General Relativity in simulations of Structure Formation

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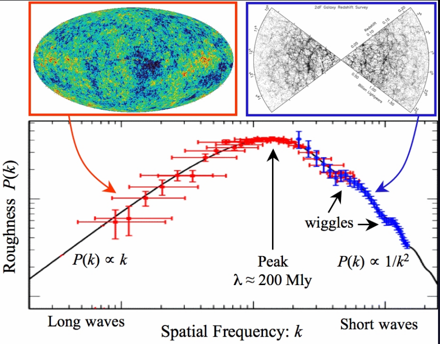
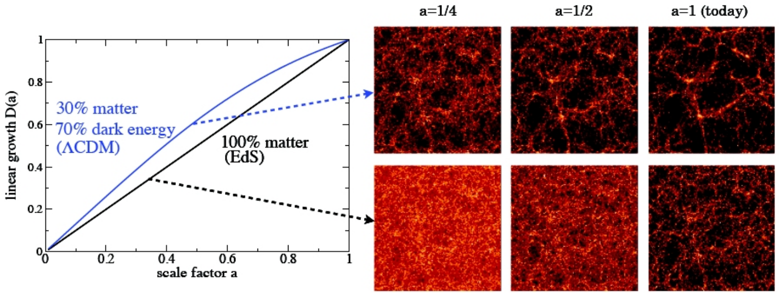
Large Scale Structure

- Statistics of Matter Fields
- Growth rate of $\delta = \delta\rho/\bar{\rho}$
- Baryon Acoustic Oscillations





Modeling the Universe

Inference + Models

- Inferences:
  - Missing matter (Dark Matter / MONDian dynamics)
  - Accelerated Expansion (Dark Energy / Λ)
  - History of Structure Formation

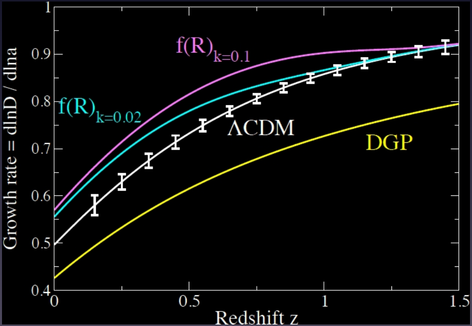
- Most accepted model: Λ-CDM
  - An early inflationary stage
  - Dark Matter ($\Omega_{DM} = 0.266$) + Cosmological Const. ($\Omega_{\Lambda} = 0.685$)
  - Flat+homogeneous+isotropic universe: FLRW

Modeling the Universe

Inference + Models

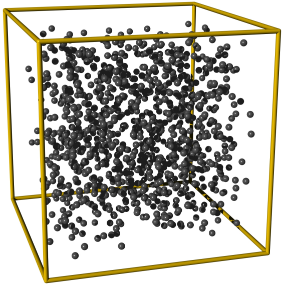
- Inferences:
  - Missing matter (Dark Matter)
  - Accelerated Expansion (Dark Energy/ Λ)
  - History of Structure Formation

- Alternative models:
  - An early and late inflation
  - Dark Energy
  - Modified Gravity
  - Modified Dynamics
  - Inhomogeneous Models



Modeling the halo distribution

- First stage: Initial Conditions Generator.



- Initial conditions in lagrangian coordinates (*Zel'dovich, 2LPT*).

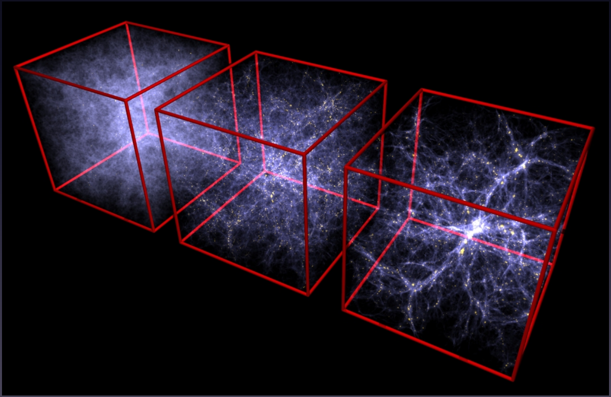
$\mathbf{x} = a(\tau)\mathbf{q}_0 + a(\tau)\Psi(q_0, \tau)$

- Ψ = Lagrangian displacement = Initial particle position.

