Perturbaciones gravitacionales en métricas con fuentes

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PERTURBACIONES EN RELATIVIDAD GENERAL

Para linealizar las Ecuaciones de Einstein, debemos realizar perturbaciones gravitacionales: **Perturbaciones Tensoriales.**

La idea es suponer que se tiene un espacio de fondo, descrito por $g^0_{\mu\nu}$ y que existe una pequeña modificación,

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu},$$

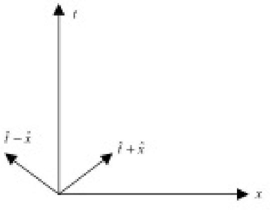
al utilizar las ecuaciones de Einstein, a primer orden, se obtiene una ecuación de onda para la perturbación en formalismo tensorial

$$g^{0\,\alpha\beta}\,\nabla_\alpha\,\nabla_\beta\,h_{\mu\nu} = 8\,\pi\,T^{(1)}_{\mu\nu},$$

que son la base para las ondas gravitacionales.

Formalismo de Newman Penrose (NP)

En esta platica hablare de las perturbaciones tensoriales, formadas por objetos masivos con fuentes. No solo hoyos negros.



A diferencia del caso tensorial, en el formalismo tetradial se proyectan dos vectores nulos reales sobre el cono de luz, y dos vectores perpendiculares sobre el plano complejo.

Por un lado, Newman y Penrose considerando que la radiación gravitacional se propaga a la velocidad de la luz, propusieron poner un sistema coordenado con los vectores base sobre el cono de luz, dieron una tétrada nula, con las siguientes propiedades:

- Tétrada $l^\mu, n^\mu, m^\mu, \overline{m}^\mu$,
- Son nulos, $l^\mu\, l_\mu = 0,\ n^\mu\, n_\mu = 0,\ m^\mu\, m_\mu = 0.$
- Orthonormales $l^\mu\, n_\mu = 1, \quad m^\mu\, \overline{m}_\mu = -1$
- Definen al tensor métrico: $g_{\mu\nu} = 2\, \left[l_{(\mu}\, n_{\nu)} - m_{(\mu}\, \overline{m}_{\nu)} \right]$.

μ y ν corren de $0, 1, 2, 3$. La sobrelinea significa complejo conjugado. Parentesis como subíndices significa índices simétricos.

Se desarrolla el formalismo:

- Se construyen los operadores derivada, proyectados sobre la tétrada: $D = l^\mu \, \partial_\mu, \Delta = n^\mu \, \partial_\mu, \delta = m^\mu \, \partial_\mu, \overline{\delta} = \overline{m}^\mu \, \partial_\mu$
- Se construyen los 12 coeficientes espinores, formados por las proyecciones de las derivadas covariantes de los vectores nulos sobre la tétrada misma: $\alpha, \beta, \pi, \tau, \rho, \mu, \gamma, \epsilon, \kappa, \sigma, \nu, \lambda$, por ejemplo: $\rho := l_{\mu;\nu} \, m^\mu \, \overline{m}^\nu$, que juegan un papel similar a los símolos de Christoffel, y más ya que permiten caracterizar al espacio-tiempo.
- Las 5 proyecciones del tensor de Weyl sobre la tétrada dan las funciones de Petrov, o escalares de Weyl, $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$, con propiedades características de los espacio-tiempos.
- Los 10 tensores de Ricci $\Phi_{00}, \Phi_{11}, \Phi_{22}, \Lambda$ son reales y $\Phi_{01}, \Phi_{10}, \Phi_{02}, \Phi_{02}, \Phi_{12}, \Phi_{21}$, son complejos.

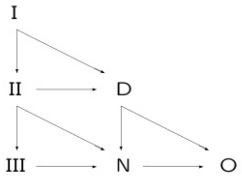
Clasificación de los espacios métricos con el formalismo de NPenrose

Algebraicamente general

- Tipo I ($\Psi_0 = 0$),

Algebraicamente especial

- Tipo II ($\Psi_0 = \Psi_1 = 0$),
- Tipo III ($\Psi_0 = \Psi_1 = \Psi_2 = 0$),
- Tipo N ($\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$)
- Tipo O ($\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$),
- Tipo D ($\Psi_2 \neq 0$).



Penrose digram

ECUACION DE TEUKOLSKY

Teukolsky [1972-73] utiliza esta formulación para estudiar las perturbaciones en agujeros negros. A partir de las identidades de Bianchi, los tensores de Riemman y Ricci, llega a una ecuación general, tanto para Ψ_0 como para Ψ_4,

$$_s\Box\, _s\psi = {}_s\mathcal{T},$$

con

$$\begin{aligned}
_s\Box = \{&\mathbf{\Delta D} - \delta^*\delta + (\mu^* - (2s+1)\,\gamma - \gamma^*)\,\mathbf{D} - ((2s+1)\rho + 2s\varepsilon)\,\mathbf{\Delta} \\
&-(2(s+1)\beta^* - (2s+1)\pi - \tau^*)\,\delta + (2s\beta + (2s+1)\tau)\,\delta^* \\
&+s\,(2s+1)[2(\rho+\varepsilon)\,\gamma - 2\beta\,\pi - 2\pi\,\tau + 2\beta^*\,\tau - \Psi_2] - \\
&2\,s\,[(\mathbf{\Delta}\,\varepsilon) - (\delta^*\beta) + \varepsilon\,(\mu^* - \gamma^*) + \beta\,\tau^*] + 4\,s\,(s+1)\,\beta\,\beta^*\},
\end{aligned}$$

donde $_2\psi = {\Psi_0}^{(1)}$ y $_{-2}\psi = \rho^{-4}\,{\Psi_4}^{(1)}$.

