Gravity localization, mass hierarchy, mass gap and corrections to Newton's law on thick branes

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Outline

Introduction

- Work in collaboration with Alfredo Herrera-Aguilar, Konstantinos Kanakoglou, Ulises Nucamendi and Israel Quiros (Gen.Rel.Grav. 46 (2014) 1631)
- Brane world scenarios have probe succes in adressing some problems in high energy problems.
- The model considered describes a thick brane world model arising in a 5D theory of gravity coupled to a self-interacting scalar field

Setup and solution

- We consider the 5D Riemannian action

$$S_5^W = \int d^5x\sqrt{|G|}\left[\frac{1}{4}R - \frac{1}{2}\left(\nabla\phi\right)^2 - V\left(\phi\right)\right] \tag{1}$$

  were ϕ is a bulk scalar field and $V(\phi)$ is a self–interacting potential and $M_*^3 = 1/8$ for the time being.

- We shall study solutions which preserve 4D Poincaré invariance with the metric

$$ds_5^2 = e^{2A(y)}\eta_{nm}dx^n dx^m + dy^2, \tag{2}$$

  where $e^{2A(y)}$ is the warp factor of the metric and depends just on the fifth dimension y; $m, n = 0, 1, 2, 3$.

Setup and solution

- Solution

$$A(y) = -b\ln\left[2\cosh(cy)\right] \tag{3}$$

$$\phi(y) = \sqrt{6b}\,\arctan\left[\tanh\left(\frac{cy}{2}\right)\right], \tag{4}$$

where the domain of the extra dimension is infinite $-\infty < y < \infty$

- The corresponding self–interacting potential is

$$V(\phi) = \frac{3c^2b}{8}\left[(1-4b) + (1+4b)\cos\left(2\sqrt{\frac{2}{3b}}\phi\right)\right]. \tag{5}$$

- This solution can be interpreted as a thick AdS domain wall located at $y_0 = 0$ with two free parameters: one for the width of the wall, given by c, and another for the AdS curvature, which is characterized by bc.

Fluctuations of the metric and gravity localization

- We study the metric fluctuations h_{mn} given by

$$ds_5^2 = e^{2A(y)} \left[\eta_{mn} + h_{mn}(x,y)\right] dx^m dx^n + dy^2. \tag{6}$$

- We perform the coordinate transformation $dz = e^{-A}dy$ in order to get a conformally flat metric and obtain the equation for the ttm of the metric fluctuations h^T_{mn}

$$\left(\partial_z^2 + 3A'\partial_z + \Box\right) h^T_{mn} = 0. \tag{7}$$

- We use the ansatz $h^T_{mn} = e^{ipx} e^{-3A/2}\Psi_{mn}(z)$ in order to recast as usual eq. (7) into a Schrödinger's equation form

$$[\partial_z^2 - V_{QM}(z) + m^2]\Psi = 0, \tag{8}$$

Fluctuations of the metric and gravity localization

- The analog quantum mechanical potential, which is completely defined by the curvature of the manifold, reads

$$V_{QM}(z) = \frac{3}{2}\partial_z^2 A + \frac{9}{4}(\partial_z A)^2. \tag{9}$$

- The warp factor determines the dynamics of the KK gravitational fluctuations
- The spectrum of eigenvalues m^2 parameterizes the spectrum of graviton masses that a 4D observer standing at $z = 0$ sees
- The analysis of the physical properties of the graviton spectrum predicted by Eq. (8), allow us to establish whether localization of 4D gravity is posible

Localization of gravity for the solution

- In the particular case when $b = 1$ we can express the variable y in terms of z by the following relation

$$2\cosh(c(y - y_0)) = \sqrt{4 + c^2 z^2}. \qquad (10)$$

- The warp factor adopts the form

$$A(z) = -\ln\sqrt{4 + c^2 z^2}; \qquad (11)$$

- The quantum mechanical potential reads

$$V_{QM}(z) = \frac{3c^2}{4}\frac{5c^2 z^2 - 8}{(4 + c^2 z^2)^2} \qquad (12)$$

Localization of gravity for the solution

- The quantum mechanical potential has the form of a volcano potential with finite bottom which asymptotically vanishes.
- Indicating that there exists a single normalizable bound state with no mass gap in the graviton spectrum of KK fluctuations for this particular case.
- And a infine tower of massive KK modes.
- For this special solution to our model the 5D Ricci scalar is regular everywhere as can be seen from its expression:

$$R = -8e^{-2A}\left(A_{zz} + \frac{3}{2}A_z^2\right) = 4a^2\frac{8-5a^2z^2}{4+a^2z^2}, \qquad (13)$$

- leading to a 5D manifold which is completely free of naked singularities.