- Displacement from matter power spectrum $\int P_0(k)k^2T(k)W(k\,8\mathrm{Mpc}) = \sigma_8^2$.

Modeling the halo distribution

- First stage: Initial Conditions Generator.
- Second stage: Large ($\sim 10^{10}$) N-body evolution from $z \sim 100$ up to t_0.

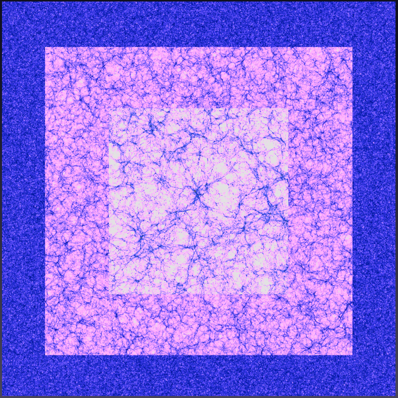


Modeling the halo distribution

- First stage: Initial Conditions Generator.
- Second stage: Large ($\sim 10^{10}$) N-body evolution from $z \sim 100$ up to t_0.

- Caveat: Newtonian dynamics not valid at horizon scales!

Teyssier et al. 2009



Compatibility problem
Newton vs. Einstein cosmologies

Strategies for consistency

- Post-Newtonian treatment (in powers of $1/c^2$).
- Dictionaries of perturbations ($\delta_N^{(1)} = \delta_{\rm com}^{(1)}$, $v_N^{(1)} = v_\ell^{(1)}$).
- Match the dynamical equations (e.g. Silent Universe).

Effects

- Tensorial perturbations $h_{ij}^{+}, h_{ij}^{\times}$.
- Extra constraints $\nabla^2\Phi \neq 4\pi G\delta\rho$.
  - Scale-dependent non-linear bias.
  - Incorporation of primordial non-Gaussianity.
  - CMB-Lensing Cross correlation.

This talk

Newton vs. Einstein cosmologies

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Newtonian Cosmology
Perturbations of irrotational dust

- Flat FRW–Universe (homogeneous+isotropic): ΛCDM

$$\rho_N = \bar{\rho}\,[1+\delta_N(\mathbf{x},\tau)]\,, \quad u^i_N(\mathbf{x},t) = x^i\mathcal{H} + \partial^i v_N\,, \qquad E_N = \bar{E}_N + \delta E_N(\mathbf{x}),$$

- Evolution: Continuity and Euler equations,

$$\frac{d\delta_N}{d\tau} = -(1+\delta_{\mathrm{N}})\nabla^2 v_N\,,$$

$$\frac{\partial \nabla^2 v_N}{\partial \tau} + \mathcal{H}\nabla^2 v_N + \nabla(\nabla v_N \cdot \nabla)\nabla v_N + \nabla^2\Phi_N = 0$$

$\Rightarrow$ Raichaudhuri Equation,

$$\frac{d(\nabla^2 v_N)}{d\tau} + \mathcal{H}\nabla^2 v_N + \partial^i\partial_k v_N \partial^k \partial_i v_N + 4\pi G a^2 \bar{\rho}\delta_N = 0$$

Relativistic Cosmology
Definitions in Synchronous Gauge

- Perturbations in a Flat FRW-universe: ΛCDM

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \gamma_{ij}(\mathbf{x},\tau)dx^i dx^j\right],$$

- Synchronous-comoving gauge: $u_\mu = [-a(\tau), 0, 0, 0]$,

- Deformation tensor

$$\vartheta^\mu_\nu \equiv a(\tau){u^\mu}_{;\nu} - \mathcal{H}\delta^\mu_\nu \Rightarrow \frac{1}{2}\gamma^{ik}{\gamma_{kj}}',$$