Los términos de materia son proyecciones del tensor de energía momento:

$$ {}_s\mathcal{T} = {}_s\mathcal{T}^{\mu\nu}\, T^{(1)}{}_{\mu\nu}, $$

con

$$ {}_{-2}\mathcal{T} = \mathcal{T}^{n\,n}\, T^{(1)}{}_{n\,n} + \mathcal{T}^{n\,\overline{m}}\, T^{(1)}{}_{n\,\overline{m}} + \mathcal{T}^{\overline{m}\,\overline{m}}\, T^{(1)}{}_{\overline{m}\,\overline{m}}, $$

donde

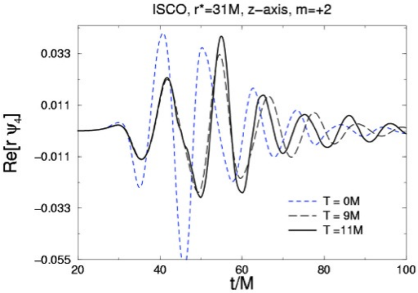
$$ \begin{aligned} \mathcal{T}^{n\,n} &= -\left(\overline{\delta} + 3\alpha + \overline{\beta} + 4\,\pi - \overline{\tau}\right)\left(\overline{\delta} + 2\alpha + 2\overline{\beta} - \overline{\tau}\right), \\ \mathcal{T}^{n\,\overline{m}} &= \left(\Delta + 4\,\mu + \overline{\mu} + 3\gamma - \overline{\gamma}\right)\left(\overline{\delta} + 2\alpha - \overline{\tau}\right) \\ &\quad + \left(\overline{\delta} + 3\alpha + \overline{\beta} + 4\,\pi - \overline{\tau}\right)\left(\Delta + 2\overline{\mu} + \gamma\right), \\ \mathcal{T}^{\overline{m}\,\overline{m}} &= -\left(\Delta + 4\,\mu + \overline{\mu} + 3\gamma - \overline{\gamma}\right)\left(\Delta + \overline{\mu} + 2\gamma - 2\overline{\gamma}\right). \end{aligned} $$

$T_{\mu\nu}$ puede ser una fuente de campo escalar, eléctrico, magnético como perturbación.

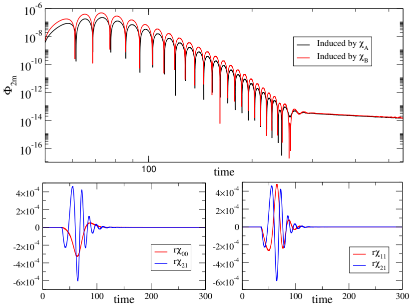
Los tensores de Ricci forman los operadores sobre la fuente:

$$
\begin{aligned}
\Phi_{01} &= \overline{\Phi}_{10} = \frac{1}{2}R_{02} = 4\pi T_{\mu\nu} l^{\mu} m^{\nu} \equiv 4\pi T_{lm}, \\
\Phi_{02} &= \overline{\Phi}_{20} = \frac{1}{2}R_{22} = 4\pi T_{\mu\nu} m^{\mu} m^{\nu} \equiv 4\pi T_{mm}, \\
\Phi_{12} &= \overline{\Phi}_{21} = \frac{1}{2}R_{12} = 4\pi T_{\mu\nu} n^{\mu} m^{\nu} \equiv 4\pi T_{nm}, \\
\Phi_{22} &= \overline{\Phi}_{22} = \frac{1}{2}R_{11} = 4\pi T_{\mu\nu} n^{\mu} n^{\nu} \equiv 4\pi T_{nn}.
\end{aligned}
$$

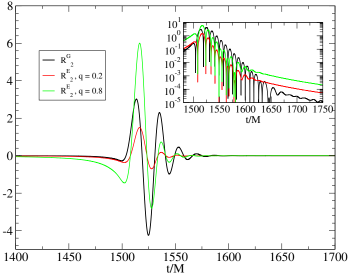
Primer frente de onda de la orbita mas estable de dos hoyos negros (Proyecto Lazarus)



Disco de acreción de un campo escalar como fuente



Disco de partículas cargadas como fuente



ECUACION DE TEUKOLSKY CON FUENTES

De las identidades de Bianchi

$$(D+4\epsilon-\rho)\Psi_4-(\overline{\delta}+4\pi+2\alpha)\Psi_3+(3\Psi_2+2\Phi_{11})\,\lambda$$
$$=(\bar{\delta}+2\alpha-2\bar{\tau})\Phi_{21}-(\Delta+\bar{\mu}+2\gamma-2\bar{\gamma})\Phi_{20}+2\nu\Phi_{10}+\bar{\sigma}\Phi_{22},$$

$$-(\delta+4\beta-\tau)\Psi_4+(\Delta+2\gamma+4\mu)\Psi_3-(3\Psi_2-2\Phi_{11})\,\nu$$
$$=-(\bar{\delta}-\bar{\tau}+2\alpha+2\bar{\beta})\Phi_{22}+(\Delta+2\bar{\mu}+2\gamma)\Phi_{21}-\bar{\nu}\Phi_{20}+2\lambda\Phi_{12}.$$

