Quantum Entanglement in Plebański-Demiański Spacetimes

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I. Introduction

II. Plebański-Demiański Spacetime

III. Frame-dragging

IV. Spin precession

V. EPR correlation

VI. Spin precession angle in Expanding and Twisting Plebański-Demiański Black Hole

I. Introduction

For many years quantum states of matter in a classical gravitational background have been of great interest in physical models. Examples:

- Neutron interferometry in laboratories on the Earth. It captures the effects of the gravitational field into quantum phases associated to the possible trajectories of a beam of neutrons, following paths with different intensity of the gravitational field. (Colella et al 1975)

- Another instance of the description of quantum states of matter in classical gravitational fields is Hawking's radiation describing the process of black hole evaporation. This process involves relativistic quantum particles and uses quantum field theory in curved spacetimes.

II. Plebański-Demiański Spacetime

The metric

$$
\begin{aligned}
ds^2 = \; & \frac{1}{\Omega^2}\Bigg[-\frac{D}{\rho^2}\left(dt-(a\sin^2\theta+2l(1-\cos\theta))d\phi\right)^2+\frac{\rho^2}{D}dr^2 \\
& +\frac{P}{\rho^2}\left(adt-(r^2+(a+l)^2)d\phi\right)^2+\rho^2\frac{\sin^2\theta}{P}d\theta^2\Bigg],
\end{aligned} \tag{1}
$$

with

$$
\begin{aligned}
\rho^2 &= r^2+(l+a\cos\theta)^2, \\
\Omega &= 1-\frac{\alpha}{\omega}(l+a\cos\theta)r, \\
P &= \sin^2\theta(1-a_3\cos\theta-a_4\cos^2\theta), \\
D &= (\kappa+e^2+g^2)-2mr+\varepsilon r^2-2n\frac{\alpha}{\omega}r^3-\left(\frac{\alpha^2}{\omega^2}\kappa+\frac{\Lambda}{3}\right)r^4,
\end{aligned} \tag{2}
$$

and where

$$
\begin{aligned}
a_3 &= 2a\frac{\alpha}{\omega}m - 4al\frac{\alpha^2}{\omega^2}(\kappa + e^2 + g^2) - 4\frac{\Lambda}{3}al, \\
a_4 &= -a^2\frac{\alpha^2}{\omega^2}(\kappa + e^2 + g^2) - \frac{\Lambda}{3}a^2, \\
\varepsilon &= \frac{\kappa}{a^2 - l^2} + 4l\frac{\alpha}{\omega}m - (a^2 + 3l^2)\left(\frac{\alpha^2}{\omega^2}(\kappa + e^2 + g^2) + \frac{\Lambda}{3}\right), \\
n &= \frac{\kappa l}{a^2 - l^2} - (a^2 - l^2)\frac{\alpha}{\omega}m + (a^2 - l^2)l\left(\frac{\alpha^2}{\omega^2}(\kappa + e^2 + g^2) + \frac{\Lambda}{3}\right), \\
\kappa &= \frac{1 + 2l\frac{\alpha}{\omega}m - 3l^2\frac{\alpha^2}{\omega^2}(e^2 + g^2) - l^2\Lambda}{\frac{1}{a^2 - l^2} + 3l^2\frac{\alpha^2}{\omega^2}}.
\end{aligned}
\tag{3}
$$

paremeters: $\alpha, \omega, m, e, g, \Lambda, \ell, a$.

Equation (1) can be represented by the line element

$$ds^2 = g_{00}dt^2 + 2g_{03}dtd\phi + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2, \qquad (4)$$

which has a non-diagonal element that represents the axial symmetry and the metric coefficients are:

$$\begin{aligned}
g_{00} = & \frac{-D + Pa^2}{\Omega^2\rho^2}, \\
g_{0i}dx^i = & \frac{1}{\Omega^2}\left[\frac{D}{\rho^2}(a\sin^2\theta + 2l(1-\cos\theta)) - \frac{P}{\rho^2}a(r^2 + (a+l)^2)\right]d\phi, \\
g_{ij}dx^i dx^j = & \frac{\rho^2}{\Omega^2 D}dr^2 + \rho^2\frac{\sin^2\theta}{\Omega^2 P}d\theta^2 + \frac{1}{\Omega^2}\left[-\frac{D}{\rho^2}(a\sin^2\theta + 2l(1-\cos\theta))^2\right. \\
& \left. + \frac{P}{\rho^2}(r^2 + (a+l)^2)^2\right]d\phi^2.
\end{aligned} \qquad (5)$$

In order to describe the motion of spinning particles in a curved spacetime, the vierbein chosen is:

$$
\begin{aligned}
e_0{}^{\mu}(x) &= \frac{1}{\sqrt{-g_{00}}}(1,0,0,0), & e^0{}_{\mu} &= \sqrt{-g_{00}}\left(1,0,0,\frac{g_{03}}{g_{00}}\right), \\
e_1{}^{\mu}(x) &= \frac{1}{\sqrt{g_{11}}}(0,1,0,0), & e^1{}_{\mu} &= \sqrt{g_{11}}(0,1,0,0), \\
e_2{}^{\mu}(x) &= \frac{1}{\sqrt{g_{22}}}(0,0,1,0), & e^2{}_{\mu} &= \sqrt{g_{22}}(0,0,1,0), \\
e_3{}^{\mu}(x) &= \sqrt{\frac{-g_{00}}{g_{03}^2 - g_{00}g_{33}}}\left(-\frac{g_{03}}{g_{00}},0,0,1\right), & e^3{}_{\mu} &= \sqrt{\frac{g_{03}^2 - g_{00}g_{33}}{-g_{00}}}(0,0,0,1.),
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
e_a{}^{\mu}(x)e_b{}^{\nu}(x)g_{\mu\nu}(x) &= \eta_{ab}, \\
e^a{}_{\mu}(x)e_a{}^{\nu}(x) &= \delta_{\mu}{}^{\nu}, \\
e^a{}_{\mu}(x)e_b{}^{\mu}(x) &= \delta^a{}_b.
\end{aligned}
\tag{7}
$$

