\rm rad} f_R\right)\right] \tag{21}$$

Which $f(R)$?

Among the infinite a priori possible choices of $f(R)$ (restricted by $f_R > 0$ so as to $G_{\rm eff} = G_0/f_R > 0$ and $f_{RR} > 0$, stable perturbations around a background), how to choose ?

– Simplicity $\rightarrow f(R) = R - 2\Lambda$. But we don't want this. We want something with $\omega_X(t)$ such that today $\omega_x \approx -1$.

– Ingeeniring, trial and error, handcraft, reconstruction, ...

– Is there any new physical principle that single out an $f(R)$ different from $f_{GR}(R)$, that match all the tested gravitational observations and yet provide new and "unexpected" predictions ? Ans. Maybe.

Specific $f(R)$ models

Given a specific $f(R)$, we integrate the differential equations forward from past to future with suitable "initial conditions". We have considered three specific $f(R)$ models which have become very popular in the literature

- Miranda *et. al.* model (PRL 102, 221101, 2009)

$$f(R)_{\rm MJW} = R - \beta R_* \ln\left(1 + \frac{R}{R_*}\right) \quad . \tag{22}$$

We used $\beta = 2$ and $R_* = H_0^2$.

- Starobinsky model (JETP Lett. 86, 157 2007)

$$f(R)_{\rm St} = R + \lambda R_S \left[\left(1 + \frac{R^2}{R_S^2}\right)^{-q} - 1\right] \quad . \tag{23}$$

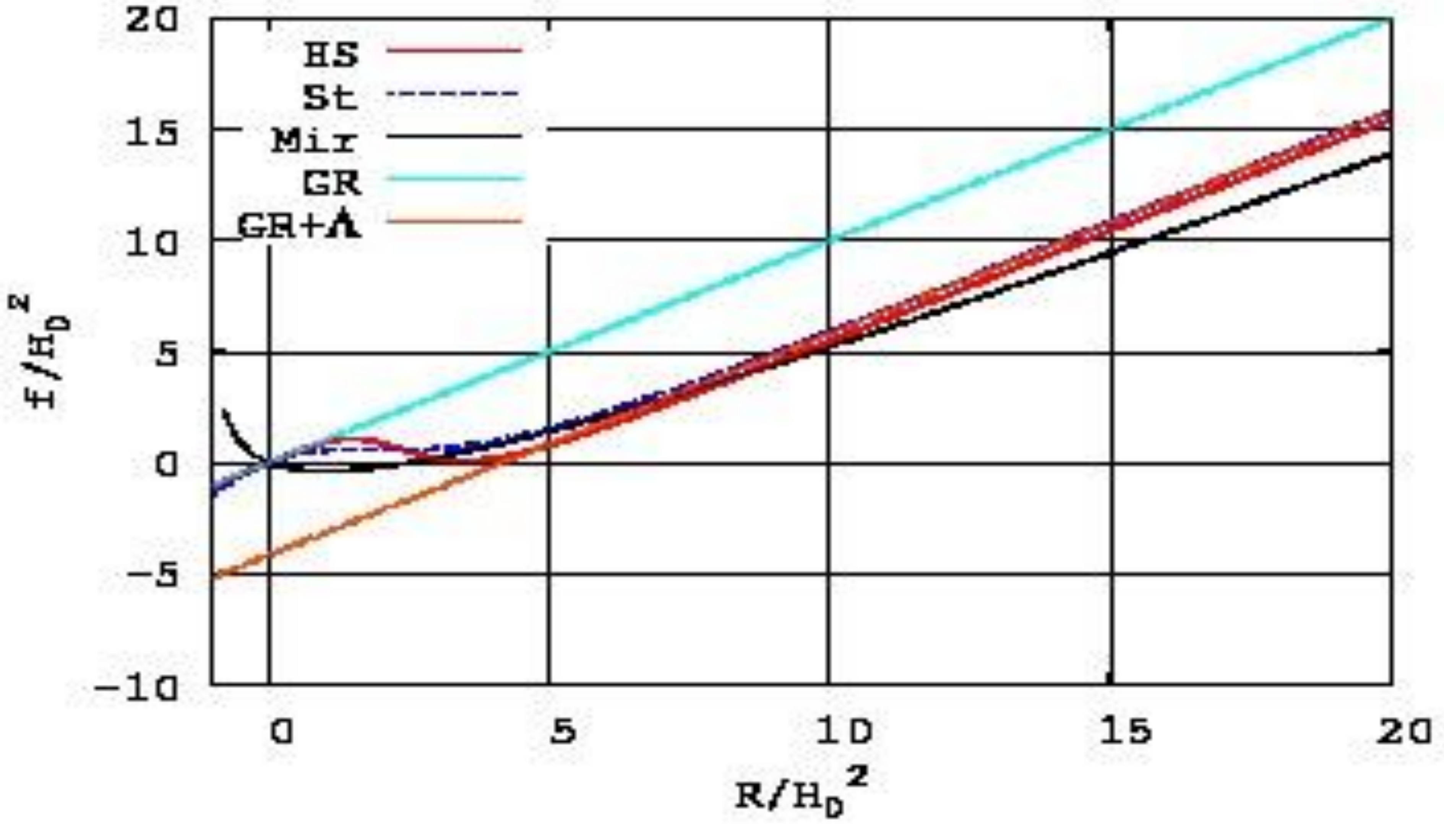
We take $q = 2$ and $\lambda = 1$, $R_S \approx 4.17 H_0^2$.

- Hu & Sawicky model (PRD 76, 064004, 2007)

$$f(R)_{HS} = R - m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1} \quad . \tag{24}$$

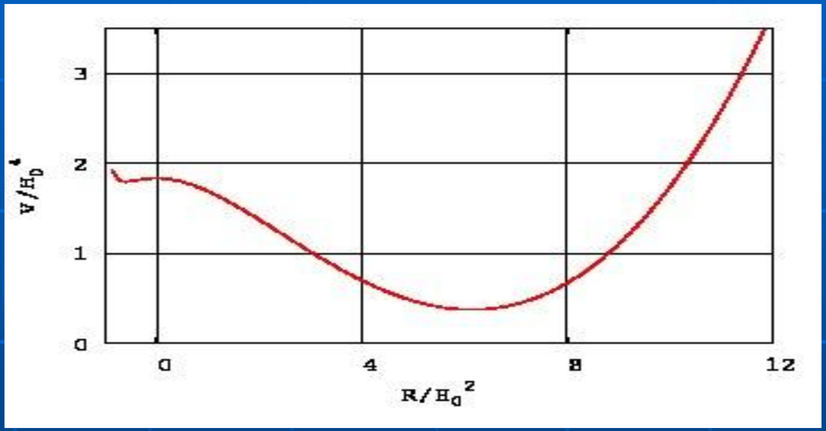
We take $n = 4$, $m^2 \approx 0.24 H_0^2$, $c_1 \approx 1.25 \times 10^{-3}$ and $c_2 \approx 6.56 \times 10^{-5}$.

$f(R)$ Models

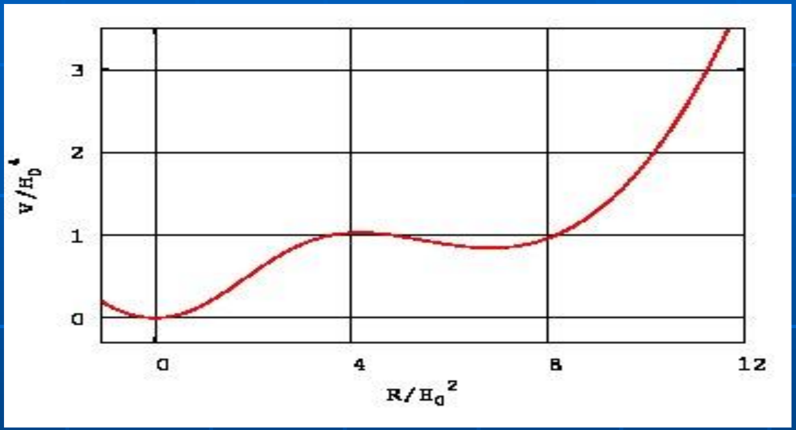


Potentials $V(R) = -Rf(R)/3 + \int^R f(x)dx$ such that $V'(R) = \frac{1}{3}(2f - Rf_R)$. At the extrema of $V(R)$ (notably at the global minimum) the de Sitter "point" is reached where the models behave as a GR plus $\Lambda_{eff} = R_1/4$, where $V'(R_1) = 0$. The specific cosmological models interpolate between a large R (at early time) and near the nontrivial minimum $R \neq 0$ at present time.