Del tensor de Riemann

$$\Psi_4+(\Delta+\mu+\overline{\mu}+3\gamma-\bar{\gamma})\lambda-(\overline{\delta}+3\alpha+\overline{\beta}+\pi-\bar{\tau})\nu=0.$$

Las identidades de Bianchi perturbadas quedan expresadas como

$$(D+4\epsilon-\rho)\Psi_4^{(1)}-(\bar{\delta}+4\pi+2\alpha)\Psi_3^{(1)}+(D+4\epsilon-\rho)^{(1)}\Psi_4-(\bar{\delta}+4\pi+2\alpha)^{(1)}\Psi_3$$
$$+\left(3\Psi_2+2\Phi_{11}\right)\lambda^{(1)}+\left(3\Psi_2+2\Phi_{11}\right)^{(1)}\lambda=(\bar{\delta}+2\alpha-2\bar{\tau})\Phi_{21}^{(1)}+(\bar{\delta}+2\alpha-2\bar{\tau})^{(1)}\Phi_{21}$$
$$-\left(\Delta+\bar{\mu}+2\gamma-2\bar{\gamma}\right)\Phi_{20}^{(1)}-\left(\Delta+\bar{\mu}+2\gamma-2\bar{\gamma}\right)^{(1)}\Phi_{20}+2\nu\Phi_{10}^{(1)}+2\nu^{(1)}\Phi_{10}+\bar{\sigma}^{(1)}\Phi_{22}+\bar{\sigma}\Phi_{22}^{(1)},$$

$$-(\delta+4\beta-\tau)\Psi_4^{(1)}+(\Delta+2\gamma+4\mu)\Psi_3^{(1)}-(\delta+4\beta-\tau)^{(1)}\Psi_4+(\Delta+2\gamma+4\mu)^{(1)}\Psi_3$$
$$-\left(3\Psi_2-2\Phi_{11}\right)\nu^{(1)}-\left(3\Psi_2-2\Phi_{11}\right)^{(1)}\nu=-(\bar{\delta}-\bar{\tau}+2\alpha+2\bar{\beta})^{(1)}\Phi_{22}-(\bar{\delta}-\bar{\tau}+2\alpha+2\bar{\beta})\Phi_{22}^{(1)}$$
$$(\Delta+2\bar{\mu}+2\gamma)\Phi_{21}^{(1)}+(\Delta+2\bar{\mu}+2\gamma)^{(1)}\Phi_{21}+2\lambda\Phi_{12}^{(1)}+2\lambda^{(1)}\Phi_{12}-\bar{\nu}\Phi_{20}^{(1)}-\bar{\nu}^{(1)}\Phi_{20},$$

Multiplicando por $(\bar{\delta}+\overline{\beta}+3\alpha+4\pi-\overline{\tau})$ y $(\Delta+3\gamma-\bar{\gamma}+4\mu+\bar{\mu})$ y la identidad

$$\left[\left(\Delta+3\gamma-\bar{\gamma}+4\mu+\bar{\mu}\right)\left(\bar{\delta}+4\pi+2\alpha\right)-\left(\bar{\delta}+\overline{\beta}+3\alpha+4\pi-\overline{\tau}\right)\left(\Delta+4\mu+2\gamma\right)\right]=0, \quad (1)$$

para eliminar $\Psi_3^{(1)}$, obtenemos

La ecuación de Teukolsky con fuentes se expresa como

$$\left[(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(D + 4\epsilon - \rho) - \left(\bar{\delta} + \overline{\beta} + 3\alpha + 4\pi - \overline{\tau}\right)(\delta + 4\beta - \tau) - 3\Psi_2\right]\Psi_4^{(1)} = T_4 + T_{4a},$$

donde

$$\begin{aligned} T_4 &= (\bar{\delta} + \overline{\beta} + 3\alpha + 4\pi - \overline{\tau})\left[(\Delta + 2\overline{\mu} + 2\gamma)\Phi_{21}^{(1)} - (\bar{\delta} - \overline{\tau} + 2\alpha + 2\overline{\beta})\Phi_{22}^{(1)}\right] \\ &+ (\Delta + 3\gamma - \overline{\gamma} + 4\mu + \overline{\mu})\left[(\bar{\delta} + 2\alpha - 2\overline{\tau})\Phi_{21}^{(1)} - (\Delta + \overline{\mu} + 2\gamma - 2\overline{\gamma})\Phi_{20}^{(1)}\right], \end{aligned}$$