III. Frame-dragging

As seen by long distance observers, the hovering position has a four-velocity defined by

$$u_{\rm h}^{\mu} = (dt/d\tau,\ 0,\ 0,\ 0) = ((-g_{00})^{-1/2},\ 0,\ 0,\ 0).$$

For a free-falling particle the four-velocity due the frame-dragging, as seen by the same distant observers, is described by

$$u_{\rm fd}^{\mu} = \sqrt{\frac{-g_{33}}{g_{00}g_{33} - (g_{03})^2}} \left(1,\ 0,\ 0, -\frac{g_{03}}{g_{33}}\right).$$

In the present work, the frame-dragging velocity has to be measured by the hovering observer as a local inertial frame velocity and it can be obtained by projecting out the four-momentum $mu^{\mu}_{\rm fd}$ of the particle over the four-vector velocity $u_{{\rm h}\mu}$ of the hovering observer

$$u^{\mu}_{\rm fd} u_{{\rm h}\mu} = -E = -\gamma_{\rm fd}, \tag{8}$$

where E is the relativistic energy per unit mass of the particle with respect to a local (hovering) observer. Here $\gamma_{\rm fd} = (1 - v^2_{\rm fd})^{-1/2}$. Moreover $u^a_h = \eta^{ab} e_b{}^{\mu} u_{{\rm h}\mu}$ and therefore

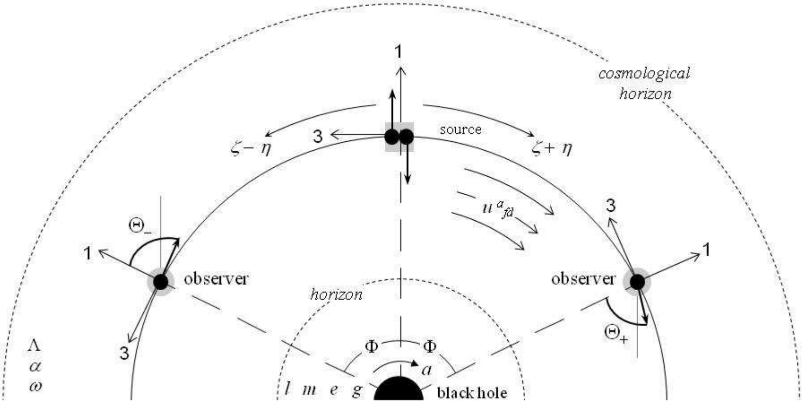
$$u^a_{\rm fd} = (\cosh\eta, 0, 0, \sinh\eta). \tag{9}$$

IV. Spin precession

Now we consider two observers and one EPR source on the equator plane $\theta = \pi/2$. The observers are placed at azimuthal angles $\phi = \pm\Phi$ and the EPR source is located at $\phi = 0$. The observers and the EPR source are assumed to be hovering.
From the perspective of a zero angular momentum observers (ZAMO), the local velocity of the entangled particles is given by

$$u^a_{\mathrm{EPR}} = (\cosh\zeta,\ 0,\ 0,\ \sinh\zeta), \tag{10}$$

where $v_{\mathrm{EPR}} = \tanh\zeta$ is the speed of particles in the local inertial frame of the ZAMO.


1
cosmological
horizon
3
source
$\zeta - \eta$
$\zeta + \eta$
$u^a{}_{f\hat{a}}$
3
Θ_-
1
1
observer
observer
horizon
Θ_+
Λ
α
ω
3
Φ
Φ
a
l m e g
black hole

After the pair of entangled spin-1/2 particles is generated at the EPR source, they leave it and follow a circular path around a black hole. In spherical coordinates on the equatorial plane $\theta = \pi/2$, the velocity of particles has two relevant components, the temporal one and the spatial one with ϕ-coordinate at constant radius r. Thus, for the hovering observer, the motion is measured by the proper-velocity with $v = \tanh \xi$. That is, $u^a = (\cosh \xi, 0, 0, \sinh \xi)$, therefore the general contravariant four-velocity is

$$\begin{aligned} u^t &= e_0{}^t \cosh \xi + e_3{}^t \sinh \xi, \\ u^\phi &= e_3{}^\phi \sinh \xi, \end{aligned} \tag{11}$$

such that $u^\mu u_\mu = -1$.

In order the particles describe circular motion, we must apply an external force that compensates both the centrifugal force and the gravity. The acceleration due to this external force is

$$a^{\mu}(x) = u^{\nu}(x)\nabla_{\nu}u^{\mu}(x). \tag{12}$$

On the equatorial plane the acceleration then becomes

$$\begin{aligned} a^{r} &= ({e_0}^{t})^2\Gamma^{1}_{0\,0}\cosh^2\xi \\ &\quad + \left[({e_3}^{t})^2\Gamma^{1}_{0\,0} + ({e_3}^{\phi})^2\Gamma^{1}_{3\,3} + 2{e_3}^{t}{e_3}^{\phi}\Gamma^{1}_{0\,3}\right]\sinh^2\xi \\ &\quad + 2{e_0}^{t}({e_3}^{\phi}\Gamma^{1}_{0\,3} + {e_3}^{t}\Gamma^{1}_{0\,0})\sinh\xi\cosh\xi, \end{aligned} \tag{13}$$

where $\Gamma^{\mu}_{\rho\,\sigma}$ are the usual Christoffel's symbols.