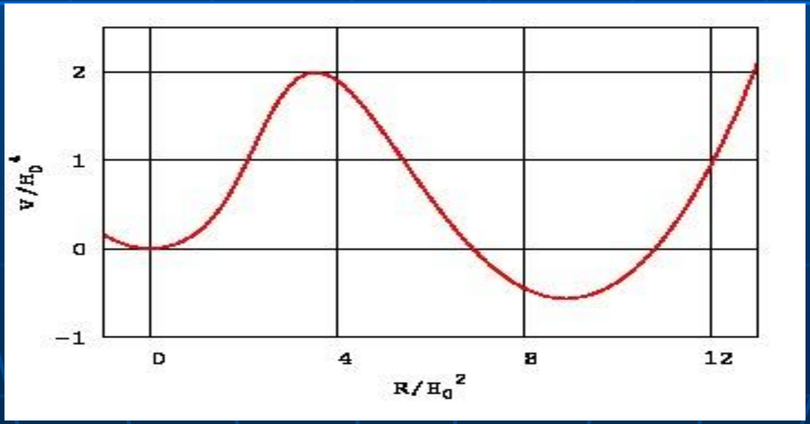
$f(R)_{MJW}$



$f(R)_{St}$



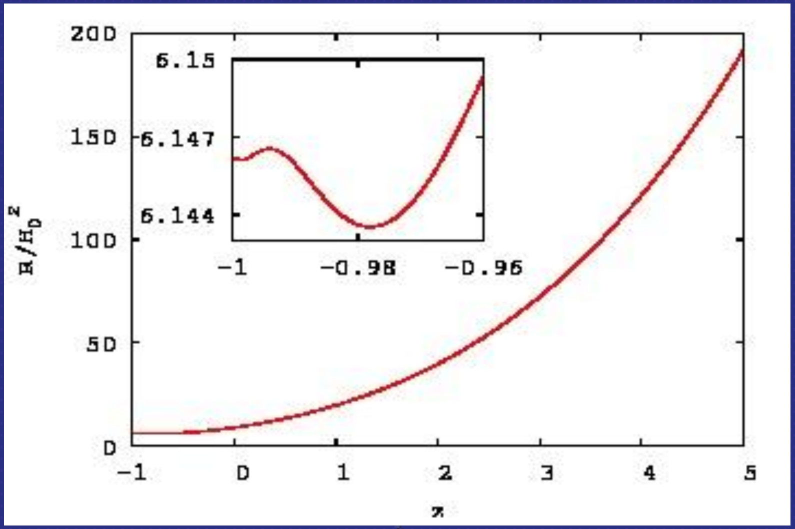
$f(R)_{HS}$



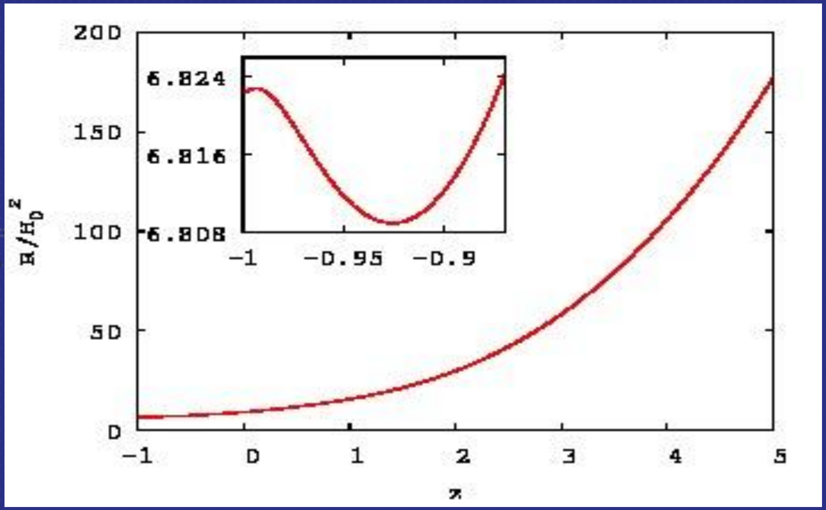
Numerical Results

Plot $\frac{R}{H_0^2}$ vs z, $z = \frac{\omega_e}{\omega_d} - 1 = \frac{\omega(t)}{\omega_0} - 1 = \frac{a_0}{a(t)} - 1$