$$\begin{aligned} T_{4a} &= 3\Psi_2[(\Delta + 4\mu + \overline{\mu} + 3\gamma - \bar{\gamma})^{(1)}\lambda - (\bar{\delta} + 3\alpha + \overline{\beta} + 4\pi - \bar{\tau})^{(1)}\nu] \\ &- \left[(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(D + 4\epsilon - \rho)^{(1)} - \left(\bar{\delta} + \overline{\beta} + 3\alpha + 4\pi - \overline{\tau}\right)(\delta + 4\beta - \tau)^{(1)}\right]\Psi_4 \\ &+ [(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(\delta + 4\pi + 2\alpha)^{(1)} - \left(\bar{\delta} + \overline{\beta} + 3\alpha + 4\pi - \overline{\tau}\right)(\Delta + 2\gamma + 4\mu)^{(1)}]\Psi_3 \\ &+ (\bar{\delta} + \overline{\beta} + 3\alpha + 4\pi - \overline{\tau})[-2\Phi_{11}\nu^{(1)} + (3\Psi_2 - 2\Phi_{11})^{(1)}\nu - \bar{\nu}\Phi_{20}^{(1)} - \bar{\nu}^{(1)}\Phi_{20} + 2\lambda\Phi_{12}^{(1)} \\ &+ 2\lambda^{(1)}\Phi_{12} - (\bar{\delta} - \bar{\tau} + 2\alpha + 2\bar{\beta})^{(1)}\Phi_{22} + (\Delta + 2\bar{\mu} + 2\gamma)^{(1)}\Phi_{21}] \\ &+ (\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})[-2\Phi_{11}\lambda^{(1)} - (3\Psi_2 + 2\Phi_{11})^{(1)}\lambda + 2\nu\Phi_{10}^{(1)} + 2\nu^{(1)}\Phi_{10} + \bar{\sigma}^{(1)}\Phi_{22} \\ &+ \bar{\sigma}\Phi_{22}^{(1)} + (\bar{\delta} + 2\alpha - 2\bar{\tau})^{(1)}\Phi_{21} - (\Delta + \bar{\mu} + 2\gamma - 2\bar{\gamma})^{(1)}\Phi_{20}] \\ &+ 3[(D + 2(\epsilon - \rho))\Psi_3 + 2\lambda\Psi_1 + \kappa\Psi_4 - (D - 2(\overline{\rho} - \epsilon))\Phi_{21} + (\delta + \overline{\pi} - 2(\overline{\alpha} - \beta))\Phi_{20} - 2\mu\Phi_{10} \\ &+ 2\pi\Phi_{11} - \overline{\kappa}\Phi_{22} - 2\overline{\delta}\Omega]\nu^{(1)} \\ &- 3[(\delta + 2(\beta - \tau))\Psi_3 + 2\nu\Psi_1 + \sigma\Psi_4 - (D - \overline{\rho} + 2(\epsilon + \overline{\epsilon}))\Phi_{22} + (\delta + 2(\overline{\pi} + \beta))\Phi_{21} - 2\mu\Phi_{11} \\ &- \overline{\lambda}\Phi_{20} + 2\pi\Phi_{12} - 2\Delta\Omega]\lambda^{(1)}. \end{aligned}$$

EJEMPLO: Métrica de un Wormhole Ultraestático

$$ds^2 = dt^2 - dl^2 - (l^2 + b_0^2)\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

La tetrada nula

$$\begin{aligned}
l^\mu &= \frac{1}{\sqrt{2}}(1, 1, 0, 0), \\
n^\mu &= \frac{1}{\sqrt{2}}(1, -1, 0, 0), \\
m^\mu &= \frac{1}{\sqrt{2}(l^2 + b_0^2)^{1/2}}(0, 0, 1, i\csc\theta), \\
\overline{m}^\mu &= \frac{1}{\sqrt{2}(l^2 + b_0^2)^{1/2}}(0, 0, 1, -i\csc\theta).
\end{aligned}$$

Las derivadas direccionales

$$D = l^\mu \partial_\mu = \frac{1}{\sqrt{2}}\left(\partial_t + \partial_l\right),$$

$$\Delta = n^\mu \partial_\mu = \frac{1}{\sqrt{2}}\left(\partial_t - \partial_l\right),$$

$$\delta = m^\mu \partial_\mu = \frac{1}{\sqrt{2}(l^2 + b_0^2)^{1/2}}\left(\partial_\theta + i\csc\theta\partial_\varphi\right),$$

$$\bar{\delta} = \overline{m}^\mu \partial_\mu = \frac{1}{\sqrt{2}(l^2 + b_0^2)^{1/2}}\left(\partial_\theta - i\csc\theta\partial_\varphi\right).$$

La tetrada nula

$$\nu = \sigma = \kappa = \lambda = \epsilon = \gamma = \tau = \pi = 0,$$

$$\rho = \mu = -\frac{l}{\sqrt{2}(l^2 + b_0^2)},$$

$$\beta = -\alpha = \frac{1}{2\sqrt{2}} \frac{\cot\theta}{(l^2 + b_0^2)^{1/2}} = \delta \ln \sin^{1/2}\theta.$$

El escalar de Weyl y los coeficientes Ricci

$$\Phi_{00} = -\frac{b_0^2}{2(l^2 + b_0^2)^2}, \qquad \Phi_{11} = \frac{b_0^2}{4(l^2 + b_0^2)^2},$$

$$\Phi_{22} = -\frac{b_0^2}{2(l^2 + b_0^2)^2}, \qquad \Lambda = -\frac{b_0^2}{12(l^2 + b_0^2)^2}, \qquad \Psi_2 = -\frac{b_0^2}{3\left(l^2 + b_0^2\right)^2}.$$

Cuya relación entre ellos es

$$\Psi_2 = \frac{2}{3}\Phi_{00} = -\frac{4}{3}\Phi_{11} = \frac{2}{3}\Phi_{22} = 4\Lambda,$$

donde Λ es proporcional al escalar de curvatura $R = g_{\mu\nu}R^{\mu\nu}$.