The change of the local inertial frame consists of a boost along the 1-axis and a rotation about the 2-axis calculated by

$$\chi^a{}_b(x) = -u^\nu \omega_\nu{}^a{}_b(x), \tag{14}$$

where the connection one-forms are defined as

$$\omega_\mu{}^a{}_b(x) = -e_b{}^\nu(x)\nabla_\mu e^a{}_\nu(x) = e^a{}_\nu(x)\nabla_\mu e_b{}^\nu(x). \tag{15}$$

In our particular situation, the connections of interest are given by:

$$\begin{aligned}
\omega_t{}^0{}_1 &= e_1{}^r e^0{}_t \Gamma^0_{0\,1} + e_1{}^r e^0{}_\phi \Gamma^3_{0\,1}, \\
\omega_t{}^1{}_3 &= e_3{}^t e^1{}_r \Gamma^1_{0\,0} + e_3{}^\phi e^1{}_r \Gamma^1_{0\,3}, \\
\omega_\phi{}^0{}_1 &= e_1{}^r e^0{}_t \Gamma^0_{1\,3} + e_1{}^r e^0{}_\phi \Gamma^3_{1\,3}, \\
\omega_\phi{}^1{}_3 &= e_3{}^t e^1{}_r \Gamma^1_{0\,3} + e_3{}^\phi e^1{}_r \Gamma^1_{3\,3}.
\end{aligned} \tag{16}$$

The relevant boost is described by

$$\chi^0{}_1 = -e_0{}^t e_1{}^r (e^0{}_t \Gamma^0_{0\,1} + e^0{}_\phi \Gamma^3_{0\,1}) \cosh\xi$$

$$-e_1{}^r \left[e_3{}^\phi (e^0{}_t \Gamma^0_{1\,3} + e^0{}_\phi \Gamma^3_{1\,3}) + e_3{}^t (e^0{}_t \Gamma^0_{0\,1} + e^0{}_\phi \Gamma^3_{0\,1}) \right] \sinh\xi,$$

while the rotation about the 2-axis is given by

$$\chi^1{}_3 = -e_0{}^t e^1{}_r (e_3{}^t \Gamma^1_{0\,0} + e_3{}^\phi \Gamma^1_{0\,3}) \cosh\xi$$

$$-e^1{}_r \left[e_3{}^t (e_3{}^t \Gamma^1_{0\,0} + e_3{}^\phi \Gamma^1_{0\,3}) + e_3{}^\phi (e_3{}^t \Gamma^1_{0\,3} + e_3{}^\phi \Gamma^1_{3\,3}) \right] \sinh\xi.$$

The infinitesimal Lorentz transformation

$$\lambda^a{}_b(x) = -\frac{1}{m}[a^a(x) p_b(x) - p^a(x) a_b(x)] + \chi^a{}_b(x). \qquad (17)$$

The boost along the 1-axis and the rotation about the 2-axis are respectively

$$
\begin{aligned}
\lambda^0{}_1 \quad &= e^1{}_r\left[(e_0{}^t)^2\Gamma^1_{0\,0}\cosh^2\xi + ((e_3{}^t)^2\Gamma^1_{0\,0} + (e_3{}^\phi)^2\Gamma^1_{3\,3} + 2e_3{}^t e_3{}^\phi\Gamma^1_{0\,3})\sinh^2\xi \right.\\
&\left. + 2e_0{}^t(e_3{}^\phi\Gamma^1_{0\,3} + e_3{}^t\Gamma^1_{0\,0})\sinh\xi\cosh\xi\right]\cosh\xi \\
&- e_0{}^t e_1{}^r(e^0{}_t\Gamma^0_{0\,1} + e^0{}_\phi\Gamma^3_{0\,1})\cosh\xi \\
&- e_1{}^r\left[e_3{}^\phi(e^0{}_t\Gamma^0_{1\,3} + e^0{}_\phi\Gamma^3_{1\,3}) + e_3{}^t(e^0{}_t\Gamma^0_{0\,1} + e^0{}_\phi\Gamma^3_{0\,1})\right]\sinh\xi, \\
\\
\lambda^1{}_3 \quad &= -e^1{}_r\left[(e_0{}^t)^2\Gamma^1_{0\,0}\cosh^2\xi + ((e_3{}^t)^2\Gamma^1_{0\,0} + (e_3{}^\phi)^2\Gamma^1_{3\,3} + 2e_3{}^t e_3{}^\phi\Gamma^1_{0\,3})\sinh^2\xi \right.\\
&\left. + 2e_0{}^t(e_3{}^\phi\Gamma^1_{0\,3} + e_3{}^t\Gamma^1_{0\,0})\sinh\xi\cosh\xi\right]\sinh\xi \\
&- e_0{}^t e^1{}_r(e_3{}^t\Gamma^1_{0\,0} + e_3{}^\phi\Gamma^1_{0\,3})\cosh\xi \\
&- e^1{}_r\left[e_3{}^t(e_3{}^t\Gamma^1_{0\,0} + e_3{}^\phi\Gamma^1_{0\,3}) + e_3{}^\phi(e_3{}^t\Gamma^1_{0\,3} + e_3{}^\phi\Gamma^1_{3\,3})\right]\sinh\xi.
\end{aligned}
\tag{18}
$$