- Three-metric Ricci curvature

$$R^i_j \equiv {}^{(3)}R^i_j(\gamma); \qquad R = R^i_i,$$

Tomita, 1967
Ellis, 1971

Relativistic Cosmology

Non-linear Equations

- Energy Conservation (Continuity equation):

$$\frac{\rho'}{\rho} = -\frac{1}{2}\gamma^{ij}\gamma'_{ij} - 3\mathcal{H}, \Rightarrow \rho = \frac{A(\mathbf{x})}{a^3\sqrt{\gamma}} \Rightarrow \boxed{\delta' = -(1+\delta)\vartheta}$$

- Evolution equation (*ij* Einstein Eqs.):

$$\vartheta^{i\prime}_j + 2\mathcal{H}\vartheta^i_j + \vartheta\vartheta^i_j + \frac{1}{4}\left(\vartheta^k_l\vartheta^l_k - \vartheta^2\right)\delta^i_j + R^i_j - \frac{1}{4}R\delta^i_j = 0\,.$$

- Energy and Momentum Constraints (00, 0*i* Einstein Eqs.):

$$\vartheta^2 - \vartheta^i_j\vartheta^j_i + 4\mathcal{H}\vartheta + R = 16\pi G a^2\bar{\rho}\delta\,, \qquad \vartheta^i_{j|i} = \vartheta_{,\,j}\,.$$

$\Rightarrow$ Relativistic Raychaudhuri equation:

$$\boxed{\vartheta' + \mathcal{H}\vartheta + \vartheta^i_j\vartheta^j_i + 4\pi G a^2\bar{\rho}\delta = 0}$$

Relativistic evolution

Vs. Newtonian evolution

Correspondence: $\vartheta^i_k = \partial^i\partial_k v_N = \partial^i\partial_k v_\ell^{(1)}, \quad \vartheta = \nabla^2 v_N = \nabla^2 v_\ell^{(1)}.$

- Continuity + Raychaudhuri:

$$\delta' = -(1+\delta)\vartheta, \qquad \vartheta' + \mathcal{H}\vartheta + \vartheta^i_j\vartheta^j_i + 4\pi G a^2 \bar{\rho}\delta = 0\,.$$

$$\Downarrow \qquad \qquad \Downarrow$$

$$\frac{d\delta_N}{d\tau} = -(1+\delta_{\rm N})\nabla^2 v_N\,,$$

$$\frac{d(\nabla^2 v_N)}{d\tau} + \mathcal{H}\nabla^2 v_N + \partial^i\partial_k v_N \partial^k\partial_i v_N + 4\pi G a^2\bar{\rho}\delta_N = 0.$$

Bardeen, 1980

Wands & Slozar, 2009

Chisari & Zaldarriaga, 2011

JCH, Christopherson, Malik, 2013

Noh & Hwang, 2004

Relativistic evolution

Vs. Newtonian evolution

Correspondence: $\vartheta^i_k = \partial^i \partial_k v_N$, $\vartheta = \nabla^2 v_N$, $\delta_{\rm com} = \delta_{\rm N}$.

- Continuity + Raychaudhuri:

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- Extra GR: Energy and Momentum Constraints,

$$\vartheta^2 - \vartheta^i_j \vartheta^j_i + 4\mathcal{H}\vartheta + R = 16\pi G a^2 \bar{\rho}\delta\,, \qquad \vartheta^i_{j|i} = \vartheta_{,\,j}\,.$$

First order evolution

$\delta_N^{(1)} = \delta_{\rm com}^{(1)}$

- Continuity + Raychaudhuri = Evolution:

$$\delta^{(1)\prime\prime} + \mathcal{H}\delta^{(1)\prime} - \frac{3}{2}\mathcal{H}^2\Omega_m\delta^{(1)} = 0\,.$$