La ecuación de Teukolsky en este ejemplo es

$$\left[(\Delta+5\rho)(D-\rho)-\left(\bar{\delta}-2\beta\right)(\delta+4\beta)-3\Psi_2\right]\Psi_4^{(1)}=T_4+T_{4a}=T_{source},$$

donde

$$\begin{aligned} T_4 &= (\bar{\delta}-2\beta)\left[(\Delta+2\rho)\,\Phi_{21}^{(1)}-(\bar{\delta})\Phi_{22}^{(1)}\right]+(\Delta+5\rho)\left[(\bar{\delta}-2\beta)\Phi_{21}^{(1)}-(\Delta+\rho)\Phi_{20}^{(1)}\right], \\ T_{4a} &= (\bar{\delta}+2\beta)[\nu^{(1)}\Phi_{22}-(\bar{\delta})^{(1)}\Phi_{22}]+(\Delta+5\rho)\,[\lambda^{(1)}\Phi_{22}+\bar{\sigma}^{(1)}\Phi_{22}]+3\lambda^{(1)}[(D-2\rho)\Phi_{22}-\frac{2}{6}\Delta\Phi_{22}]. \end{aligned}$$

Donde hemos usado $\pi=-\overline{\tau}$, $\lambda=-\overline{\sigma}$ y $D\Phi_{22}=4\rho\Phi_{22}$, y la identidad de Bianchi

$$\delta^{(1)}\Phi_{22}=\delta\Phi_{22}^{(1)}-(D-3\rho)\Phi_{12}^{(1)}-(\Delta+3\rho)\Phi_{01}^{(1)}+\bar{\delta}\Phi_{02}^{(1)}, \qquad (2)$$

La expresión final para la fuente es

$$\begin{aligned} T_{source} &= 4\pi\{\left[(\bar{\delta}-2\beta)(\Delta+2\rho)+(\Delta+5\rho)(\bar{\delta}-2\beta)+(\bar{\delta}+2\beta)(D-3\rho)\right]T_{n\overline{m}}^{(1)} \\ &- [(\Delta+5\rho)(\Delta+\rho)-(\bar{\delta}+2\beta)\delta]T_{\overline{mm}}^{(1)}-2\overline{\delta\delta}T_{nn}^{(1)}+(\bar{\delta}+2\beta)(\Delta+5\rho)T_{l\overline{m}}^{(1)}\} \\ &+ \frac{3}{2}\Psi_2\Psi_4^{(1)} \end{aligned}$$

Haciendo uso del método de operadores adjuntos

$$
\begin{aligned}
&\mathcal{O}_G\left(\Psi_4^{(1)}\right) = \\
&4\pi\{\left[(\bar{\delta}-2\beta)\left(\Delta+2\rho\right)+\left(\Delta+5\rho\right)\left(\bar{\delta}-2\beta\right)+(\bar{\delta}+2\beta)(D-3\rho)\right]T^{(1)}_{n\overline{m}} \\
&-[(\Delta+5\rho)\left(\Delta+\rho\right)-(\bar{\delta}+2\beta)\delta]T^{(1)}_{\overline{mm}}-2\overline{\delta\delta}T^{(1)}_{nn} \\
&+(\bar{\delta}+2\beta)(\Delta+5\rho)T^{(1)}_{l\overline{m}}\},
\end{aligned}
$$

donde

$$
\mathcal{O}_G \equiv \left[\left(\Delta+5\rho\right)\left(D-\rho\right)-\left(\bar{\delta}-2\beta\right)\left(\delta+4\beta\right)-\frac{9}{2}\Psi_2\right].
$$

Encontramos la perturbación métrica con fuentes

$$
\begin{aligned}
h_{\mu\nu} &= 2\{n_{(\mu}\overline{m}_{\nu)}\left[\Delta(\bar{\delta}+2\beta)+\left(\bar{\delta}+2\beta\right)\left(\Delta-3\rho\right)+(D+\rho)(\bar{\delta}-2\beta)\right] \\
&- \overline{m}_\mu\overline{m}_\nu\left[\left(\Delta+\rho\right)\left(\Delta-3\rho\right)+\delta(\bar{\delta}-2\beta)\right] \\
&- n_\mu n_\nu\overline{\delta\delta}-l_{(\mu}\overline{m}_{\nu)}[(\Delta-3\rho)(\bar{\delta}-2\beta)]\}\Psi_4+c.c.
\end{aligned}
$$

En términos de la notación de Newman-Penrose, los adjuntos para los operadores diferenciales de una tetrada, se transforman de la siguiente manera

$$
\begin{aligned}
\mathcal{D}^{\dagger} &= -\left(\mathcal{D} + \varepsilon + \overline{\varepsilon} - \rho - \overline{\rho}\right), \\
\tilde{\Delta}^{\dagger} &= -\left(\tilde{\Delta} - \gamma - \overline{\gamma} + \mu + \overline{\mu}\right), \\
\delta^{\dagger} &= -\left(\delta - \overline{\alpha} + \beta + \overline{\pi} - \overline{\tau}\right), \\
\overline{\delta}^{\dagger} &= -\left(\overline{\delta} - \alpha + \overline{\beta} + \pi - \overline{\tau}\right).
\end{aligned}
\tag{3}
$$

Para el caso de una función f (e.g. los coeficientes de espin o las componentes de los vectores tetradas l^{μ}, n^{μ}, m^{μ}, $\overline{m}^{\mu}$), se tiene que

$$f^{\dagger} = f. \tag{4}$$

La perturbación de la métrica en este formalismo es

$$h_{\mu\nu} = 2l^{(1)}_{(\mu}n_{\nu)} + 2l_{(\mu}n^{(1)}_{\nu)} - 2m^{(1)}_{(\mu}\overline{m}_{\nu)} - 2m_{(\mu}\overline{m}^{(1)}_{\nu)}.$$