Localization of gravity for the solution

- In order to analytically study the spectrum of massive KK modes for this solution of the model we exactly solve the following Schrödinger equation.

$$[\partial_z^2 - \frac{3c^2}{4}\frac{5c^2z^2-8}{(4+c^2z^2)^2} + m^2]\Psi = 0. \tag{14}$$

- The way to obtain the solution for this equation with arbitrary m is the following: we go to the complex realm and solve.
- Then come back to the original real variables and analyze in closed form the corrections to Newton's law an solve the hierarchy problem.

Confluent Heun equation in the ince limit

- The confluent Heun equation is a generalization of the Hypergeometric equation.
- Some particular solutions have been found in the literature.
- We will look at the Ince's limit of the confluent Heun equation, or the generalized spheroidal wave equation given by

$$w(w-w_0)\frac{d^2U}{dw^2}+(B_1+B_2w)\frac{dU}{dw}+[B_3+q(w-w_0)]U=0 \quad (15)$$

- were $q \neq 0$, if $q = 0$ the Heun equation can be transformed in to a hypergeometric equation.

Exact solutions for the localization of gravity in the solution

- By applying the following transformations for the parameter $a = i\alpha$, the fifth coordinate $w = \alpha^2 z^2/4$, where the domain of the new coordinate is $0 \leq w < \infty$, and the wave function

$$\Psi = (1 - w)^{-3/4} U(w), \tag{16}$$

- We recast the Schrödinger equation (8) into the Ince's limit of the confluent Heun equation, or of the generalized spheroidal wave equation with $w_0 = 1$, $B_1 = -1/2$, $B_2 = -1$, $B_3 = 0$ and $q = m^2/\alpha^2$;

Exact solutions for the localization of gravity in B solution

- The equation adopts the form

$$w(w-1)\frac{d^2U}{dw^2} - \left(\frac{1}{2}+w\right)\frac{dU}{dw} + \frac{m^2}{\alpha^2}(w-1)\,U = 0, \qquad (17)$$

  where $w = 0$ and $w = w_0 = 1$ are regular singularities and infinity is an irregular one.
- We require that solutions (with arbitrary mass $m > 0$) behave at infinity as the so-called subnormal Thomé solutions

$$\lim_{w\to\infty} U(w) \sim e^{\pm 2i\sqrt{qw}} w^{(1/4)-(B_2/2)} = e^{\pm 2i\frac{m}{|\alpha|}\sqrt{w}} w^{3/4}, \qquad (18)$$

- This particular behaviour of the solutions corresponds to the description of plane waves at spatial infinity in the extra dimension by the wave function (16).

Explicit solutions

- In our case, the pair of solutions adopts the following form:

$$U^0 = \sum_{n=-\infty}^{\infty} b_n \; F\left(-n-\nu-\frac{3}{2}, n+\nu-\frac{1}{2}; -\frac{3}{2}; 1-w\right),$$

$$U^\infty = w \sum_{n=-\infty}^{\infty} b_n \; K_{2n+2\nu+1}\left(\pm 2i\frac{m}{|\alpha|}\sqrt{w}\right), \tag{19}$$

- The coefficients b_n obey three–term recurrence relations

$$\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \tag{20}$$

Explicit solutions

- The coefficients can be calculated with the aid of the following quantities

$$\alpha_n = \frac{m^2\left(n+\nu+\frac{5}{2}\right)(n+\nu+2)}{\alpha^2(n+\nu+1)\left(n+\nu+\frac{3}{2}\right)},$$
$$\beta_n = -2\frac{m^2}{\alpha^2} + 4\left(n+\nu+\frac{3}{2}\right)\left(n+\nu-\frac{1}{2}\right) - \frac{3m^2}{\alpha^2(n+\nu)(n+\nu+1)},$$
$$\gamma_n = \frac{m^2\left(n+\nu-\frac{3}{2}\right)(n+\nu-1)}{\alpha^2(n+\nu)\left(n+\nu-\frac{1}{2}\right)}.$$

Explicit solution

- Computing an explicit solution like (19) involves the calculation of the coefficients b_n, and possess a dominant and a minimal solution.
- We produce two minimal solutions for the recurrence relations.
- These two solutions are pasted in the origin and normalized accordingly.
- The minimal solution thus constructed guarantees the convergence of the two–sided infinite series and must be chosen as a physical solution to the Ince's limit of the confluent Heun equation of our problem.

Explicit solution

- The zero mode can easily be computed by looking at the solution in the massless case $m = 0$ $(q = 0)$.
- The parameters α_n and γ_n vanish and the remaining parameters β_n are all non–zero, a fact which implies that all the coefficients $b_n = 0$.
- The only bound state of the system is given by the following eigenfunction

$$\Psi_0 = \frac{k_0}{(4 + c^2 z^2)^{3/4}}, \qquad k_0 = \text{const.} \tag{21}$$

- This mode corresponds to the normalizable 4D graviton, free of tachyonic instabilities, as expected from the solution of the massless Schrödinger equation $\Psi_0 \sim e^{3A/2}$.