- Energy constraint = first integral:

$$4\mathcal{H}\delta^{(1)\prime} + 6\mathcal{H}^2\Omega_m\delta^{(1)} - R^{(1)} = 0.$$

$$\Rightarrow \qquad \boxed{\delta^{(1)}(\tau, \mathbf{x}) = \delta^{(1)}(\mathbf{x})D_+(\tau) + \delta_-^{(1)}(\mathbf{x})D_-(\tau)}$$

First order evolution

$\delta_N^{(1)} = \delta_{\rm com}^{(1)}$

- Continuity + Raychaudhuri = Evolution:

$$\delta^{(1)\prime\prime} + \mathcal{H}\delta^{(1)\prime} - \frac{3}{2}\mathcal{H}^2\Omega_m\delta^{(1)} = 0\,.$$

- Energy constraint ($D_+{}' = f_1(\Omega_m)\mathcal{H}D_+(\tau)$):

$$4\mathcal{H}^2 D_+ f_1 + 6\mathcal{H}^2 D_+ - 4\nabla^2\mathcal{R}_{\rm c} = 0.$$

$$\Rightarrow \boxed{\delta^{(1)}(\tau,\mathbf{x}) = \nabla^2\mathcal{R}_{\rm c}(\mathbf{x})\left[\frac{3}{2} + \frac{f_1(\Omega_{mIN})}{\Omega_{mIN}}\right]^{-1}\frac{D_+(\tau)}{\mathcal{H}_{IN}^2 D_{+IN}}}$$

$\Rightarrow$ Consistent Newtonian and Relativistic description!

Second order evolution
Initial conditions differ

- Continuity + Raychaudhuri:

$$\delta^{(2)\prime\prime}+\mathcal{H}\delta^{(2)\prime}-\frac{3}{2}\mathcal{H}^2\Omega_m\delta^{(2)} = -2\left(\delta^{(1)}\vartheta^{(1)}\right)^{\prime} - 2\mathcal{H}\delta^{(1)}\vartheta^{(1)} + 2\vartheta^{(1)i}{}_j\vartheta^{(1)j}{}_i.$$

- Constraint equations:

$$2\mathcal{H}\delta^{(2)\prime} + 3\mathcal{H}^2\Omega_m\delta^{(2)} - \frac{1}{2}R^{(2)} = \vartheta^{(1)2} - \vartheta^{(1)i}_j\vartheta^{(1)j}_i - 4\mathcal{H}\delta^{(1)}\vartheta^{(1)}.$$

$$R^{(2)\prime} = -4\vartheta^{(1)}{}^i{}_j R^{(1)}{}^j{}_i$$

- For Einstein-De Sitter (dust):

$$\boxed{\delta^{(2)}(\tau,\mathbf{x}) = \delta_h^{(2)}(\mathbf{x})D_+(\tau) + \delta_p^{(2)}(\mathbf{x})D_+(\tau)^2}$$

Relativistic evolution

homogeneous solution $\delta_h^{(2)}$

Bruni, JCH, Meures, Wands, (arxiv:1307.1478)

- Homogeneous part:

$$4\mathcal{H}\delta_h^{(2)\prime} + 6\mathcal{H}^2\Omega_m\delta_h^{(2)} - R_h^{(2)} = 0$$

- Primordial Non-Gaussianity $\mathcal{R}_c^{(2)} = \left(\frac{6}{5}f_{\rm NL} + 2\right)\mathcal{R}_c^2$ + G.R. contributions.

$$\begin{aligned}
\frac{1}{2}R_h^{(2)} &= -2\nabla^2\left[\left(\frac{6}{5}f_{\rm NL} + 2\right)\mathcal{R}_c^2\right] + 16\mathcal{R}_c\nabla^2\mathcal{R}_c + 6\partial^k\mathcal{R}_c\partial_k\mathcal{R}_c \\
&- 2\partial^k\nabla^2\chi^{(1)}\partial_k\mathcal{R}_c - \left[\partial^i\partial_j\chi^{(1)}\partial^j\partial_i\mathcal{R}_c + \nabla^2\chi^{(1)}\nabla^2\mathcal{R}_c\right] \\
&+ \frac{1}{4}\left[\partial^i\partial^j\partial^k\chi^{(1)}\partial_i\partial_j\partial_k\chi^{(1)} - \partial^k\nabla^2\chi^{(1)}\partial_k\nabla^2\chi^{(1)}\right]. \\
&= \mathrm{C}(\mathrm{x})
\end{aligned}$$

