Sasakian Manifolds, a new link between Thermodynamics and Gravity

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Contents

Why Geometry + Thermodynamics?

Several Reasons

- Infer microscopic properties
  Ruppeiner, Rev. Mod. Phys. **67**, *605, 1995*
- Emergence of Gravity
  Zhao, Commun. Theor. Phys. **54**, *641, 2010*
- Non-equilibrium TD
  Salamon and Berry, Phys. Rev. Lett. **51**, *1127, 1983*
  Crooks, Phys. Rev. Lett. **99**, *100602, 2007*
- AdS/CFT correspondence
  Martelli, Sparks and Yau, Commun. Math. Phys. **280**, *611-673, 2008*
  Boyer, Galicki, Matzeu, Commun. Math. Phys. **262**, *177-208, 2006*

Geometry of Thermodynamics

Geometry of Thermodynamics
Thermodynamic Phase Space

$\mathcal{T}^{2n+1}$ with coordinates $\{\tilde{w}, x^a, p_a\}$ and

$$\eta = \mathrm{d}\tilde{w} - p_a \mathrm{d}x^a$$

- Maximally non-integrable $\quad \eta \wedge (\mathrm{d}\eta)^n \neq 0$
- Contact distribution $\quad \mathcal{D} = \ker(\eta)$

Geometry of Thermodynamics
Space of equilibrium states

$\varphi : \mathcal{E} \longrightarrow \mathcal{T}$, Legendre submanifold defined by

$$\varphi^*(\eta) = 0 \qquad \mathrm{d}\tilde{w} = p_a \mathrm{d}x^a$$

- Fundamental equation $\tilde{w} = \tilde{w}(x^a)$
- EoS $p_a = \frac{\partial \tilde{w}}{\partial x^a}$
- Convexity $\frac{\partial^2 \tilde{w}}{\partial x^a \partial x^b} > 0$

Information Theory

Gibbs Distribution

$$S(\rho) = -\frac{\int \rho \ln \rho \, \mathrm{d}\Gamma}{\int \rho \, \mathrm{d}\Gamma} \tag{1}$$

$$\int \rho \, \mathrm{d}\Gamma = 1 \tag{2}$$

$$\frac{\int F^i \rho \, \mathrm{d}\Gamma}{\int \rho \, \mathrm{d}\Gamma} = x^i \quad i = 1, \ldots, n \tag{3}$$

$$\rho_G(\Gamma; w, p_1, \ldots, p_n) = \mathrm{e}^{-w + p_i F^i} \tag{4}$$

Microscopic Entropy Moments $\lambda_i = w, p_i$

$$s = -\ln\rho \tag{5}$$

$$\langle s(\rho)\rangle = \frac{\int s(\rho)\,\rho\,\mathrm{d}\Gamma}{\int \rho\,\mathrm{d}\Gamma} = S(\rho) \tag{6}$$

$$\langle \mathrm{d}s\rangle = -\left\langle \frac{\partial \ln\rho}{\partial\lambda_i}\right\rangle \mathrm{d}\lambda_i \tag{7}$$

$$\langle (\mathrm{d}s)^2\rangle = \left\langle \frac{\partial \ln\rho}{\partial\lambda_i}\frac{\partial \ln\rho}{\partial\lambda_j}\right\rangle \mathrm{d}\lambda_i \mathrm{d}\lambda_j \tag{8}$$

Microscopic Entropy Moments: $\rho = \rho_G$

$$s = w - p_i F^i \tag{9}$$

$$\langle s(\rho_G)\rangle_G = S(\rho_G) = w - p_i x^i \tag{10}$$

$$\langle \mathrm{d}s\rangle_G = \mathrm{d}w - x^i \mathrm{d}p_i \tag{11}$$

$$\begin{aligned}\langle (\mathrm{d}s)^2\rangle_G \;=\; & \left(\mathrm{d}w - x^i\,\mathrm{d}p_i\right)^2 \\ &+ \langle \left(F^i - x^i\right)\left(F^j - x^j\right)\rangle_G\,\mathrm{d}p_i\,\mathrm{d}p_j\end{aligned} \tag{12}$$

Relative Entropy

$$S(\rho, \rho_G) = -\int \rho_G \ln(\rho_G/\rho) = \langle \Delta s \rangle_G \qquad (13)$$

To first order (near $\rho = \rho_G$)

$$S(\rho, \rho_G) = \langle \mathrm{d}s \rangle_G \quad (= 0) \qquad (14)$$

To second order

$$S(\rho, \rho_G) = \frac{1}{2} \left\langle \frac{\partial \ln \rho}{\partial \lambda_i} \frac{\partial \ln \rho}{\partial \lambda_j} \right\rangle_G \mathrm{d}\lambda_i \mathrm{d}\lambda_j \qquad (15)$$