Las direcciones nulas son

$$\begin{aligned}
l^{\mu(1)} &= \frac{1}{2}h_{ln}l^{\mu} + \frac{1}{2}h_{ll}n^{\mu}, \\
n^{\mu(1)} &= \frac{1}{2}h_{nn}l^{\mu} + \frac{1}{2}h_{nl}n^{\mu}, \\
m^{\mu(1)} &= -\frac{1}{2}h_{mm}\overline{m}^{\mu} - \frac{1}{2}h_{m\overline{m}}m^{\mu} + h_{ml}n^{\mu} + h_{mn}l^{\mu},
\end{aligned}$$

donde por ejemplo $h_{mm} = h_{\mu\nu}m^{\mu}m^{\nu}$.

Las derivadas direccionales perturbadas son

$$\begin{aligned}
D^{(1)} &= l^{\mu(1)}\partial_{\mu} = -\frac{1}{2}h_{ln}D - \frac{1}{2}h_{ll}\Delta, \\
\Delta^{(1)} &= n^{\mu(1)}\partial_{\mu} = -\frac{1}{2}h_{nn}D - \frac{1}{2}h_{nl}\Delta, \\
\delta^{(1)} &= m^{\mu(1)}\partial_{\mu} = \frac{1}{2}h_{mm}\overline{\delta} + \frac{1}{2}h_{m\overline{m}}\delta - h_{ml}\Delta - h_{mn}D.
\end{aligned}$$

Coeficientes de spin perturbados

$$
\begin{aligned}
\kappa^{(1)} &= (D-\overline{\rho}-2\epsilon)h_{lm}-\frac{1}{2}(\delta-2\overline{\alpha}-2\beta+\overline{\pi}+\tau)h_{ll}-\sigma_{l\overline{m}}+\frac{1}{2}\overline{\kappa}h_{mm}+\frac{1}{2}\kappa h_{m\overline{m}},\\
\sigma^{(1)} &= -(\overline{\pi}+\tau)h_{lm}+\frac{1}{2}(D-\overline{\rho}+\rho-2\epsilon+2\overline{\epsilon})h_{mm}+\frac{1}{2}\overline{\lambda}h_{ll}-\frac{1}{2}\sigma h_{ln},\\
\nu^{(1)} &= -(\Delta+\overline{\mu}+2\gamma)h_{n\overline{m}}+\frac{1}{2}(\overline{\delta}+2\alpha+2\overline{\beta}-\pi-\overline{\tau})h_{nn}-\lambda h_{nm}+\frac{1}{2}\nu h_{m\overline{m}}+\overline{\nu}h_{\overline{m}\overline{m}},\\
\lambda^{(1)} &= -(\overline{\tau}+\pi)h_{n\overline{m}}-\frac{1}{2}(\Delta+\overline{\mu}-\mu+2\gamma-2\overline{\gamma})h_{\overline{m}\overline{m}}-\frac{1}{2}\lambda h_{ln}+\frac{1}{2}\overline{\sigma}h_{nn},\\
2\mu^{(1)} &= \rho h_{nn}-(\delta+2\beta+\tau)h_{n\overline{m}}+(\overline{\delta}+2\overline{\beta}-2\pi-\overline{\tau})h_{nm}-(\Delta+\overline{\mu}-\mu+\gamma-\overline{\gamma})h_{m\overline{m}}-\overline{\mu}h_{ln}-\nu h_{lm}\\
2\rho^{(1)} &= \mu h_{ll}+(\rho-\overline{\rho})h_{nl}+(D+\rho-\overline{\rho})h_{m\overline{m}}+(\overline{\delta}-2\alpha-\pi)h_{lm}-(\delta+2\tau-2\overline{\alpha}+\overline{\pi})h_{l\overline{m}}+\overline{\kappa}h_{nm}-\kappa\\
4\epsilon^{(1)} &= (D+\rho-\overline{\rho}-2\epsilon)h_{nl}+(\overline{\delta}-2\alpha-3\pi-2\overline{\tau})h_{lm}-(\delta-2\overline{\alpha}+\overline{\pi}+2\tau)h_{l\overline{m}}\\
&+ (\rho-\overline{\rho})h_{m\overline{m}}-(\Delta-\mu+\overline{\mu}-2\overline{\gamma})h_{ll}+\overline{\kappa}h_{nm}-\kappa h_{n\overline{m}}-\sigma h_{\overline{m}\overline{m}}+\overline{\sigma}h_{mm},\\
2\pi^{(1)} &= -(D-\rho+2\epsilon)h_{n\overline{m}}+(\overline{\delta}-\overline{\tau}-\pi)h_{nl}-(\Delta+\overline{\mu}-2\overline{\gamma})h_{l\overline{m}}-\overline{\tau}h_{m\overline{m}}-\tau h_{\overline{m}\overline{m}}+\overline{\sigma}h_{nm}-\lambda h_{lm},\\
2\tau^{(1)} &= (D-\overline{\rho}+2\overline{\epsilon})h_{nm}-(\delta+\overline{\pi}+\tau)h_{nl}+(\Delta+\mu-2\gamma)h_{lm}-\overline{\pi}h_{m\overline{m}}-\pi h_{mm}+\overline{\lambda}h_{l\overline{m}}-\sigma h_{n\overline{m}},\\
4\alpha^{(1)} &= (D-2\overline{\rho}-\rho-2\epsilon)h_{n\overline{m}}-(\Delta+4\gamma-2\mu+\overline{\mu}-2\overline{\gamma})h_{l\overline{m}}-(\overline{\delta}+\pi+\overline{\tau})h_{nl}+\nu h_{ll}-3\lambda h_{lm}+\overline{\kappa}h_{n}\\
&+ (\overline{\delta}+2\alpha-\pi-\overline{\tau})h_{m\overline{m}}-(\delta-2\overline{\alpha}+\overline{\pi}+\tau)h_{\overline{m}\overline{m}},\\
4\beta^{(1)} &= (D-\overline{\rho}-4\epsilon+2\rho+2\overline{\epsilon})h_{nm}-(\Delta+\mu+2\overline{\mu}+2\gamma)h_{lm}\\
&- (\delta+\overline{\pi}+\tau)h_{nl}-(\delta-2\beta+\overline{\pi}+\tau)h_{m\overline{m}}+(\overline{\delta}+2\overline{\beta}-\pi-\overline{\tau})h_{mm}+\overline{\nu}h_{ll}+\overline{\lambda}h_{l\overline{m}}+\kappa h_{nn}-3\sigma h\\
4\gamma^{(1)} &= -(\Delta-\mu+\overline{\mu}+2\gamma)h_{nl}+(D+\rho-\overline{\rho}+2\overline{\epsilon})h_{nn}\\
&- (\delta+2\beta+2\overline{\pi}+3\tau)h_{n\overline{m}}+(\overline{\delta}+2\overline{\beta}-2\pi-\overline{\tau})h_{nm}+(3\overline{\mu}-2\mu+\gamma-\overline{\gamma})h_{m\overline{m}}-\nu h_{lm}+\overline{\nu}h_{l\overline{m}}-
\end{aligned}
$$