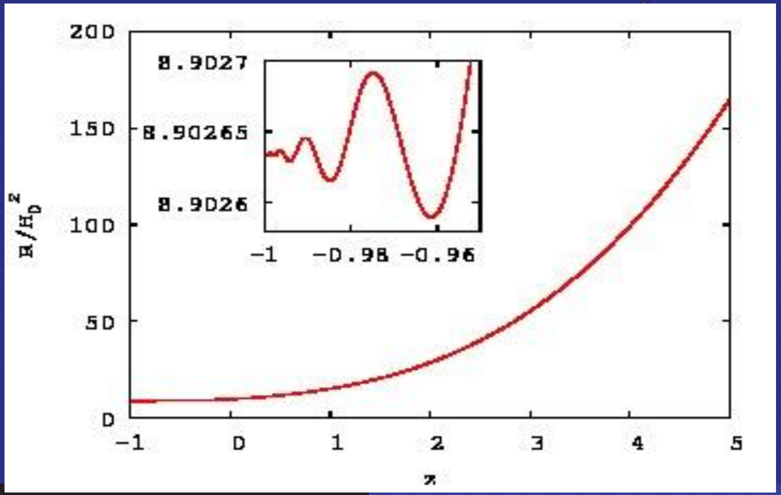
$f(R)_{MJW}$



$f(R)_{St}$

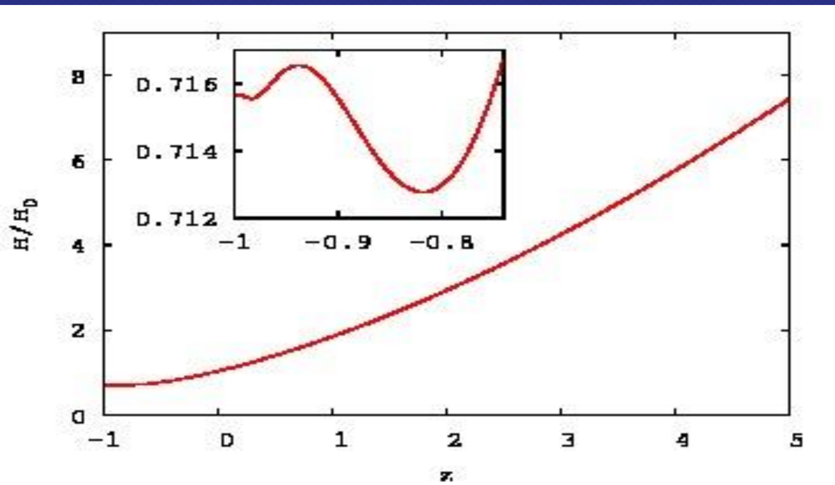


$f(R)_{HS}$

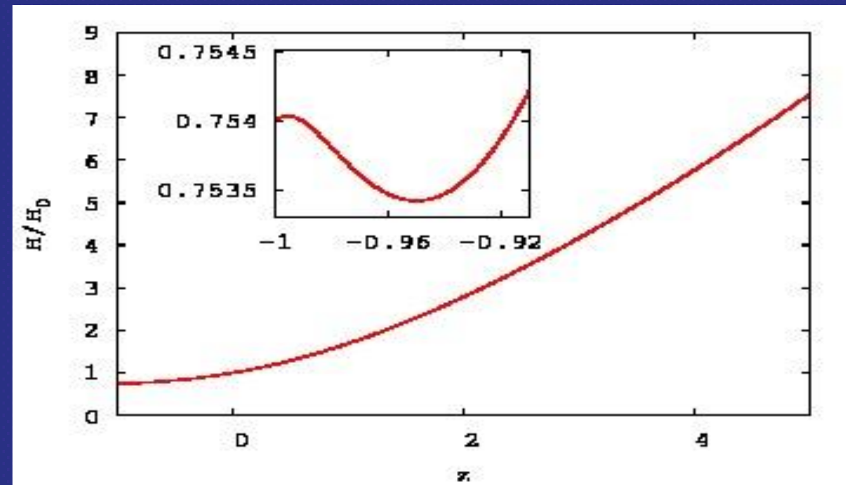


Plot $\frac{H}{H_0}$ vs z

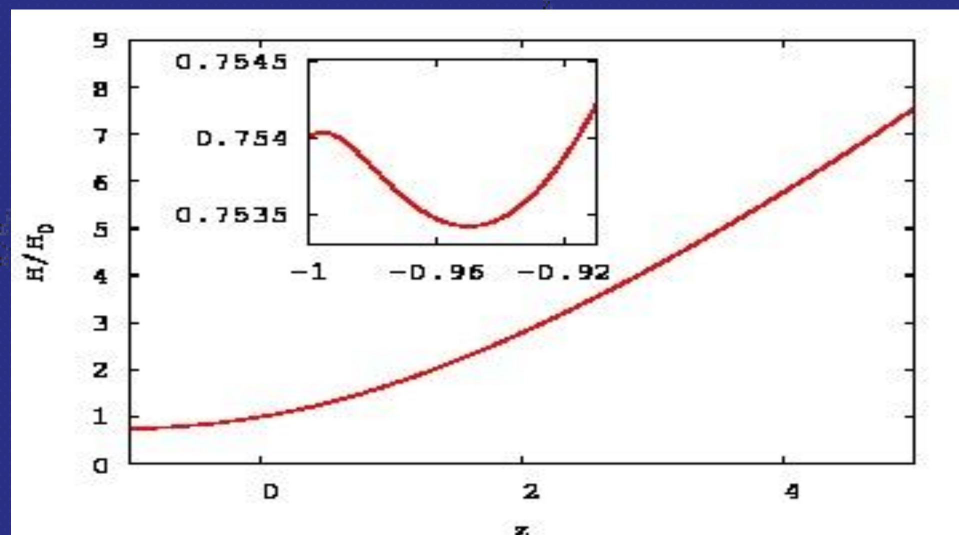
$f(R)_{MJW}$



$f(R)_{St}$



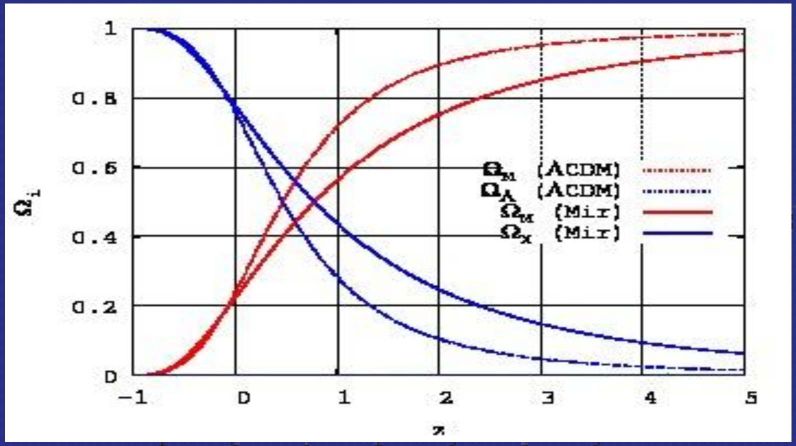
$f(R)_{HS}$



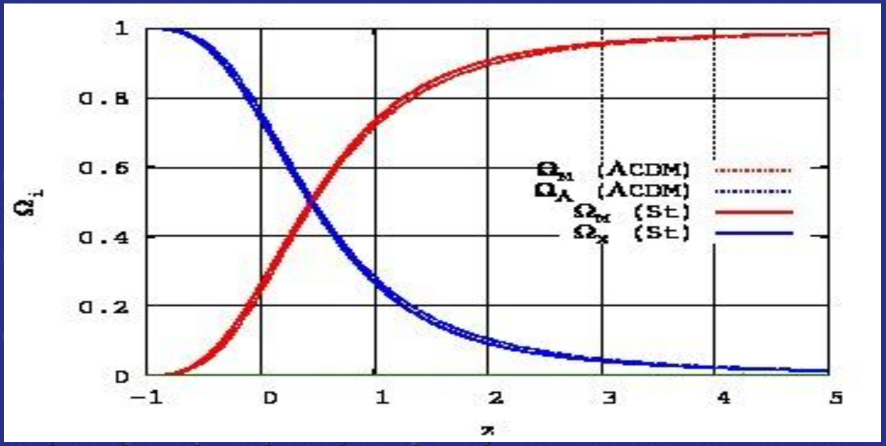
Plot Ω_i vs z

$\Omega_i = \kappa\rho_i/(3H^2)$, $i = \text{matt}, \text{rad}, X$; $\text{matt} = baryons + DM$.

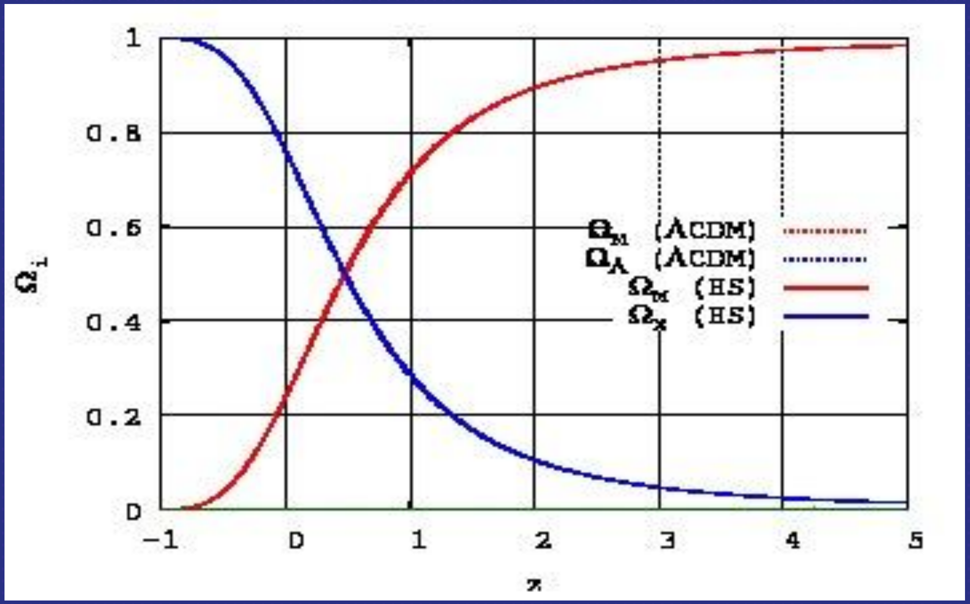
$f(R)_{MJW}$



$f(R)_{St}$

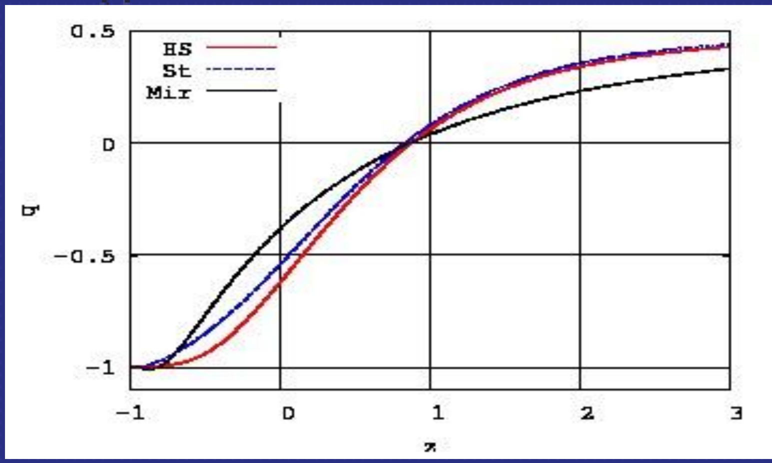


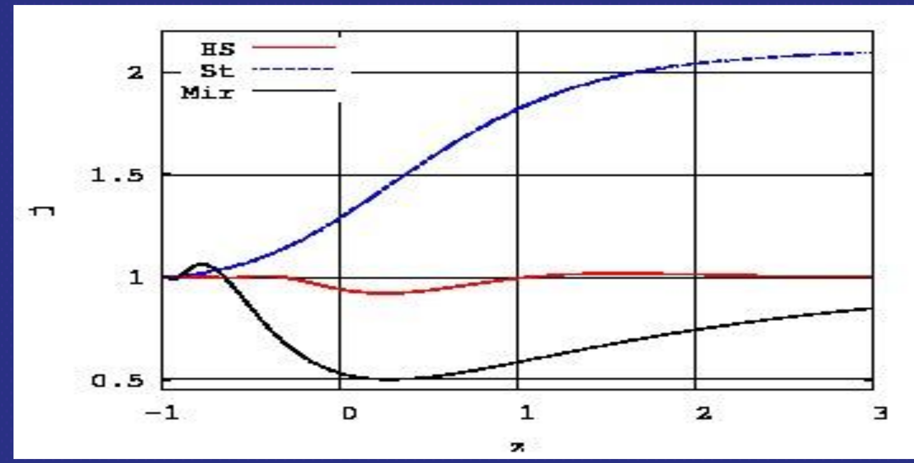
$f(R)_{HS}$



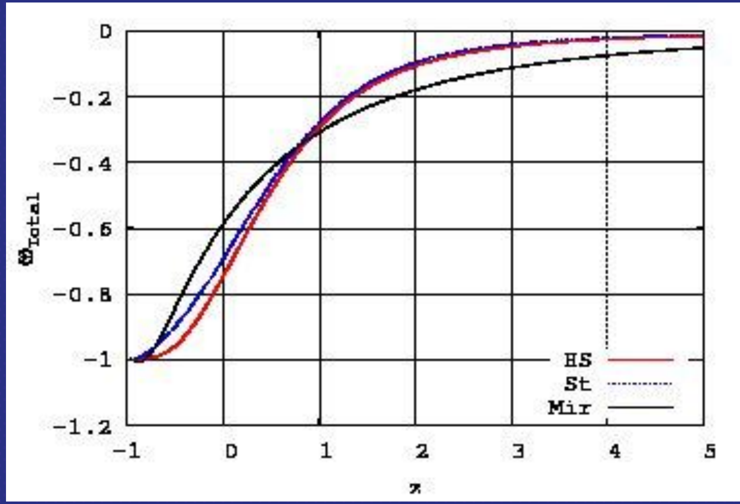
Deceleration parameter: $q := -\frac{\ddot{a}}{aH^2} = -\frac{H^2+\dot{H}}{H^2} = 1 - \frac{R}{6H^2} = \frac{1}{2}\left(1 + 3\omega_{\rm tot}\right)$,