$$\chi^{(1)} = -\frac{4}{5}\frac{\mathcal{R}_c}{\mathcal{H}^2}$$

Relativistic evolution

homogeneous solution $\delta_h^{(2)}$

Bruni, JCH, Meures, Wands, (arxiv:1307.1478)

- Homogeneous part:

$$4\mathcal{H}\delta_h^{(2)\prime} + 6\mathcal{H}^2\Omega_m\delta_h^{(2)} - R_h^{(2)} = 0$$

- Primordial Non-Gaussianity + G.R. contributions.

$$\begin{aligned}
\delta_h^{(2)} &= -\frac{8}{5\mathcal{H}_0^2}\Bigg\{\left(\frac{3}{5}f_{\rm NL}+\frac{1}{4}\right)\partial^j\mathcal{R}_{\rm c}\partial_j\mathcal{R}_{\rm c} + \left(\frac{3}{5}f_{\rm NL}-1\right)\mathcal{R}_{\rm c}\nabla^2\mathcal{R}_{\rm c} \\
&+ \frac{1}{10\mathcal{H}_0^2}\left[\partial^k\nabla^2\mathcal{R}_{\rm c}\partial_k\mathcal{R}_{\rm c} + \frac{1}{2}\left(\partial^l\partial_j\mathcal{R}_{\rm c}\partial_l\partial^j\mathcal{R}_{\rm c} + \nabla^2\mathcal{R}_{\rm c}\nabla^2\mathcal{R}_{\rm c}\right)\right] \\
&+ \frac{1}{40\mathcal{H}_0^4}\left[\partial^l\partial^j\partial_k\mathcal{R}_{\rm c}\partial_l\partial_j\partial^k\mathcal{R}_{\rm c} - \partial^k\nabla^2\mathcal{R}_{\rm c}\partial_k\nabla^2\mathcal{R}_{\rm c}\right]\Bigg\}D_+(\tau)
\end{aligned}$$

Initial conditions
Relativistic vs Newtonian

- *Poisson* Equation at second order

$$\delta_N^{(2)} = \frac{\nabla^2 \Phi_N^{(2)}}{4\pi G a^2 \bar{\rho}}.$$

$$\frac{1}{2}\delta_{IN}^{(2)} = -\frac{6}{25}\frac{D_+(\tau_{IN})}{\mathcal{H}_0^2 \Omega_{m0}} \nabla^2 \left[f_{\rm NL} \mathcal{R}_{\rm c}^2 \right].$$

$$\delta_h^{(2)} = \tfrac{2}{5} R_h^{(2)}(x) D_+(\tau_{IN}).$$

$$\frac{1}{2}\delta_{IN}^{(2)} = -\frac{12}{25}\frac{D_+(\tau_{IN})}{\mathcal{H}_0^2 \Omega_{m0}} \times \left[\left(f_{\rm NL} - \frac{5}{3} \right) \mathcal{R}_{\rm c} \nabla^2 \mathcal{R}_{\rm c} + \left(f_{\rm NL} + \frac{5}{12} \right) \partial^k \mathcal{R}_{\rm c} \partial_k \mathcal{R}_{\rm c} \right]$$

- Relativistic contribution $\boxed{f_{\rm NL}^{GR} = -5/3}$ (Local N-G)

Consistency with: Bartolo et al. 2004,2010;