The change of the spin is obtained by computing the infinitesimal Wigner rotation

$$\vartheta^i{}_k(x) = \lambda^i{}_k(x) + \frac{\lambda^i{}_0(x)p_k(x) - \lambda_{k0}(x)p^i(x)}{p^0(x) + m}. \tag{19}$$

In particular, the rotation about the 2-axis through a certain angle reads:

$$\begin{aligned}
\vartheta^1{}_3 \quad &= -e^1{}_r\left[(e_0{}^t)^2\Gamma^1_{0\,0}\cosh^2\xi + ((e_3{}^t)^2\Gamma^1_{0\,0} + (e_3{}^\phi)^2\Gamma^1_{3\,3} + 2e_3{}^t e_3{}^\phi\Gamma^1_{0\,3})\sinh^2\xi\right.\\
&\left. +\ 2e_0{}^t(e_3{}^\phi\Gamma^1_{0\,3} + e_3{}^t\Gamma^1_{0\,0})\sinh\xi\cosh\xi\right]\sinh\xi\\
&- e_0{}^t e^1{}_r(e_3{}^t\Gamma^1_{0\,0} + e_3{}^\phi\Gamma^1_{0\,3})\cosh\xi\\
&- e^1{}_r\left[e_3{}^t(e_3{}^t\Gamma^1_{0\,0} + e_3{}^\phi\Gamma^1_{0\,3}) + e_3{}^\phi(e_3{}^t\Gamma^1_{0\,3} + e_3{}^\phi\Gamma^1_{3\,3})\right]\sinh\xi\\
&+ \left(\frac{\sinh\xi}{\cosh\xi+1}\right)\left\{e^1{}_r\left[(e_0{}^t)^2\Gamma^1_{0\,0}\cosh^2\xi + ((e_3{}^t)^2\Gamma^1_{0\,0} + (e_3{}^\phi)^2\Gamma^1_{3\,3} + 2e_3{}^t e_3{}^\phi\Gamma^1_{0\,3})\sinh^2\xi\right.\right.\\
&\left. +\ 2e_0{}^t(e_3{}^\phi\Gamma^1_{0\,3} + e_3{}^t\Gamma^1_{0\,0})\sinh\xi\cosh\xi\right]\cosh\xi\\
&- e_0{}^t e_1{}^r(e^0{}_t\Gamma^0_{0\,1} + e^0{}_\phi\Gamma^3_{0\,1})\cosh\xi\\
&\left. -\ e_1{}^r\left[e_3{}^\phi(e^0{}_t\Gamma^0_{1\,3} + e^0{}_\phi\Gamma^3_{1\,3}) + e_3{}^t(e^0{}_t\Gamma^0_{0\,1} + e^0{}_\phi\Gamma^3_{0\,1})\right]\sinh\xi\right\}.
\end{aligned} \tag{20}$$

Finally, from the tetrad given in Eq. (6), it can be shown, after some algebra, that the previous expression (20), can be expressed as

$$\begin{aligned}\vartheta^{1}{}_{3} &= -\frac{\cosh(2\xi)}{2g_{00}\sqrt{g_{11}[(g_{03})^2 - g_{00}g_{33}]}}\left(g_{03}\frac{\partial g_{00}}{\partial r} - g_{00}\frac{\partial g_{03}}{\partial r}\right) \\ &- \frac{\sinh(2\xi)}{4g_{00}[(g_{03})^2 - g_{00}g_{33}]\sqrt{g_{11}}} \\ \times &\left[g_{00}\left(g_{33}\frac{\partial g_{00}}{\partial r} - g_{00}\frac{\partial g_{33}}{\partial r}\right) + 2g_{03}\left(g_{03}\frac{\partial g_{00}}{\partial r} - g_{00}\frac{\partial g_{03}}{\partial r}\right)\right].\end{aligned} \tag{21}$$

V. EPR correlation

First of all we recall that in the case of the curved spacetime, the one-particle quantum states $|p^a(x), \sigma; x\rangle$ transforms under a local Lorentz transformation as

$$U(\Lambda(x))|p^a(x), \sigma; x\rangle = \sum_{\sigma'} D^{(1/2)}_{\sigma'\sigma}(W(x))|\Lambda p^a(x), \sigma'; x\rangle, \tag{22}$$

where $W^a{}_b(x) \equiv W^a{}_b(\Lambda(x), p(x))$ is the so called local Wigner rotation.

If the frame-dragging is taken into account on the local inertial frame velocity u^a, it will affect the previous local velocity transformation and then the total velocity will be written as $u^a_\pm = (\cosh\xi_\pm, 0, 0, \sinh\xi_\pm)$, where $\xi_\pm = \zeta \pm \eta$. After a proper time $\Phi/u^\phi_\pm$, each particle reaches the corresponding observer. Thus the finite Wigner rotation can be written as

$$W^a{}_b(\pm\Phi, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta_\pm & 0 & \pm\sin\Theta_\pm \\ 0 & 0 & 1 & 0 \\ 0 & \mp\sin\Theta_\pm & 0 & \cos\Theta_\pm \end{pmatrix}, \tag{23}$$

where $\Theta_\pm = \frac{\Phi \vartheta^1{}_3}{u^\phi_\pm}$ or

$$\Theta_\pm = \frac{\Phi}{2\sqrt{-(g_{00})^3 g_{11}}} \left\{ \left(g_{03}\frac{\partial g_{00}}{\partial r} - g_{00}\frac{\partial g_{03}}{\partial r} \right) \frac{\cosh(2\zeta \pm 2\eta)}{\sinh(\zeta \pm \eta)} \right.$$

$$+ \left[g_{00}\left(g_{00}\frac{\partial g_{33}}{\partial r} - g_{33}\frac{\partial g_{00}}{\partial r} \right) + 2g_{03}\left(g_{03}\frac{\partial g_{00}}{\partial r} - g_{00}\frac{\partial g_{03}}{\partial r} \right) \right]$$

$$\left. \times \frac{\cosh(\zeta \pm \eta)}{\sqrt{(g_{03})^2 - g_{00}g_{33}}} \right\}.$$