Escalares de Ricci Perturbados

$$
\begin{aligned}
\Phi_{00}^{(1)} &= D^{(1)}\rho - \overline{\delta}^{(1)}\kappa - (\rho^2 + \sigma\overline{\sigma})^{(1)} - \rho^{(1)}(\epsilon + \overline{\epsilon}) + \overline{\kappa}^{(1)}\tau + \kappa^{(1)}(3\alpha + \overline{\beta} - \pi) \\
&+ D\rho^{(1)} - \overline{\delta}\kappa^{(1)} - \rho(\epsilon + \overline{\epsilon})^{(1)} + \overline{\kappa}\tau^{(1)} - \kappa(3\alpha + \overline{\beta} - \pi)^{(1)}, \\
\Phi_{11}^{(1)} &= \delta^{(1)}\alpha - \overline{\delta}^{(1)}\beta - (\mu\rho - \lambda\sigma)^{(1)} - \alpha^{(1)}\overline{\alpha} - \beta^{(1)}\overline{\beta} + 2\alpha^{(1)}\beta - \gamma^{(1)}(\rho - \overline{\rho}) - \epsilon^{(1)}(\mu - \overline{\mu}) + \Psi_2^{(1)} \\
&+ \delta\alpha^{(1)} - \overline{\delta}\beta^{(1)} - \alpha\overline{\alpha}^{(1)} - \beta\overline{\beta}^{(1)} + 2\alpha\beta^{(1)} - \gamma(\rho - \overline{\rho})^{(1)} - \epsilon(\mu - \overline{\mu})^{(1)} + \Psi_2 - \Lambda - \Lambda^{(1)}, \\
\Phi_{01}^{(1)} &= D^{(1)}\tau - \Delta^{(1)}\kappa - \rho^{(1)}(\tau + \overline{\pi}) - \sigma^{(1)}(\overline{\tau} + \pi) - \tau^{(1)}(\epsilon - \overline{\epsilon}) - \kappa^{(1)}(3\gamma + \overline{\gamma}) - \Psi_1 - \Psi_1^{(1)} \\
&+ D\tau^{(1)} - \Delta\kappa^{(1)} - \rho(\tau + \overline{\pi})^{(1)} - \sigma(\overline{\tau} + \pi)^{(1)} - \tau(\epsilon - \overline{\epsilon})^{(1)} - \kappa(3\gamma + \overline{\gamma})^{(1)}, \\
\Phi_{21}^{(1)} &= D^{(1)}\nu - \Delta^{(1)}\pi - \mu^{(1)}(\pi + \overline{\tau}) - \lambda^{(1)}(\overline{\pi} + \tau) - \pi^{(1)}(\gamma - \overline{\gamma}) + \nu^{(1)}(3\epsilon + \overline{\epsilon}) \\
&+ D\nu^{(1)} - \Delta\pi^{(1)} - \mu(\pi + \overline{\tau})^{(1)} - \lambda(\overline{\pi} + \tau)^{(1)} - \pi(\gamma - \overline{\gamma})^{(1)} + \nu(3\epsilon + \overline{\epsilon})^{(1)} - \Psi_3 - \Psi_3^{(1)}, \\
\Phi_{02}^{(1)} &= \delta^{(1)}\tau - \Delta^{(1)}\sigma - (\mu\sigma + \overline{\lambda}\rho)^{(1)} - \tau^{(1)}(\tau + \beta - \overline{\alpha}) + \sigma^{(1)}(3\gamma - \overline{\gamma}) + \kappa^{(1)}\overline{\nu} \\
&+ \delta\tau^{(1)} - \Delta\sigma^{(1)} - \tau(\tau + \beta - \overline{\alpha})^{(1)} + \sigma(3\gamma - \overline{\gamma})^{(1)} + \kappa\overline{\nu}^{(1)}, \\
\Phi_{22}^{(1)} &= \delta^{(1)}\nu - \Delta^{(1)}\mu - \mu^{(1)}(\gamma + \overline{\gamma}) + \overline{\nu}^{(1)}\pi - \nu^{(1)}(\tau - 3\beta - \overline{\alpha}) \\
&+ \delta\nu^{(1)} - \Delta\mu^{(1)} - (\mu^2 + \lambda\overline{\lambda})^{(1)} - \mu(\gamma + \overline{\gamma})^{(1)} + \overline{\nu}\pi^{(1)} - \nu(\tau - 3\beta - \overline{\alpha})^{(1)},
\end{aligned}
$$