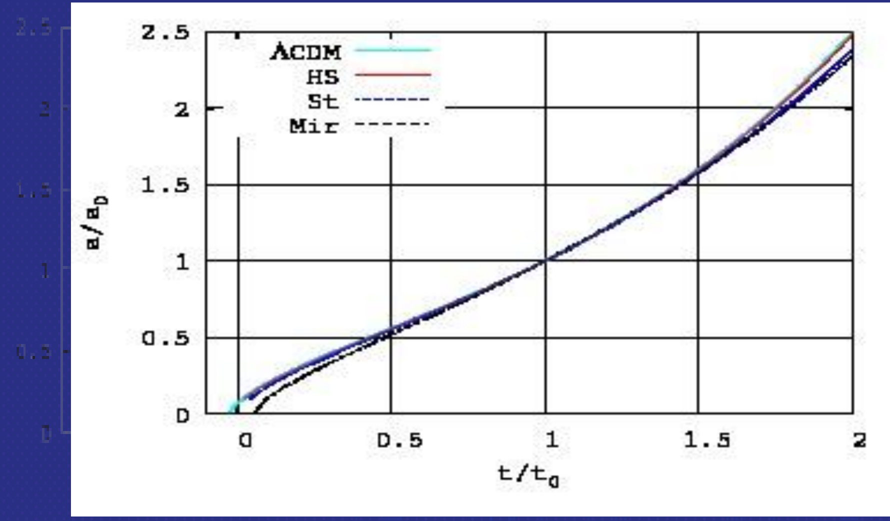
$\omega_{\rm tot} = -\frac{1}{3}\left[\frac{\frac{1}{2}\left(f_R R + f\right) + 3f_{RR}H\dot{R} - \kappa\rho}{\frac{1}{2}\left(f_R R - f\right) - 3f_{RR}H\dot{R} + \kappa\rho}\right]$. Jerk: $j := \frac{\dddot{a}}{aH^3} = \frac{\dot{R}}{6H^3} - \frac{\dot{H}}{H^2} + 1 = \frac{\dot{R}}{6H^3} + q + 2$



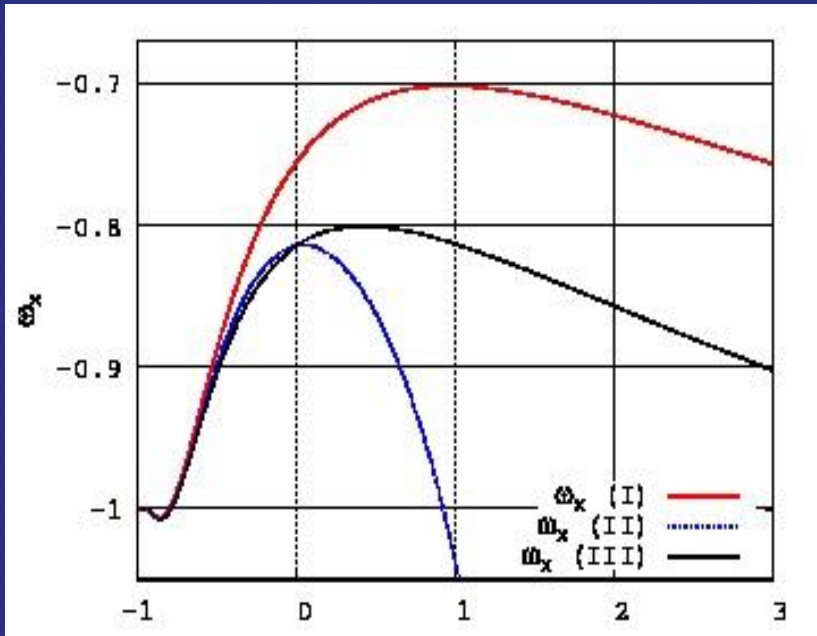


The age of the Universe is $\sim t_0 = H_0^{-1} \approx 9.78h^{-1} \times 10^9\,\mathrm{y} \sim 13.97 \times 10^9\,\mathrm{y}$ (with $h = 0.7$)