Then the required Wigner rotation is given in the following form

$$D^{(1/2)}_{\sigma'\sigma}(W(\pm\Phi,0)) = \exp\left(\mp i\frac{\sigma_y}{2}\Theta_\pm\right), \qquad (24)$$

where σ_y is the Pauli matrix. Now we can define the four-momentum of the particle as seen by each hovering observer. Thus, the spin-singlet state for entangled particles is given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|p^a_+,\uparrow;0\rangle|p^a_-,\downarrow;0\rangle - |p^a_+,\downarrow;0\rangle|p^a_-,\uparrow;0\rangle], \qquad (25)$$

Therefore after the finite Wigner rotation, the new total quantum state is given by $|\psi'\rangle = W(\pm\Phi)|\psi\rangle$. Consequently in the local inertial frames at the corresponding positions $\phi = \Phi$ and $-\Phi$, each particle state can be written as

$$|p^a_\pm, \uparrow; \pm\Phi\rangle' = \cos\frac{\Theta_\pm}{2}|p^a_\pm, \uparrow; \pm\Phi\rangle \pm \sin\frac{\Theta_\pm}{2}|p^a_\pm, \downarrow; \pm\Phi\rangle, \quad (26)$$

$$|p^a_\pm, \downarrow; \pm\Phi\rangle' = \mp\sin\frac{\Theta_\pm}{2}|p^a_\pm, \uparrow; \pm\Phi\rangle + \cos\frac{\Theta_\pm}{2}|p^a_\pm, \downarrow; \pm\Phi\rangle. \quad (27)$$

Thus the entangled state is described by the combination

$$|\psi\rangle' = \frac{1}{\sqrt{2}}\left[\cos\left(\frac{\Theta_+ + \Theta_-}{2}\right)\left(|p^a_+, \uparrow; \Phi\rangle|p^a_-, \downarrow; -\Phi\rangle - |p^a_+, \downarrow; \Phi\rangle|p^a_-, \uparrow; -\Phi\rangle\right) + \sin\left(\frac{\Theta_+ + \Theta_-}{2}\right)\left(|p^a_+, \uparrow; \Phi\rangle|p^a_-, \uparrow; \Phi-\rangle + |p^a_+, \downarrow; \Phi\rangle|p^a_-, \downarrow; -\Phi\rangle\right)\right]. \quad (28)$$

It is easy to see that the final quantum state reads

$$|\psi\rangle'' = \frac{1}{\sqrt{2}}[\cos\Delta(|p_+^a,\uparrow;\Phi\rangle'|p_-^a,\downarrow;-\Phi\rangle' - |p_+^a,\downarrow;\Phi\rangle'|p_-^a,\uparrow;-\Phi\rangle') + \sin\Delta(|p_+^a,\uparrow;\Phi\rangle'|p_-^a,\uparrow;\Phi-\rangle' + |p_+^a,\downarrow;\Phi\rangle'|p_-^a,\downarrow;-\Phi\rangle')]. \tag{29}$$

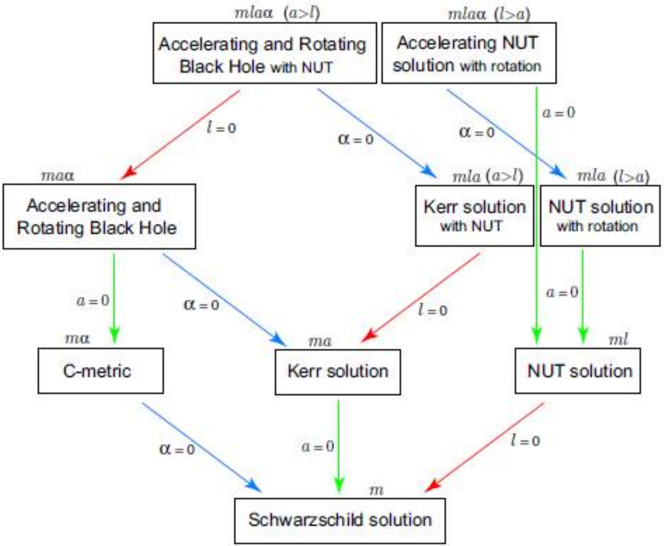
Here $\Delta = (\Theta_+ + \Theta_-)/2 - \Phi$ and Δ is given by

$$\Delta = \Phi\left[(2A\sinh\zeta + B\cosh\zeta)\cosh\eta - A\frac{\sinh\zeta\cosh\eta}{\cosh^2\eta - \cosh^2\zeta} - 1\right], \tag{30}$$

where

$$A = \frac{1}{2\sqrt{-(g_{00})^3 g_{11}}}\left(g_{03}\frac{\partial g_{00}}{\partial r} - g_{00}\frac{\partial g_{03}}{\partial r}\right),$$
$$B = \frac{1}{2\sqrt{-(g_{00})^3 g_{11}[(g_{03})^2 - g_{00}g_{33}]}} \times\left[g_{00}\left(g_{00}\frac{\partial g_{33}}{\partial r} - g_{33}\frac{\partial g_{00}}{\partial r}\right) + 2g_{03}\left(g_{03}\frac{\partial g_{00}}{\partial r} - g_{00}\frac{\partial g_{03}}{\partial r}\right)\right]. \tag{31}$$