Escalares de Weyl perturbados

$$
\begin{aligned}
\Psi_0^{(1)} &= (D - 3\epsilon + \bar{\epsilon} - \rho - \bar{\rho})\sigma^{(1)} - (\delta - \bar{\alpha} - 3\beta + \bar{\pi} - \tau)\kappa^{(1)} \\
&+ (D - 3\epsilon + \bar{\epsilon} - \rho - \bar{\rho})^{(1)}\sigma - (\delta - \bar{\alpha} - 3\beta + \bar{\pi} - \tau)^{(1)}\kappa, \\
\Psi_1^{(1)} &= (D + \bar{\epsilon} - \bar{\rho})\beta^{(1)} - (\delta - \bar{\alpha} + \bar{\pi})\epsilon^{(1)} - (\alpha + \pi)\sigma^{(1)} + (\gamma + \mu)\kappa^{(1)}, \\
&+ (D + \bar{\epsilon} - \bar{\rho})^{(1)}\beta - (\delta - \bar{\alpha} + \bar{\pi})^{(1)}\epsilon - (\alpha + \pi)^{(1)}\sigma + (\gamma + \mu)^{(1)}\kappa, \\
\Psi_2^{(1)} &= [(\bar{\delta} - 2\alpha + \bar{\beta} - \pi - \bar{\tau})\beta^{(1)} - (\delta - \bar{\alpha} + \bar{\pi} + \tau)\alpha^{(1)} + (D + \epsilon + \bar{\epsilon} + \rho - \bar{\rho})\gamma^{(1)} + 2(\nu\kappa - \lambda\sigma)^{(1)}]/3 \\
&\quad -(\Delta - \bar{\gamma} - \gamma + \bar{\mu} - \mu)\epsilon^{(1)} + (\bar{\delta} - \alpha + \bar{\beta} - \bar{\tau} - \pi)\tau^{(1)} - (\Delta - \bar{\gamma} - \gamma + \bar{\mu} - \mu)\rho^{(1)}, \\
&+ [(\bar{\delta} - 2\alpha + \bar{\beta} - \pi - \bar{\tau})^{(1)}\beta - (\delta - \bar{\alpha} + \bar{\pi} + \tau)^{(1)}\alpha + (D + \epsilon + \bar{\epsilon} + \rho - \bar{\rho})^{(1)}\gamma]/3 \\
&\quad -(\Delta - \bar{\gamma} - \gamma + \bar{\mu} - \mu)^{(1)}\epsilon + (\bar{\delta} - \alpha + \bar{\beta} - \bar{\tau} - \pi)^{(1)}\tau - (\Delta - \bar{\gamma} - \gamma + \bar{\mu} - \mu)^{(1)}\rho, \\
\Psi_3^{(1)} &= (\bar{\delta} + \bar{\beta} - \bar{\tau})\gamma^{(1)} - (\Delta - \bar{\gamma} + \bar{\mu})\alpha^{(1)} + (\epsilon + \rho)\nu^{(1)} - (\beta + \tau)\lambda^{(1)}, \\
&+ (\bar{\delta} + \bar{\beta} - \bar{\tau})^{(1)}\gamma - (\Delta - \bar{\gamma} + \bar{\mu})^{(1)}\alpha + (\epsilon + \rho)^{(1)}\nu - (\beta + \tau)^{(1)}\lambda, \\
\Psi_4^{(1)} &= (\bar{\delta} + 3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu^{(1)} - (\Delta - \bar{\gamma} + 3\gamma + \mu + \bar{\mu})\lambda^{(1)} \\
&+ (\bar{\delta} + 3\alpha + \bar{\beta} + \pi - \bar{\tau})^{(1)}\nu - (\Delta - \bar{\gamma} + 3\gamma + \mu + \bar{\mu})^{(1)}\lambda,
\end{aligned}
\tag{6}
$$

CONCLUSIONES

- Haciendo uso del formalismo de Newman-Penrose y ayudados del algebra ya establecida en este formalismo, pudimos obtener un ecuación tipo Teukolsky cuando los escalares de Ricci no son nulos, que no sean nulos implica que el objeto astrofísico contenga materia y por lo tanto aparecen términos extras de la ecuación de Teukolsky obtenida para hoyos negros.
- Con esta ecuación podremos obtener las ondas gravitacionales generadas por objetos astrofísicos con materia.