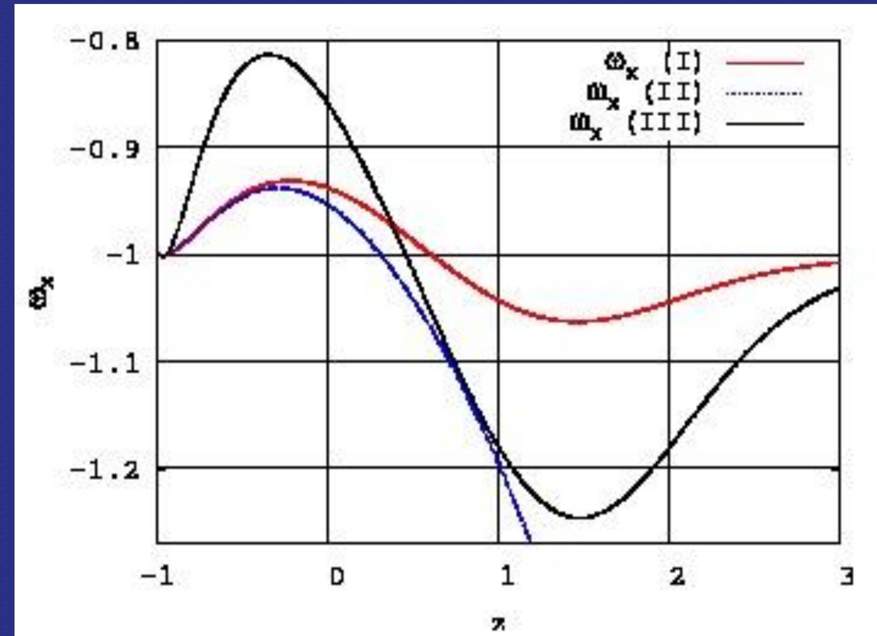




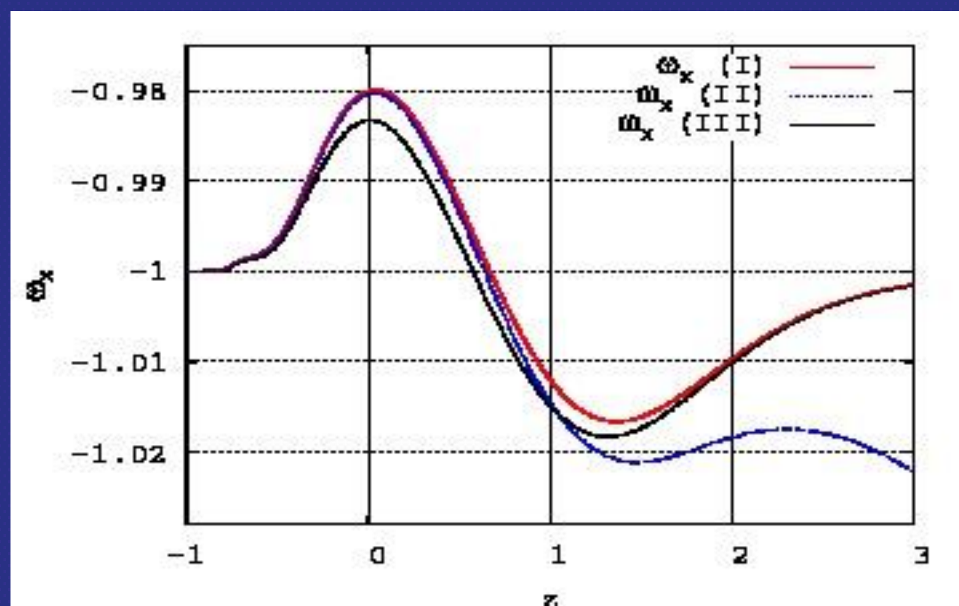
$f(R)_{MJW}$



$f(R)_{St}$



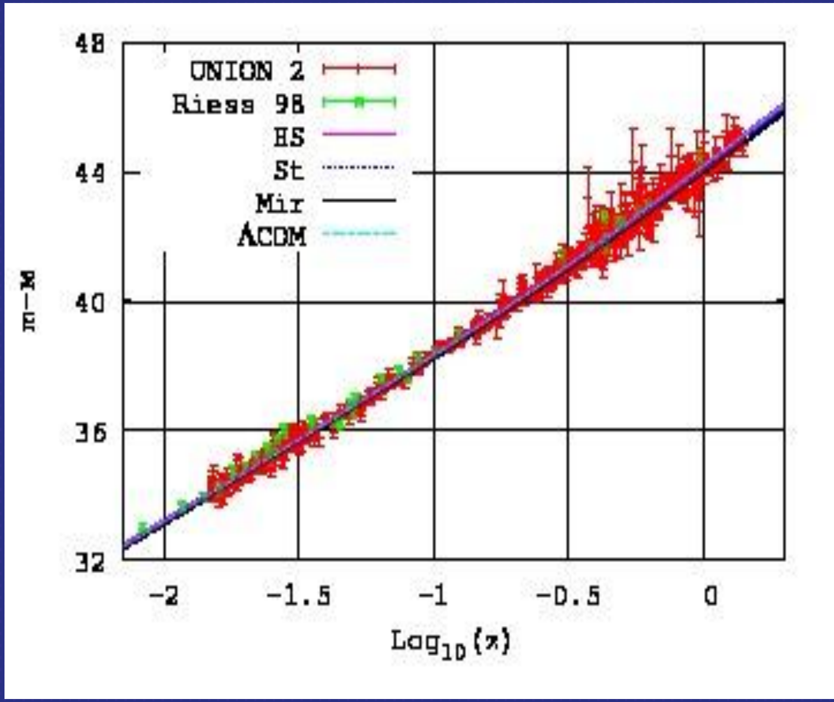
$f(R)_{HS}$

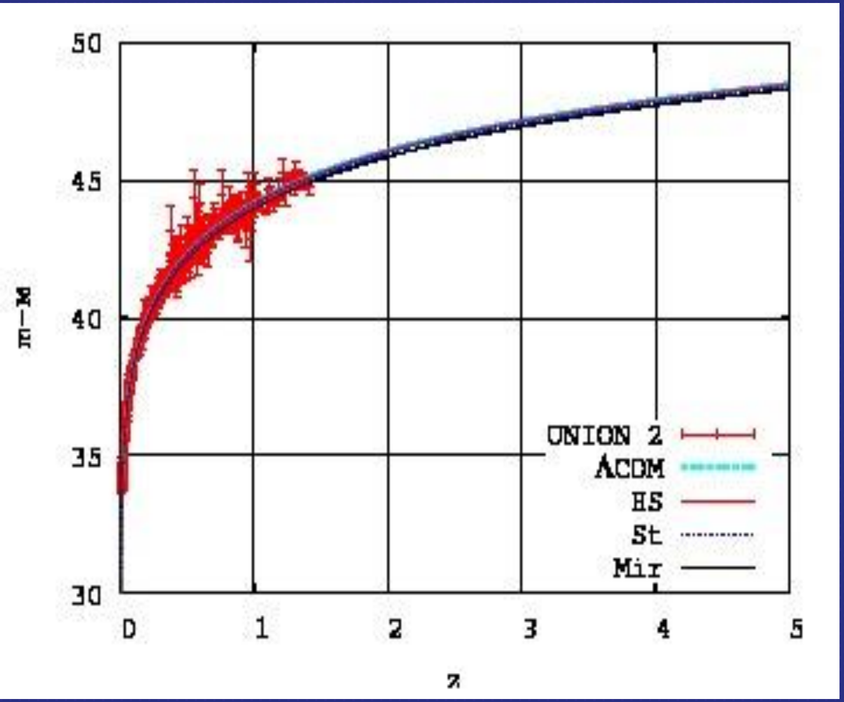


Luminosity distance and SNIa data confrontation ($k = 0$): $d_L^{\rm flat} = \frac{\zeta(\bar{a})}{\bar{a}}$, where $\zeta = c\, H_0^{-1} \int_{\bar{a}}^{1} \frac{d\bar{a}^*}{\bar{a}^{*\,2} H(\bar{a}^*)}$, $z = \frac{1}{\bar{a}} - 1$.

The luminous distance in log-scale (modulus distance) is given by

$\mu := m - M = 5\log_{10}(d_L^{\rm flat}/\mathrm{Mpc}) + 25.$

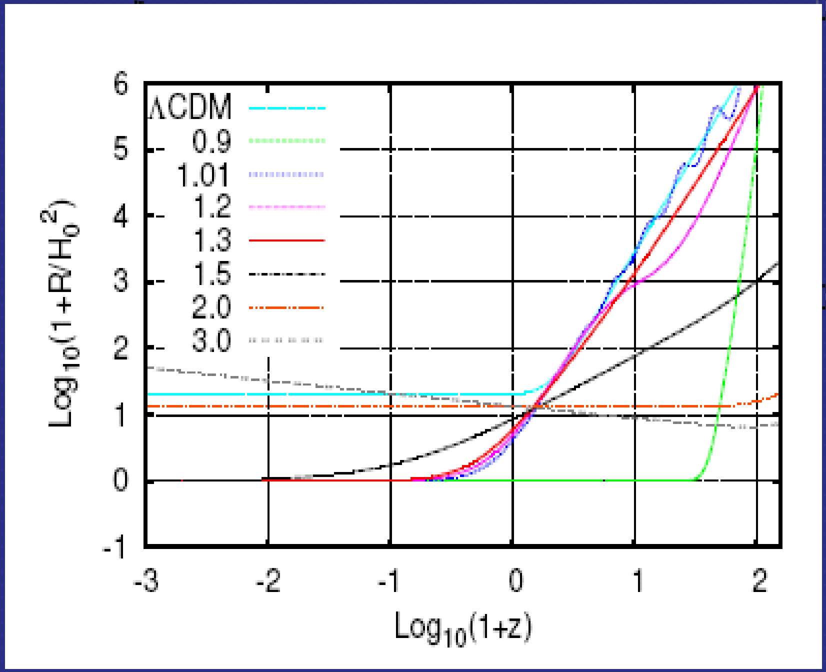


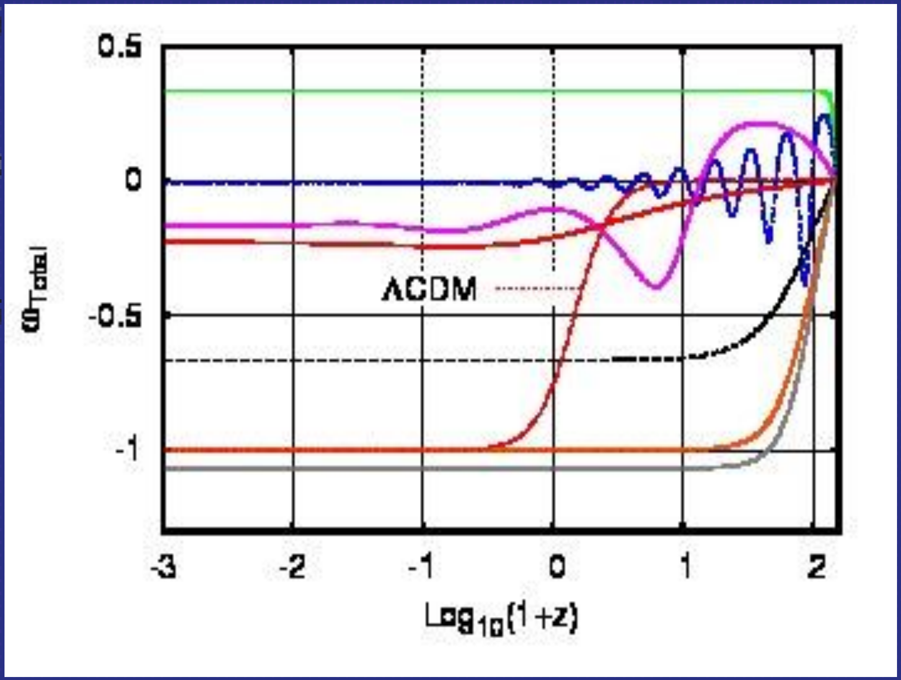


Other $f(R)$ models

The prototype model $f(R) = \lambda R_n (R/R_n)^n$ (where $\lambda R_n = const. = \alpha_n H_0^2$, the dimensionless constant α_n is some kind of "normalization factor" which is fixed so as that for all the models, we have that $H = H_0$ today, when integrating from the matter domination epoch to the future) was one of the first to be analyzed so that it produced a late accelerated expansion. Recently it was the object of debate between several authors (S. Capozziello et al., PLB **639**, 135, 2006; PLB **664**, 135, 2008; GRG **40**, 357, 2008; Carloni et al., CQG **22**, 4839, 2005; GRG **41**, 1757, 2009) and the results of L. Amendola et al. (PRL **98**, 131302, 2007; PRD **75**, 083504, 2007; IJMPD **16**, 1555, 2007). The orange group claimed that this kind of models were ruled out because wheter the produced a late time acceleration but an inadequate matter domination epoch or the opposite. The green group criticized their analysis on two grounds: 1) They resorted to the scalar-tensor approach, which the Capozziello et al. group raised "doubts"; 2) The phase-space (dynamical system) analysis was "incomplete" (Carloni et al. group).
As concerns the first criticisms Amendola et al. repeated the analysis in the original frame and recovered the same conclusions. They have not address the second criticism.
We have performed a full numerical analysis based upond the equations presented before, and we confirmed the same findings of Amendola et al., namely, these models

$q := -\frac{\ddot{a}}{aH^2} = -\frac{H^2+\dot{H}}{H^2} = 1 - \frac{R}{6H^2} = \frac{1}{2}\left(1 + 3\omega_{\rm tot}\right)$. So if $\omega_{\rm tot} < -1/3$ the Universe start accelerating. The figure on the right summarized our findings:





We have also analyzed the so called *exponential gravity* model $f(R) = R_*[\tilde{R} - \lambda(1 - e^{-\tilde{R}})]$, where $\tilde{R} := R/R_*$ and $R_* \sim H_0^2$ (see arXiv:1211.0015: Proc. 100 years after Einstein in Prague). This model seems to be also cosmologically viable:

This model have been studied in more detail (perturbations) by Linder (PRD 80, 123528, 2009) who showed that is a potentially viable model.