VI. Spin precession angle in Expanding and Twisting Plebański-Demiański Black Hole



Now we study the spin precession angle of the spin-1/2 systems of entangled particles in the spacetime described by the Plebański-Demiański metric with frame-dragging.
Thus, it is easy to show that the coefficients A and B from the spin precession angle Δ on the equator ($\theta = \pi/2$), given by Eq. (30), are written as

$$\begin{aligned} A_{\rm PD} = & \frac{a\sqrt{D}}{2(r^2+l^2)(D-a^2)^{3/2}}[(r^2+l^2)D' - 2r(D-a^2)], \\ B_{\rm PD} = & \frac{1}{2(r^2+l^2)(D-a^2)^{3/2}} \\ & \times[4Dr(D-a^2) - (a^2r^2 + Dr^2 + Dl^2 + a^2l^2)D'], \end{aligned} \tag{32}$$

where D' is defined by

$$D' = \frac{\partial D}{\partial r} = -4\left(\frac{\alpha^2\kappa}{\omega^2} + \frac{\Lambda}{3}\right) r^3 - \frac{6n\alpha r^2}{\omega} + 2\varepsilon r - 2m. \tag{33}$$

The frame-dragging local inertial frame velocity is given by

$$\cosh\eta_{\rm PD} = (r^2+l^2)\sqrt{\frac{D}{(D-a^2)\left[(r^2+a^2+2al+l^2)^2 - (a+2l)^2D\right]}}.$$

Non-accelerating Kerr-Newman-(anti)de Sitter-NUT black hole

The KN(A)dSNUT spacetime represents a non-accelerating ($\alpha = 0$) black hole with mass m, electric and magnetic charges e and g, a rotation parameter a and a NUT parameter l in a de Sitter or anti-de Sitter background with non-vanishing cosmological constant Λ. This case contains in turn the two limits: $|a| > |l|$ and $|a| < |l|$, that correspond to the Kerr solution with NUT and the NUT solution with rotation respectively.

After setting the acceleration parameter equals zero i.e. $\alpha = 0$, the parameters in relation (3) become

$$\begin{aligned}\kappa &= (1 - l^2\Lambda)(a^2 - l^2),\\ \varepsilon &= 1 - \left(\frac{1}{3}a^2 + 2l^2\right)\Lambda,\\ n &= l + \frac{1}{3}(a^2 - 4l^2)l\Lambda.\end{aligned} \tag{35}$$

Thus, the metric (1) is reduced to

$$ds^2 = -\frac{D}{\rho^2}[dt - (a\sin^2\theta + 4l\sin^2\frac{\theta}{2})d\phi]^2 + \frac{\rho^2}{D}dr^2$$
$$+\frac{P}{\rho^2}[adt - (r^2 + (a+l)^2 d\phi)]^2 + \frac{\rho^2}{P}\sin^2\theta d\theta^2,$$

where

$$\begin{aligned}\rho^2 &= r^2 + (l + a\cos\theta)^2,\\ P &= \sin^2\theta\left(1 + \frac{4}{3}\Lambda al\cos\theta + \frac{1}{3}\Lambda a^2\cos^2\theta\right),\\ D &= a^2 - l^2 + e^2 + g^2 - 2mr + r^2 - \Lambda\left[(a^2 - l^2)l^2 + (\frac{1}{3}a^2 + 2l^2)r^2 + \frac{1}{3}r^4\right].\end{aligned}$$

(36)

In the case when $e = g = \Lambda = 0$ the spin precession angle explicitly written in terms of the physical parameters is given by

$$\Delta_{\mathrm{Kerr-NUT}} = \left\{ \frac{\cosh \eta_{\mathrm{Kerr-NUT}}}{(r^2 + l^2)(r^2 - 2mr - l^2)^{3/2}} \left[2a\sqrt{r^2 + a^2 - l^2 - 2mr}(mr^2 + 2l^2 r - ml^2) \sinh \zeta \right.\right. \\ + [r^5 - 5mr^4 + (6m^2 - 4l^2)r^3 + (10l^2 m - 2a^2 m)r^2 \\ + (-4a^2 l^2 + 3l^4 - 2m^2 l^2)r + 2a^2 l^2 m - l^4 m] \cosh \zeta \\ \left.\left. + a\sqrt{r^2 + a^2 - l^2 - 2mr}(mr^2 + 2l^2 r - ml^2) \frac{\sinh \zeta}{\cosh^2 \eta_{\mathrm{Kerr-NUT}} - \cosh^2 \zeta} \right] - 1 \right\} \Phi, \tag{37}$$

where

$$\cosh \eta_{\mathrm{Kerr-NUT}} = \frac{(r^2 + l^2)\sqrt{r^2 - 2mr + a^2 - l^2}}{\sqrt{(r^2 - 2mr - l^2)[r^4 + (a^2 - 2l^2)r^2 + (8mal + 8l^2 m + 2a^2 m)r + 5l^4 + 3a^2 l^2 + 8al^3]}}. \tag{38}$$