$\mathcal{T}^{2n+1}$ Sasakian manifold

$\mathcal{T}^{2n+1}$ Contact Riemannian manifold

Contact structure

$$\eta = \langle \mathrm{d}s \rangle_G = \mathrm{d}w - x^i \mathrm{d}p_i \tag{16}$$

Riemannian structure

$$\begin{aligned} G &= \langle (\mathrm{d}s)^2 \rangle_G \\ &= \eta \otimes \eta + \frac{1}{2}(\mathrm{d}x^i \otimes \mathrm{d}p_j + \mathrm{d}p_j \otimes \mathrm{d}x^i) \end{aligned} \tag{17}$$

$\mathcal{E}^n$ Legendre sub-manifold

First Law

$$\eta = \langle \mathrm{d}s \rangle_G = 0 \tag{18}$$

Thermodynamic metric

$$g = \langle (\mathrm{d}s)^2 \rangle_G \,|_{\langle \mathrm{d}s \rangle_G = 0} = \frac{\partial^2 w}{\partial p_a \partial p_b} \mathrm{d}p_a \otimes \mathrm{d}p_b \tag{19}$$

Reeb vector field

$$\eta(\xi) = 1 \quad \text{and} \quad \mathrm{d}\eta(\xi, X) = 0 \tag{20}$$

Almost contact structure

$$\Phi^2 = -I + \eta \otimes \xi \tag{21}$$

Associated metric

$$G(\Phi X, \Phi Y) = G(X, Y) - \eta(X)\,\eta(Y) \tag{22}$$

Sasakian and $\eta-$Einstein manifolds

Sasakian manifold

$$(C(\mathcal{T}), \bar{G}) \equiv (\mathbb{R}^+ \times \mathcal{T}, \mathrm{d}r^2 + r^2\, G) \text{ Kähler}$$

η-Einstein manifold

$$Ric_G = \lambda G + \nu \eta \otimes \eta$$

$\mathcal{T}^5$ Sasakian manifold

Theorem

$$(\mathcal{T}^5, \langle \mathrm{d}s \rangle_G, \langle (\mathrm{d}s)^2 \rangle_G)$$

is a Sasaki and η–Einstein manifold,

with $\lambda = -2$ and $\nu = 6$

Conclusions

Summary of new results

- First Law $\longleftrightarrow$ $\langle \mathrm{d}s \rangle_G = 0$
  (c.f. *Blanco, Casini, Hung, Myers, Relative entropy and holography, JHEP* **060**, *1308, 2013*)
- $(\mathcal{T}^5, \langle \mathrm{d}s \rangle_G, \langle (\mathrm{d}s)^2 \rangle_G)$ is Sasakian
- $(\mathcal{T}^5, \langle \mathrm{d}s \rangle_G, \langle (\mathrm{d}s)^2 \rangle_G)$ is η-Einstein

(Some) Future Directions

- Non-Equilibrium thermodynamics
(Phase Space for near-equilibrium systems is also a Sasaki manifold and the Sasaki-Ricci flow converges to a S-η-E)

- Applications in AdS/CFT
(S-η-E manifolds solutions of Type IIB supergravity with a cosmological constant)

- Applications in cosmology
(S-$\eta-$E manifolds can provide natural solutions to EFE with a cosmological constant, in five dimensions)

$AdS_5 \times Y^5$ with Y^5 S-E $\longleftrightarrow$ Type IIB supergravity

1. EFE with $\Lambda = 0$ require Y^5 to be Einstein
2. Supersymmetry requires Y^5 to be Sasaki
3. Symmetries of Y^5 $\longleftrightarrow$ Symmetries of the dual CFT

$AdS_5 \times \mathcal{T}^5$ with $\mathcal{T}^5$ S-η-E $\longleftrightarrow$ Type IIB supergravity

1. EFE with $\Lambda \neq 0$ require $\mathcal{T}^5$ to be η-Einstein
2. GB gravity requires $\mathcal{T}^5$ to be η-Einstein

Applications in Cosmology

$\mathcal{T}^5$ Sasaki and η-Einstein with $\lambda = -2$ and $\nu = 6$, then

$$R_{\mu\nu} + 2G_{\mu\nu} - 6\eta_\mu\eta_\nu = 0 \quad (\eta\text{-Einstein})$$

$$R_{\mu\nu} - \frac{1}{2}R\,G_{\mu\nu} + \Lambda G_{\mu\nu} = T_{\mu\nu} \quad (\text{EFE with } \Lambda)$$

$$R = 2n(1+\lambda) = -4 \quad (\text{Sasaki})$$

$$\Lambda G_{\mu\nu} + 6\eta_\mu\eta_\nu = -pG_{\mu\nu} + (p+\rho)u_\mu u_\nu$$

$$\eta_\mu = u_\mu \qquad p = -\Lambda \qquad \rho \sim -p \qquad (\Lambda\text{CDM})$$

GRACIAS!