The following models have been ruled out in one way or another (cosmology, solar system, etc.):

$$f(R) = R - \frac{\mu^4}{R} , \quad (25)$$

$$f(R) = R - \frac{\mu_1^4}{R} + \mu_2^4 R , \quad (26)$$

$$f(R) = \alpha R^{-n} , \quad (27)$$

$$f(R) = R + \alpha R^{-n} , \text{ (possibly viable for } \alpha < 0, n \approx 1) \quad (28)$$

$$f(R) = R^p e^{qR} , \quad (29)$$

$$f(R) = R^p (\log \alpha R)^2 , \text{ (might succed for } p = 1, q > 0, q \neq 1) \quad (30)$$

$$f(R) = R^p e^{q/R} , \quad (31)$$

$$f(R) = R + \alpha R^2 , \quad (32)$$

$f(R)$ GRAVITY AND THE CMB

In order to analyze the angular anisotropies in the CMB, and all its accompanied features, within the framework of $f(R)$ gravity, a linear perturbation analysis similar to the one of GR has to be performed. In practice, everything is more-less the same, except that instead of having an EMT of matter in the r.h.s. of the Einstein equations, one has an effective EMT that includes the geometrical parts due to the modifications of gravity. So, the perturbation procedure proceeds as follows:

$$g_{ab} = g^0_{ab} + \delta g_{ab} \ , \ \delta g_{ab} \ll g^0_{ab} \quad (33)$$

$$\phi = \phi^0 + \delta\phi \ , \ \delta\phi \ll \phi^0 \quad (34)$$

$$T_{ab} = T^0_{ab} + \delta T_{ab} \ , \ \delta T_{ab} \ll T^0_{ab} \ , \quad (35)$$

where g^0_{ab} stands for the unperturbed FRW metric, and δg_{ab} is the metric perturbation which will describe the inhomogeneities and anisotropies associated with the perturbed spacetime. Here ϕ is any scalar associated with $f(R)$ gravity, like R, f_R, f_{RR}, and f_{RRR}; finally the last equation describe the pertubed EMT of matter (baryons, photons and DM, as in GR). This analysis is not new and dates back since the Starobinsky (1981) analysis of inflation and the Mukhanov *et al.* formalism (Phys. Rep. 215, 1992). One obtains then (modulo gauges) a set of field equations for the perturbation δg_{ab} and the scalar field δR or δf_R.

Many articles treat $f(R)$ gravity like an example of a scalar-tensor theory. In that instance, the formalism of perturbations have been developed in the past. More recently, this formalism has been revisited in many articles (e.g. Hu & Sawicky, PRD, **76**, 104030, 2007; Pogosian & Silvestri, PRD; **76**, 023503, 2008) specifically for $f(R)$ gravity. Indeed, a so called *Parametrized post-Friedmannian framework* for modified gravity, was devised in the previous papers, allowing to parametrize the deviations with respect to GR-cosmology independently of the metric theory at hand. This framework is reminiscent of the *Parametrized Post-Newtonian formalism* intended to parametrize the deviations of GR with respect to other metric theories of gravity, but within the context of the solar system experiments and binary pulsar.

So, for instance, when considering only scalar metric pertubations in the Newtonian gauge around a FRW metric with Euclidean (flat) 3-slices one has

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1-2\Phi)\delta_{ij}dx^i dx^j \quad . \tag{36}$$

In the early Universe in pure GR, one has

$$\Phi = \Psi \tag{37}$$

since $\delta T^i{}_j \approx 0$ (for $i \neq j$). However, in $f(R)$ gravity, the corresponding components diagonal components of δT^{eff}_{ab} (which includes the modifications of gravity) are not zero, then $\Phi \neq \Psi$.

So one of the PPF quantities is

$$\gamma = \frac{\Phi}{\Psi} \ . \tag{38}$$

which, like in the PPN formalism, parametrize the deviations with respect to GR. The fact that in modified gravity $\gamma \neq 1$, affects the primordial (plateau) Sachs-Wolfe effect (small ℓ: large angular scales) , which is related to the CMB temperature anisotropies produced by the gravitational shifts of light when the latter traverses well potentials producen by the inhomogeneities of matter.

$$\left.\frac{\delta T}{T}\right|_{t_e}^{t_d} = \phi(\vec{x_e}, t_e) - \phi(\vec{x_d}, t_d) + \int_{t_e}^{t_d} \frac{\partial[\Phi(\vec{x}(t), t) + \Psi(\vec{x}(t), t)]}{\partial t} dt$$

where $t_e =$ time at recombination (last scattering surface) and $t_d =$ today
The term $\phi(\vec{x_d}, t_d)$ gives an isotropic contribution around the observer (i.e. the probe), while the temperature anisotropies at different points of the last scattering surface $\frac{\delta T}{T}|_{t_e}$ combined with the correponding gravitational potential $\phi(\vec{x_e}, t_e)$ gives the known term of $\phi(\vec{x_e}, t_e)/3$. The last term corresponds to the ISW (see Merlin & Salgado, GRG 43, 2701, 2011 for a simple and geometrical derivation)

$f(R)$ models vs CMB

From "Cosmological constraints on $f(R)$ accelerating models", Y.S. Song, H. Peiris, and W. Hu, PRD vol.76, 063517 (2007)

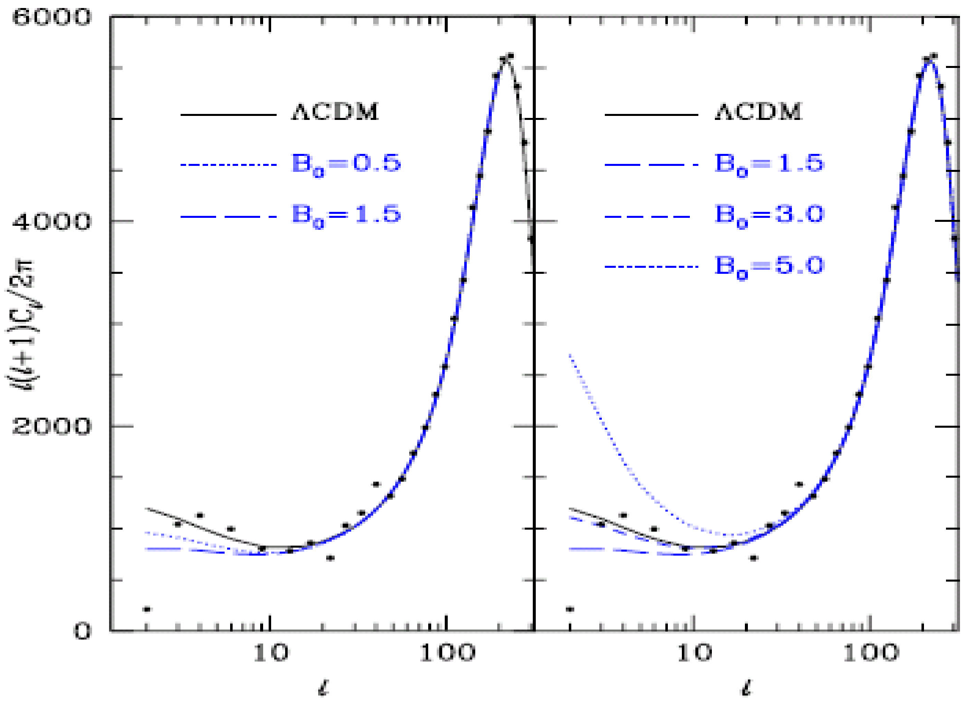


FIG. 2: CMB angular power spectrum C_ℓ for $f(R)$ models with the Compton wavelength parameter $B_0 = 0$ (ΛCDM), 0.5, 1.5, 3.0, 5.0. As B_0 increases, the ISW contributions to the low multipoles decrease, change sign, and then increase. WMAP3 data with noise error bars are overplotted and rule out $B_0 \geq 4.3$ (95% CL).

Solar system tests: weak field limit. Consider static and spherically symmetric perturbations ($|\phi|, |\psi| \ll 1$) around a De Sitter background:

$$ds^2 = -(1 - \phi - \Lambda_{eff} r^2)dt^2 + (1 + \psi - \Lambda_{eff} r^2)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (39)$$

In GR+Λ

$$\phi = 2M/r \quad (40)$$

$$\psi = 2M/r \quad (41)$$

$$\Lambda_{eff} = \Lambda \quad (42)$$

$$\gamma = \frac{\psi}{\phi} = 1 \quad (43)$$

where γ is one of the Post-Newtonian parameters. At solar system scales we can in fact neglect the term $\Lambda_{eff} r^2$. Now, in $f(R)$ gravity

$$\phi = 2M/r \quad (44)$$

$$\psi = 2\gamma M/r \quad (45)$$

$$\Lambda_{eff} = R_1/4 \quad (46)$$

$$\gamma \neq 1 \quad (47)$$

In fact γ depends on the the parameters of the theory $f(R)$ and on the global properties of the Sun, like $R_\odot$ and $M_\odot$. According to the observations (Cassini probe: Bertotti *et al.* Nature **425**, 2003, 474)

$$|\gamma - 1| \sim 10^{-5} \qquad (48)$$

It turns out that (Faulkner et al., PRD **76**, 063505, 2007)

$$|\gamma - 1| = \frac{2\Delta}{3+\Delta} \qquad (49)$$

where Δ is the so-called *thin shell parameter* which is related to the *chameleon:* (Khoury & Weltman, PRL **93**, 171104, 2004); PRD **69**, 044026, 2004) the scalar field degree of freedom f_R is supressed in regions of "high" density (the Sun) and at low density (cosmological scales) has noticeable effects, like the cosmic acceleration. This phenomenon is highly dependent on the contrast density between the central object and the surrounding environment and also on the details of the specific $f(R)$ theory. When the chamelon effect takes place, the scalar field f_R behaves like the electric potential within a conductor: inside the object $f_R \approx$ *const.* except within a thin shell $\delta R_\odot$ with $\Delta = \delta R_\odot / R_\odot \ll 1$, where the gradient of f_R is large (*screening effect* like within a conductor).