Kerr black hole

From equation (30) with coefficients (32) and parameters $e = g = l = \Lambda = 0$ the spin precession angle is reduced to

$$\Delta_{\rm K} = \Phi\left[\frac{-2\sqrt{D}am\sinh\zeta + (H - mD)\cosh\zeta}{(r^2 - 2mr)^{3/2}}\cosh\eta_{\rm K} + \frac{\sqrt{D}am}{(r^2 - 2mr)^{3/2}}\frac{\sinh\zeta\cosh\eta_{\rm K}}{\cosh^2\eta_{\rm K} - \cosh^2\zeta} - 1\right], \quad (39)$$

where

$$D = r^2 - 2mr + a^2, \quad (40)$$

$$H = r^3 - 4mr^2 + 4m^2 r - a^2 m, \quad (41)$$

$$\cosh\eta_{\rm K} = \frac{r\sqrt{D}}{\sqrt{(r-2m)(r^3 + a^2 r + 2ma^2)}} \neq 1, \quad (42)$$

with the coefficients A and B being

$$A_{\rm K} = -\frac{\sqrt{D}am}{(r^2 - 2mr)^{3/2}}, \qquad B_{\rm K} = \frac{H - mD}{(r^2 - 2mr)^{3/2}}. \quad (43)$$

Schwarzschild-NUT Black Hole

This is more clearly stated when the Eq. (30) is simplified by setting the parameters $e = g = a = \Lambda = 0$

$$\Delta_{\mathrm{NUT}} = \Phi\left(\frac{r^3 - 3mr^2 - 3l^2 r + ml^2}{(r^2+l^2)\sqrt{r^2 - 2mr - l^2}}\cosh\zeta\cosh\eta_{\mathrm{NUT}} - 1\right), \tag{44}$$

where

$$\cosh\eta_{\mathrm{NUT}} = \frac{r^2 + l^2}{\sqrt{r^4 - 2l^2 r^2 + 8ml^2 r + 5l^4}} \neq 1.$$

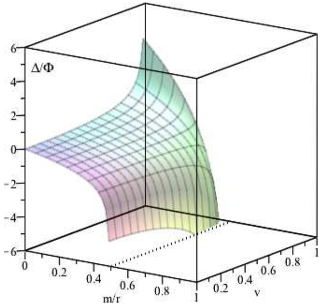
The coefficients A and B are given by

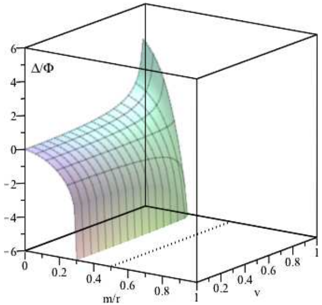
$$A_{\mathrm{NUT}} = 0, \qquad B_{\mathrm{NUT}} = \frac{r^3 - 3mr^2 - 3l^2 r + ml^2}{(r^2+l^2)\sqrt{r^2 - 2mr - l^2}}. \tag{45}$$

The precession angle Δ_{NUT} diverges precisely at $r = r_{\mathrm{NUT}}$, where

$$r_{\mathrm{NUT}} = m \pm \sqrt{m^2 + l^2}. \tag{46}$$

The positive root represents the outer Schwarzschild-NUT horizon.


Δ/Φ
m/r
v


Δ/Φ
m/r
v

Schwarzschild-(Anti)de Sitter Black Hole

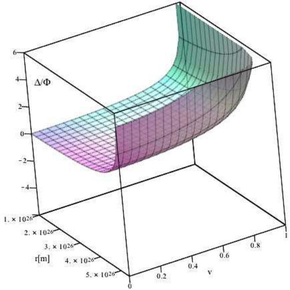
The spin precession angle in this case is given by

$$\Delta_{\mathrm{(A)dS}} = \Phi \left(\frac{r - 3m}{\sqrt{r^2 - 2mr - \frac{1}{3}\Lambda r^4}} \cosh\zeta - 1 \right), \tag{47}$$

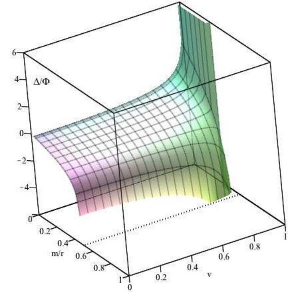
with A and B of the following form

$$A_{\mathrm{(A)dS}} = 0, \qquad B_{\mathrm{(A)dS}} = \frac{r - 3m}{\sqrt{r^2 - 2mr - \frac{1}{3}\Lambda r^4}} \tag{48}$$

and $\cosh\eta_{\mathrm{(A)dS}} = 1$.



6
4
2
0
-2
-4
Δ/Φ
1. × 10^26
2. × 10^26
3. × 10^26
4. × 10^26
5. × 10^26
r[m]
0
0.2
0.4
0.6
0.8
1
v



6
4
2
0
-2
-4
Δ/Φ
0
0.2
0.4
0.6
0.8
1
m/r
0
0.2
0.4
0.6
0.8
1
v

Reissner-Nordström Black Hole

This case corresponds to a Schwarzschild black hole with non-vanishing charges e and g, after setting l, Λ and a to zero.
The Reissner-Nordström spacetime is also a spherically symmetric solution.
The spin precession angle is then reduced to

$$\Delta_{\mathrm{RN}} = \Phi \left(\frac{r^2 - 3mr + 2e^2 + 2g^2}{r\sqrt{r^2 - 2mr + e^2 + g^2}} \cosh\zeta - 1 \right) , \qquad (49)$$

where the functions A and B are

$$A_{\mathrm{RN}} = 0, \qquad B_{\mathrm{RN}} = \frac{r^2 - 3mr + 2e^2 + 2g^2}{r\sqrt{r^2 - 2mr + e^2 + g^2}} \qquad (50)$$

and $\cosh\eta_{\mathrm{RN}} = 1$.
This result reproduces completely our previous result after adding the magnetic charge g.