≈ Tomita 1967, Hwang *et* al. 2012, JCH, Christopherson, Malik, 2013

Initial conditions

Relativistic vs Newtonian

M. Bruni, JCH, N. Meures, D. Wands, (arXiv:1307.1478)

1) *Poisson* Equation at second order

$$\delta_N^{(2)} = \left(\frac{1}{4\pi G a^2 \bar{\rho}}\right) \nabla^2 \Phi_N^{(2)} .$$

$$\delta_{IN}^{(2)} = \frac{12}{25} \frac{1}{4\pi G a^2 \bar{\rho}} \nabla^2 \left[f_{\rm NL} \zeta^2 \right] .$$

$$\delta^{(2)} \supset \frac{1}{4\pi G a^2 \bar{\rho}} R_h^{(2)}(x) .$$

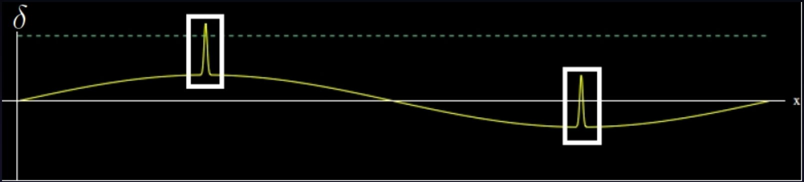
$$\delta_{IN}^{(2)} \supset \frac{12}{25} \frac{D_+(\tau_{IN})}{\mathcal{H}_0^2 \Omega_{m0}} \times \left[\left(f_{\rm NL} - \frac{5}{3} \right) \zeta \nabla^2 \zeta + \left(f_{\rm NL} + \frac{5}{12} \right) \partial^k \zeta \partial_k \zeta \right] .$$

- Relativistic contribution to Non-Gaussianity $\boxed{f_{\rm NL}^{GR} = -5/3}$

Consistency with: Bartolo et al. 2004,2010;
≈ Tomita 1967, Hwang *et* al. 2012, JCH, Christopherson, Malik, 2013

Observable Effects

General Relativity in Powerspectrum



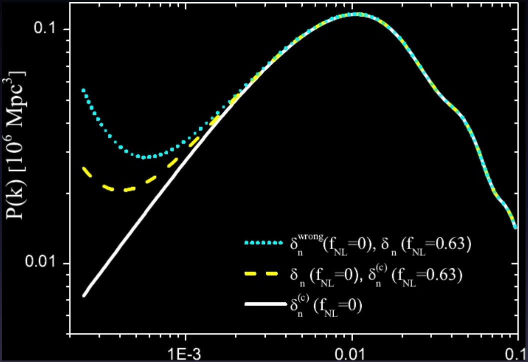
Effects

- Scale-dependent non-linear bias

$$\delta_{gal} = b\delta_{\rm DM} = (b_s + b_\ell)\delta_{\rm DM},$$

$$b = b_s + f_{\rm NL}[b-1]\left(\frac{8\pi G a^2 \bar{\rho}}{k^2 T(k)}\right)\delta_{\rm DM}\,.$$

Observable Effects
General Relativity in Powerspectrum



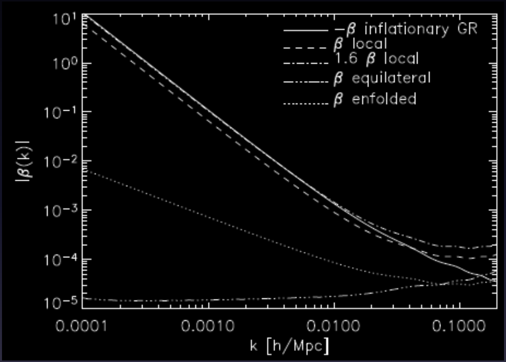
Dent & Dutta 2009,
Baldauf et al. 2011,
Bruni et al. 2012

Effects

- Scale-dependent non-linear bias

$$\delta_{gal} = b\,\delta_{\mathrm{DM}}\,, \qquad b = b_s + f_{\mathrm{NL}}[b-1]\left(\frac{8\pi G a^2\bar{\rho}}{k^2 T(k)}\right)\delta_{\mathrm{DM}}\,.$$