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Chameleon Fields: Awaiting Surprises for Tests of Gravity in Space

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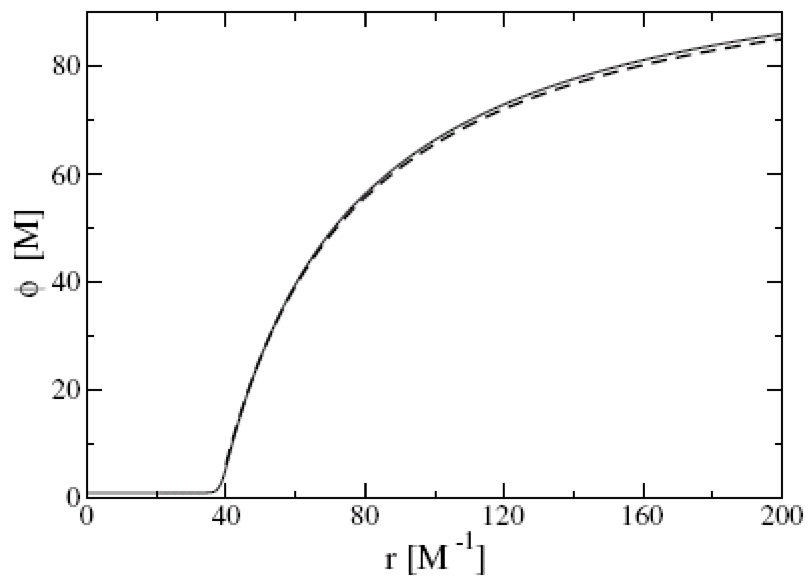

FIG. 2. Example of solution with thin shell.

Outside the object $f_R \propto M_\odot/r$. In this instance it is possible to satisfy the bound (Faulkner et al., PRD 76, 063505, 2007)

$$|\gamma - 1| = \frac{2\Delta}{3+\Delta} < 10^{-5} \qquad (50)$$

if $\Delta = \delta R_\odot / R_\odot \ll 1$. The thin shell parameter depends on the two minima of the effective potential $V_{\rm eff}(f_R, \rho_{\rm in,out})$ whose respective values inside the extended object (e.g. the Sun) where and outside depend on $\rho_{\rm in}$ and $\rho_{\rm out}$ and the bulk properties of the object (e.g. $M_\odot, R_\odot$).
However, when the chameleon does not ensue, f_R behaves like the electric potential within a dielectric: it has important variations within the object and the "thin" shell disapears: $\Delta = \delta R_\odot / R_\odot \sim 1$ and therefore

$$|\gamma - 1| = \frac{2\Delta}{3+\Delta} \sim \frac{1}{2} >> 10^{-5} \qquad (51)$$

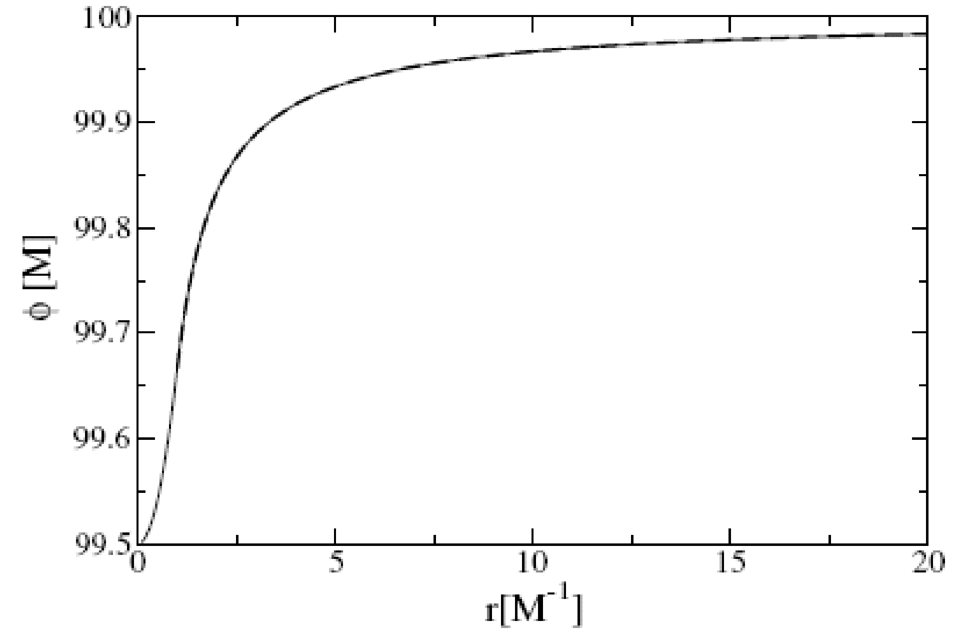

FIG. 3. Example of solution without thin shell.

Models of $f(R)$ Cosmic Acceleration that Evade Solar-System Tests

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(Dated: February 11, 2013)

In Fig. 8 we show $|\gamma - 1|$ for the same $n = 4$ models. The deviations peak at $\sim 10^{-15}$. Such deviations easily pass the stringent solar system tests of gravity from the Cassini mission [95]

$$|\gamma - 1| < 2.3 \times 10^{-5} \qquad (63)$$

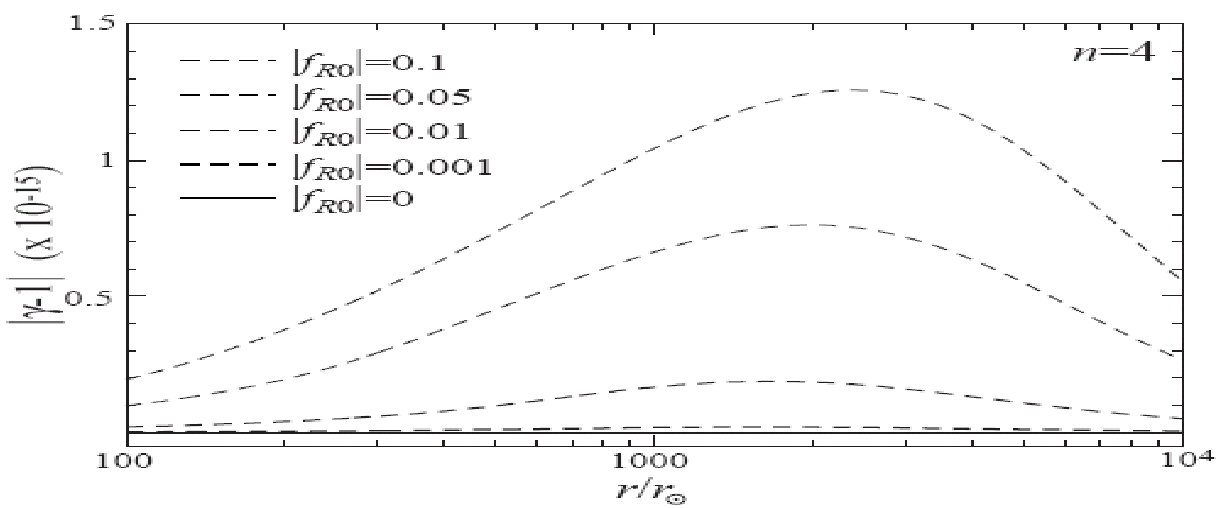

FIG. 8: Metric deviation parameter $|\gamma - 1|$ for $n = 4$ models and a series of cosmological field amplitudes f_{R0} with a galactic field that minimizes the potential. These deviations are unobservably small for the whole range of amplitudes.

Summary of standard inflation

In one of the simples scenarios, (primordial) inflation (i.e. the primordial accelerated expansion of the Universe, as opposed to the late acceleration expansion $\rightarrow$ SNIa) is produced by a single scalar field (as opposed to multiple scalar fields, or a modification of gravity $\rightarrow$ Starobinsky inflation $f(R) = R + \alpha R^2$; I'll come back to this point later) in the so called roll-over approximation:

$$
\begin{aligned}
H^2 &= \frac{\kappa}{3}\left(\frac{1}{2}\dot{\phi}_0^2 + V(\phi_0)\right) \\
&= \approx \frac{\kappa}{3}V(\phi_0)\,, \qquad (52)
\end{aligned}
$$

where $\kappa := 8\pi G$. That is we assume

$$\dot{\phi}_0^2 \ll V(\phi_0)\;, \qquad (53)$$

the potential dominates over the kinetic term.

As concerns the KG equation, one has

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi_0) = 0 \qquad (54)$$

where we assume

$$\ddot{\phi}_0 \ll 3H\dot{\phi}_0\;, \qquad (55)$$

the friction dominates over the acceleration term.

The roll-over parameters are given by

$$\epsilon := \frac{\kappa}{2}\left(\frac{\dot{\phi}_0}{H}\right)^2 , \tag{56}$$

$$\approx \frac{1}{2\kappa}\left(\frac{V'(\phi_0)}{V(\phi_0)}\right)^2 , \tag{57}$$

$$\eta := -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} , \tag{58}$$

such that

$$\epsilon \ll 1 \; , \; \eta \ll 1 \tag{59}$$

When analysing the origin of perturbations due the quantum fields, it is better to work with the conformal time τ:

$$ds^2 = \bar{a}^2(\tau)\left[-d\tau^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \tag{60}$$

$$= \bar{a}^2(\tau)\left[-d\tau^2 + \delta_{ij}dx^i dx^j\right] \tag{61}$$

where $\tau = \int dt/a(t)$, $\bar{a}(\tau) = a(t(\tau))$ (in the following we drop the overline in all the variables that depend on conformal time).

When considering perturbations we assume that the perturbed scalar-field is given by

$$\Phi(\tau,\vec{x}) = \phi_0(\tau) + \phi(\tau,\vec{x}) \ , \tag{66}$$

$$\phi(\tau,\vec{x}) \ll \phi_0(\tau) \tag{67}$$

where $\phi_0(\tau)$ is the background (unperturbed) scalar field and $\phi(\tau,\vec{x})$ is a perturbation.