Explicit solution

- The pair of solutions of the Schrödinger equation in the language of the parameter a and the original coordinate z, transforms into

$$\Psi^0 = \left(1+\frac{a^2z^2}{4}\right)^{-3/4} \times \tag{22}$$

$$\sum_{n=-\infty}^{\infty} b_n \; F\left(-n-\nu-\frac{3}{2}, n+\nu-\frac{1}{2}; -\frac{3}{2}; 1+\frac{a^2z^2}{4}\right),$$

$$\Psi^\infty = -\frac{a^2z^2}{4}\left(1+\frac{a^2z^2}{4}\right)^{-3/4} \sum_{n=-\infty}^{\infty} b_n \; K_{2n+2\nu+1}\left(\pm imz\right) \tag{23}$$

with the same coefficients α_n, β_n and γ_n.

- The asymptotic behaviour of these solutions, as physically expected, corresponds to plane waves:

$$\lim_{z\to\infty} \Psi(z) \sim e^{\pm imz} z^{3/2} \left(4+a^2z^2\right)^{-3/4} \sim e^{\pm imz}. \tag{24}$$

KK corrections to Newton's law

- We consider the thin brane limit $\beta \to \infty$, where we can locate two test bodies at the center of the brane in the transverse direction.
- The corrections to Newton's law in 4D flat spacetime coming from the fifth dimension

$$U(r) \sim \frac{M_1 M_2}{r}\left(G_4 + M_*^{-3}\sum_i e^{-m_i r}|\Psi_i(z_0)|^2 + \right. \tag{25}$$

$$\left. + M_*^{-3}\int_{m_0}^{\infty} dm\, e^{-mr}|\Psi^{\mu(m)}(z_0)|^2\right)$$

$$= \frac{M_1 M_2}{r}\Big(G_4 + \Delta G_4\Big), \tag{26}$$

where G_4 is the 4D gravitational coupling, Ψ_i represents the wave functions of the discrete excited states with mass m_i, and $\Psi^{\mu(m)}$ denotes the continuous eigenfunctions.

KKcorrections to Newton's law

- In the case of the presented solution there are no discrete massive modes ence no contribution from excited discrete states.
- We evaluate the massive wave functions (solutions to the Heun equations)at $z_0 = 0$, and compute the corresponding series.
- We get the following corrections to Newton's law in this particular case:

$$\Delta G_4 \sim \quad M_*^{-3} \frac{1}{2\pi a^2 r^2} \left(1 + O\left(\frac{1}{(ar)^2}\right)\right), \tag{27}$$

a result that coincides with the RS correction up to a factor of $1/2\pi$.

Mass Hierarchy Problem

- To adress the mass hierarchy problem we add a thin positive tension probe brane some distance away from the location of the thick (Planck) brane
- The 4D gravity is bound on the planck brane and the SM particles are trapped in the (thin, positive tension) probe brane.
- We look at the curvature termfrom wich one can derive the scale of the gravitational interactions

$$S_{eff} \supset 2M_*^3 \int d^4x \int_{-\infty}^{\infty} dz \sqrt{|\overline{g}|} e^{3A} \overline{R}, \tag{28}$$

where we performed the coordinate transformation $dy = e^A dz$, and $\overline{R}$ is the 4D Ricci scalar.

Mass Hierarchy Problem

- We stablish the connection between the planck scales in 4D and 5D.

$$M_{pl}^2 = 2M_*^3 \int_{-\infty}^{\infty} dz e^{3A} = \frac{M_*^3}{\beta}, \tag{29}$$

- Which is finite and for the presented solution we have $\beta = c/2$.
- If we take $M_* \sim \beta \sim M_{pl}$, then the zero mode Ψ_0 is coupled correctly to generate 4D (Newtonian) gravity.
- Also the thickness of the brane is inversely proportional to the Planck mass $\Delta \sim 1/c \approx M_{pl}^{-1}$ it is extremely small to be resolved by 4D observers located at the TeV probe brane.

Conclusions

- We consider a particular scalar thick brane generalizations of the RS model in which 4D gravity is localized.
- For the linear metric perturbations we obtain analytic expressions for the lowest energy eigenfunction which represents a single bound state that can be interpreted as a stable 4D graviton free of tachyonic modes.
- The continuum spectrum of massive modes of KK excitations are explicitly given in terms of of two–sided infinite series of hypergeometric functions and modified Bessel functions, allowing for analytical computations of corrections to Newton's law.
- The presented solution represents an original application of the Ince's limit of the confluent Heun equation within the framework of thick braneworld models, generated by gravity coupled to a bulk scalar field.