Observable Effects
General Relativity in Powerspectrum



Matarrese & Verde 2009,
Yoo & Seljak 2012

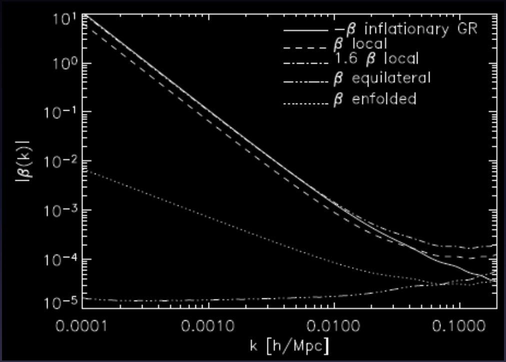
Effects

- Scale-dependent non-linear bias in CMB-Lensing

$$\delta_{gal} = b\,\delta_{\rm DM}\,, \qquad b = b_s + f_{\rm NL}[b-1]\left(\frac{8\pi G a^2 \bar{\rho}}{k^2 T(k)}\right)\delta_{\rm DM}\,.$$

Observable Effects

General Relativity in LSS



Dent & Dutta 2009, Matarrese & Verde 2009, Bruni et al. 2012

Effects

- Extra constraint $\nabla^2\Phi \neq 4\pi G\delta\rho$.
- Scale-dependent non-linear bias (in CMB-Lensing?)

$$\delta_n = b\delta_m, \quad b = b_1 + f_{\mathrm{NL}}\,(b-1)\frac{3\delta_\star \Omega_m \mathcal{H}_0^2}{k^2 T(k)}$$

Initial conditions

Relativistic effects on large scales

- *Solution* at all perturbative orders

$$\delta_h^{(n)} \supset \frac{2}{5} R_h^{(n)}(x) \left(\frac{1}{4\pi G a^2 \bar{\rho}} \right) .$$

- Split curvature in short modes ζ_s and long modes ζ_ℓ and drop gradients of long modes

$$g^{ij} = \exp(-2\zeta_\ell)\delta_s^{ij} + \mathcal{O}(\nabla\zeta_\ell) , \qquad R_h = \exp(-2\zeta_\ell) R_s + \mathcal{O}(\nabla\zeta_\ell)$$

$$\Rightarrow \boxed{\delta = \exp(-2\zeta_\ell)\delta_s + \mathcal{O}(\nabla\zeta_\ell)} .$$

- Recover Second-order result + Higher-order N-G:

$$\boxed{g_{\rm NL}^{GR} = \frac{50}{27} + \frac{20}{9} f_{\rm NL}} \quad \boxed{h_{\rm NL}^{GR} = -\frac{125}{81} + \frac{625}{243} f_{\rm NL}^2}$$

Bruni, JCH & Wands arXiv:1405.7006

Current and future work

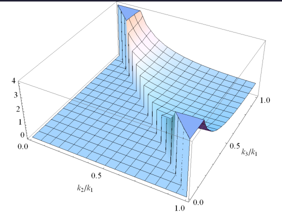
Relativistic constraints

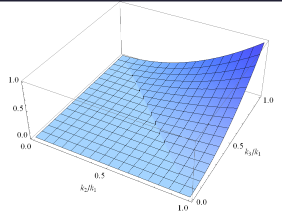
non-Gaussian correlations

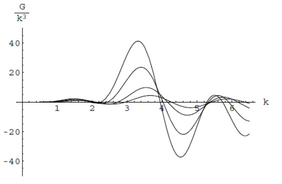
with Dagoberto Malagón

- Characterisation of the three point function (LTB)

$$\langle\delta(\boldsymbol{x_1})\delta(\boldsymbol{x_2})\delta(\boldsymbol{x_3})\rangle = \int d^3\boldsymbol{k_1}d^3\boldsymbol{k_2}d^3\boldsymbol{k_3}\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B(k_2/k_1,k_3/k_1)e^{i(\sum_\ell \boldsymbol{k_\ell x_\ell})}$$