Schwarzschild Black Hole

Finally it is easy to recover the Schwarzschild spin precession by setting $a, e, g, l, \Lambda = 0$. The coefficients and frame-dragging are reduced to $A_{\rm S} = 0$, $B_{\rm S} = (r - 3m)/\sqrt{r^2 - 2mr}$ and $\cosh \eta_{\rm S} = 1$. Consequently, the expression (30) is given by

$$\Delta_{\rm S} = \Phi \left(\frac{r - 3m}{\sqrt{r^2 - 2mr}} \cosh \zeta - 1 \right) , \qquad (51)$$

which is precisely Eq. (51) of Terashima-Ueda paper.

Accelerating and rotating black holes

In order to simplify our analysis we shall consider in this subsection the case of vanishing parameters $\Lambda = e = g = l = 0$.
with an arbitrary α and using the remaining scaling freedom to put $\omega = a$, then the Plebański-Demiański metric is reduced to

$$ds^2 = \frac{1}{\Omega^2}\left(-\frac{D}{\rho^2}[dt - a\sin^2\theta d\phi]^2 + \frac{\rho^2}{D}dr^2 + \frac{P}{\rho^2}(adt - (r^2+a^2)d\phi)^2 + \rho^2\frac{\sin^2\theta}{P}d\theta^2\right), \quad (52)$$

where the parameters (2) and (3) are given by

$$\begin{aligned} &\varepsilon = 1 - a^2\alpha^2, \\ &n = -a\alpha m, \\ &P = \sin^2\theta\left(1 - 2\alpha m\cos\theta + a^2\alpha^2\cos^2\theta\right), \end{aligned} \quad (53)$$

and

$$\begin{aligned} &\rho^2 = r^2 + a^2\cos^2\theta, \\ &\Omega = 1 - \alpha r\cos\theta, \\ &D = a^2 - 2mr + (1 - a^2\alpha^2)r^2 + 2\alpha^2 mr^3 - \alpha^2 r^4. \end{aligned} \quad (54)$$

The previous metric has four singularities when $\theta = \pi/2$, that is, we can factorize D as

$$D = (r - r_+)(r - r_-)(1 - \alpha^2 r^2), \tag{55}$$

where

$$r_\pm = m \pm \sqrt{m^2 - a^2}. \tag{56}$$

The acceleration horizon:

$$r_{\rm acc} = \frac{1}{\alpha}. \tag{57}$$

After some easy manipulations, the coefficients for the spin precession angle are given by

$$\begin{aligned} A_{\rm AccRot} &= \frac{a\sqrt{D}}{2r(D - a^2)^{3/2}}[rD' - 2(D - a^2)], \\ B_{\rm AccRot} &= \frac{1}{2r(D - a^2)^{3/2}}[4D(D - a^2) - r(a^2 + D)D'], \end{aligned} \tag{58}$$

where

$$D' = \frac{\partial D}{\partial r} = -2m + 2(1 - a^2\alpha^2)r + 6\alpha^2 m r^2 - 4\alpha^2 r^3 \qquad (59)$$

and the frame-dragging velocity is

$$\cosh \eta_{\mathrm{AccRot}} = r^2 \sqrt{\frac{D}{(D - a^2)\left[(r^2 + a^2)^2 - a^2 D\right]}}. \qquad (60)$$

One can show that we can recover the Kerr spacetime results reviewed in previous section, after setting vanishing acceleration ($\alpha = 0$). Therefore, we shall consider the effect of acceleration over the spin precession angle.

C-metric

From the pair of accelerated and rotating black holes represented by the metric, we can consider the limit in which $a \to 0$. In this case, the metric has the form of the C-metric and thus the coefficients reduce to

$$A_{\mathrm{C-metric}} = 0, \qquad B_{\mathrm{C-metric}} = \frac{\alpha^2 m r^2 + r - 3m}{\sqrt{(r^2 - 2mr)(1 - \alpha^2 r^2)}}, \tag{61}$$

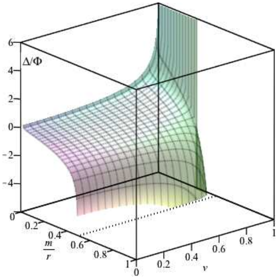
and

$$\cosh \eta_{\mathrm{C-metric}} = 1. \tag{62}$$

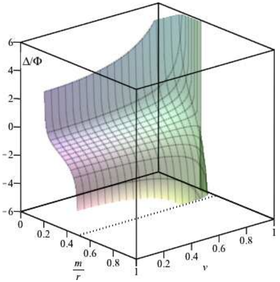
Then, the spin precession angle for the C-metric is

$$\Delta_{\mathrm{C-metric}} = \Phi \left(\frac{\alpha^2 m r^2 + r - 3m}{\sqrt{(r^2 - 2mr)(1 - \alpha^2 r^2)}} \cosh \zeta - 1 \right). \tag{63}$$

Moreover it is easy to see that this equation reduces to Schwarzschild case when $\alpha = 0$. In addition, we can see from Eq. (63) that it is divergent at the Schwarzschild radius $r = 2m$ and at the acceleration horizon, that is $\Delta_{\mathrm{C-metric}} \to \infty$ as $r \to \alpha^{-1}$.



Δ/Φ
m/r
v



Δ/Φ
m/r
v