In order to consider quantum effects we promote $\phi(\tau,\vec{x}) \to \hat{\phi}(\tau,\vec{x})$, where $\hat{\phi}(\tau,\vec{x})$ is the scalar-field operator acting upon quantum states defined on a Hilbert space (the generalization of fock space in a curved spacetime). So the field is decomposed in Fourier modes:

$$\hat{\phi}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\phi_{\vec{k}}(\tau)\hat{b}_{\vec{k}} e^{\imath\vec{k}\cdot\vec{x}} + \phi^*_{\vec{k}}(\tau)\hat{b}^\dagger_{\vec{k}} e^{-\imath\vec{k}\cdot\vec{x}} \right] \ . \tag{68}$$

where the creation and anhilation operators obey the commutation rules

$$[\hat{b}_{\vec{k}}, \hat{b}^\dagger_{\vec{k}'}] = \delta^3(\vec{k} - \vec{k}') \tag{69}$$

Bunch-Davis vacuum $\hat{b}_{\vec{k}}|0_b >= 0$.

The equation of motion that satisfies the coefficients $\phi_{\vec{k}}(\tau)$ is

$$\phi''_{\vec{k}}(\tau) + 2\frac{a'}{a}\phi'_{\vec{k}}(\tau) + k^2\phi_{\vec{k}}(\tau) = 0 \tag{70}$$

Or in terms of the new variable

$$u_{\vec{k}} = a(\tau)\phi_{\vec{k}} \ , \tag{71}$$

$$u''_{\vec{k}}(\tau) + \left(k^2 - \frac{a''}{a}\right)u_{\vec{k}}(\tau) = 0 \tag{72}$$

There are two limits at which this equation can be solved exactly:
Short-wave limit (ultraviolet limit) (SWL) (Minkowski limit): $a''/a \ll k$

$$u''_{\vec{k}}(\tau) + k^2 u_{\vec{k}}(\tau) = 0 \tag{73}$$

Then, the normalized solution is in general

$$u_{\vec{k}}(\tau) = \frac{1}{\sqrt{k}}\left[A_{\vec{k}}e^{-\imath\tau} + B_{\vec{k}}e^{\imath\tau}\right] \tag{74}$$

Bunch-Davis vacuum $A_{\vec{k}} = 1$, $B_{\vec{k}} = 0$.

Long-wave limit (mode freezing) (LWL): $k \ll a''/a$

$$u''_{\vec{k}}(\tau) - \frac{a''}{a} u_{\vec{k}}(\tau) = 0 \tag{75}$$

The normalized solution is

$$u_{\vec{k}}(\tau) = \frac{1}{\sqrt{2k}} \left(\frac{-\imath}{k\tau} \right) = \frac{1}{\sqrt{2k}} \left(\frac{aH}{k} \right) , \tag{76}$$

$$|\phi_{\vec{k}}(\tau)| = \frac{|u_{\vec{k}}(\tau)|}{a} = \frac{H}{\sqrt{2}k^{3/2}} \tag{77}$$

where $\tau = -1/[aH(1-\epsilon)]$. Remember $\epsilon \ll 1$. The simplest case $\epsilon = 0$ gives $\tau = -1/[aH] = -d_H$, where d_H, is the comoving distance at the horizon.
The power spectrum is defined in terms of the 2-point correlation function

$$< 0|\hat{\phi}(\tau,\vec{x})\hat{\phi}(\tau,\vec{x}')|0 > = \int \frac{d^3k}{(2\pi)^{3/2}} \left|\frac{u_{\vec{k}}(\tau)}{a}\right|^2 e^{\imath\vec{k}\cdot(\vec{x}-\vec{x}')} = \int \frac{dk}{k} P(k) e^{\imath\vec{k}\cdot(\vec{x}-\vec{x}')} , \tag{78}$$

$$\Longrightarrow P(k) = \frac{k^3}{2\pi^2} \left|\frac{u_{\vec{k}}(\tau)}{a}\right|^2 \tag{79}$$

In the LWL one has

$$P(k) \approx \left(\frac{H}{2\pi}\right) = \sqrt{\frac{\kappa V}{12\pi^2}} \approx const. \tag{80}$$

That is, the primordial power spectrum $P(k)$ is approximatively scale invariant.
The number of e-folds $N = -\ln a$, with $a = e^{-N}$ and $H = -dN/dt$, lead to

$$N = -\int H dt = -\sqrt{\kappa}\int \frac{1}{\sqrt{\epsilon(\phi)}} = -\kappa \int^{\phi} \frac{V(\phi)}{V'} d\phi \ , \tag{81}$$

so $N = N(\phi)$, and then $\phi(N)$, and therefore $N_{\vec{k}} = N(\phi_{\vec{k}})$. So

$$P^{1/2}(k) \approx \sqrt{\frac{\kappa V(N_{\vec{k}})}{12\pi^2}} \tag{82}$$

$N_{\vec{k}}$ is the number of e-folds of mode k at the horizon $k = aH$. A better approximation of all this consists in considering $\epsilon \ll 1$ but $\epsilon = const. \neq 0$. Then $u_{\vec{k}}(\tau)$ is given in terms of Bessel functions, and the spectrum depends slightly on k: $P(k) \approx k^n$, $n \approx -2\epsilon$. And even a more accurate description is when $\epsilon(\phi)$. But for our purpose it's sufficient to consider the above approximations.

Now, at the linear limit we have mainly two kind of perturbations for the metric and the matter fields: scalar and tensor.
The tensor perturbations associates with the metric is

$$\delta g_{ij} = \sqrt{4\kappa}(\phi_+ \hat{e}_{ij}^{+} + \phi_x \hat{e}_{ij}^{x}) \quad (83)$$

At the quantum level one can prove that the tensor perturbations ϕ_+ and ϕ_x can be treated as two scalars. Therefore, the power spectrum due to the tensor modes is

$$P_T \sim 2 \times const. \left(\frac{H}{2\pi}\right)^2 \quad (84)$$

$$= \frac{2\kappa H^2}{\pi^2} , \quad (85)$$

The power spectrum for scalar perturbations in the matter and the metric (the "Newtonian" and/or post-Newtonian potentials Φ and Ψ) is

$$P_S \sim const. \left(\frac{H}{2\pi}\right)^2 \quad (86)$$

$$= \frac{2\kappa H^2}{8\pi^2 \epsilon} , \quad (87)$$

Therefore the ratio bewteen the above power-spectrum is given by

$$r := \frac{P_T}{P_S} \approx 16\epsilon \ , \tag{88}$$

For instance, taking Linde's model $V(\phi) = \lambda\phi^4$ and inserting this into the slow-roll parameter ϵ

$$\epsilon = \frac{1}{2\kappa}\left(\frac{V'(\phi_0)}{V(\phi_0)}\right)^2 \tag{89}$$

and using the relationship between ϕ_0 and N (e-folds), one obtains

$$r = \frac{16}{N} \ . \tag{90}$$

For instance with $N \approx 60$, one gets $r \approx 0.25$.

Now, all this has been done in the framework of a single-field inflation and within GR. What as I mentioned before, $f(R)$ theories can produce a late-time accelerated expansion. Can one use this alternative theory to produce an early inflationary period "without" introducing explicitly a scalar-field ?. The answer is YES ! → Starobinsky model $f(R) = R + \alpha R^2$. One can re-write $f(R)$ theories into a kind of scalar-tensor theories (STT) (Jordan or Einstein frames) where one obtains a scalar-field potential $V(\phi)$ where $\phi = f_R$.
But in addition, STT and/or $f(R)$ theories produce an extra degree of freedom which produce scalar-gravitational waves ("breathing" mode). Therefore, one expects that the two modified power spectrums $P_T^{Mod}(f_{RR})$ and $P_S^{Mod}(f_{RR})$, will depend on f_{RR}. So that in pure GR $f(R) = R$ (without an extra scalar-field) the two power spectrums vanish. Then measuring the ratio $r(f_{RR})$ can constraint the form of $f(R)$. As mentioned, the Starobinsky model is just one example that can constraint the value α.

Conclusions

- $f(R)$ theories are alternative theories of gravity that can produce an accelerated expansion of the universe "without" the introduction of Λ. Some specific $f(R)$ models can pass several gravitational tests (e.g. the Solar System tests). They have some predictions different from GR+ Λ (e.g. variable EOS of dark energy, new gravitational-wave modes – breathing mode – –, different Sachs-Wolfe effect, ...)
- However, in my opinion they introduce more troubles than solutions. There is no fundamental principle that allows to single out one function $f(R)$. Simplicity favors: $f(R) = R - 2\Lambda$ (i.e. GR+ Λ). Time will tell if models different from GR will be taken seriously in the future.
- The are other issus concerning the EOS associated with $f(R)$ as geometric dark energy (however, I didn't have time to discuss them). Observationally further experiments will determine if such EOS is vaiable or not (e.g BigBOSS–DESI–, EUCLID, PanSTARR, WFIRST, etc.).
- Finally, the BICEP, PLANCK or other future experiments can constraint the inflationary model. In particular, the inflationary models arising from modifications of gravity $\rightarrow$ an extra scalar degree of freedom propagates and it affects the tensor-to-scalar-spectrum-amplitudes ratio.