Manera et al. 2012, Gil et al. 2014

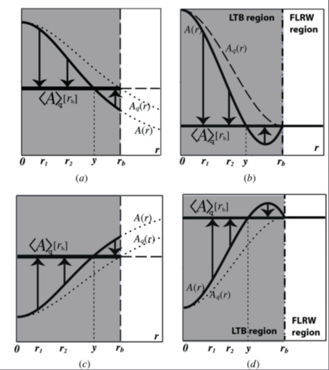
Relativistic evolution

exact solutions LTB

with R. Sussman and G. German. arXiv:1408.xxxx

- Inhomogeneous Solutions show non-linear regime of collapse (LTB)

$$ds^2 = -dt^2 + \frac{R(r,t)'^2}{1+K(r)r^2}dr^2 + R(r,t)^2 d\Omega^2 \,.$$



$$\rho_q = \frac{\int_0^r \rho R^2 R' dr}{\int_0^r R^2 R' dr}, \quad \delta_q^{(\rho)} = \frac{\rho - \rho_q}{\rho_q}.$$

$$\delta_{\rm CPT} = \frac{\rho_q}{\bar{\rho}}\left[1 + \delta_q^{(\rho)}\right] - 1.$$

Sussman 2013a,2013b

Relativistic evolution

exact solutions LTB

with R. Sussman and G. German

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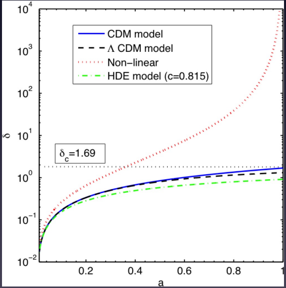
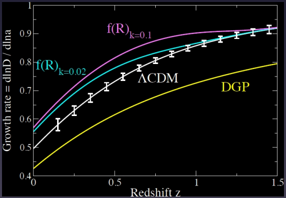




Linder & Cahn 2007. Naderik & Malekjani 2014

N-body simulations

Relativistic Input

with K. Malik

- Lagrangian Coordinates:

$$\mathbf{x} = \mathbf{q} \quad + \quad \mathbf{\Psi}(\mathbf{x}, \tau),$$

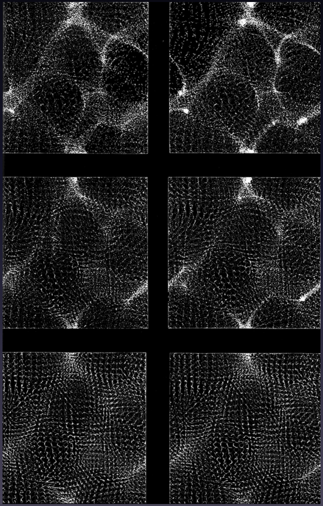
$$\Rightarrow 1 + \delta_N \quad = \quad \frac{\rho_0(\mathbf{x})}{\mathrm{Det}|1 + \partial_i \Psi_k(q, \tau)|}$$

- Relativistic solution:

$$1 + \delta = \frac{\rho_0(\mathbf{x})}{\sqrt{\mathrm{Det}[g_{ij}]}}$$

$$= \frac{\rho_0(\mathbf{x})}{\mathrm{Det}[e_i^{(j)}]}$$

- Add Dictionary entry? $e_i^{(j)} = 1 + \Psi^j{}_{,i}$



Relativistic Process of Structure Formation

Summary

- Presented relativistic non-linear solutions to ΛCDM inhomogeneities.
- Found consistency with the Newtonian cosmology at non-linear order (synchronous-comoving gauge), and solutions beyond the Zel'dovich approximation.
- Determined additional, purely relativistic, constraints to non-linear/non-Gaussian initial conditions,
  $f_{NL}^{GR} = -5/3, \quad g_{NL}^{GR} = -50/27$
- Propose relativistic initial conditions for n-body simulations and look at evolution of inhomogeneous cosmologies in this framework *In progress*

The Relativistic Process of Structure formation.

Modelling non-dust fluids in cosmology.
Adam J. Christopherson, JCH, Karim A. Malik; JCAP 1301 (2013) 002

The Poisson equation at second order in relativistic cosmology.
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