Chapter 1
Development of Linear Canonical Transforms: A Historical Sketch

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Abstract Linear canonical transformations (LCTs) were introduced almost simultaneously during the early 1970s by Stuart A. Collins Jr. in paraxial optics, and independently by Marcos Moshinsky and Christiane Quesne in quantum mechanics, to understand the conservation of information and of uncertainty under linear maps of phase space. Only in the 1990s did both sources begin to be referred jointly in the growing literature, which has expanded into a field common to applied optics, mathematical physics, and analogic and digital signal analysis. In this introductory chapter we recapitulate the construction of the LCT integral transforms, detailing their Lie-algebraic relation with second-order differential operators, which is the origin of the metaplectic phase. Radial and hyperbolic LCTs are reviewed as unitary integral representations of the two-dimensional symplectic group, with complex extension to a semigroup for systems with loss or gain. Some of the more recent developments on discrete and finite analogues of LCTs are commented with their concomitant problems, whose solutions and alternatives are contained the body of this book.

1.1 Introduction

The discovery and development of the theory of linear canonical transforms (LCTs) during the early seventies was motivated by the work on two rather different physical models: paraxial optics and nuclear physics. The integral LCT kernel was written as a descriptor for light propagation in the paraxial régime by Stuart A. Collins Jr., working in the ElectroScience Laboratory of Electrical Engineering at Ohio State University. On the other hand, Marcos Moshinsky and his postdoctoral associate Christiane Quesne, theoretical physicists at the Institute of Physics of the Universidad Nacional Autónoma de México, while working among other problems on the alpha clustering and decay of radioactive nuclei, saw LCTs as

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the key to understand the conservation of uncertainty as a matter of intrinsic mathematical interest. Some two decades elapsed before the two currents of research acknowledged each other. For this reason alone, the 45-year history of LCTs could provide an interesting case study on the intertwining of basic and applied endeavors. The more recent trend towards the analysis of discrete and finite data sets such as computers can handle also evinces a bifurcation between the search for efficient algorithms and the quest for subtler constructions based on symmetry. Usually mathematics yields more results than can be useful for applications. Applications have generated admirable technology, while symmetry catches the eye and pleases the mind.

The two seminal papers on LCTs, of Collins [1], and of Moshinsky and Quesne [2–4], are highly referenced ($>$657 and $>$390 joint citations, respectively, 11/10/15). Yet closer analysis shows that the authors who cited each of them have been mostly disjoint up to recent years: there was an optics community and a theoretical physics community, each with its own preferred journals, interests, and working styles [5]. The author's [6] grievously omits Collins' work—and any reference to optics as well. Fortunately, during the early eighties a mathematician colleague brought to my attention a series of papers by Nazarathy, Shamir, and Hardy on linear systems with loss or gain [7–11], and the work of Alex J. Dragt (University of Maryland) and several of his collaborators [12, 13] who had been developing techniques to control charged particle beams for the Superconducting Supercollider project [14, 15], which started a learning process on optical systems seen as a group-theoretical construct.

It should not be a matter of apology to focus this introductory chapter toward a review of LCTs seen from a more mathematical perspective. Section 1.2 contains the Collins and Moshinsky–Quesne approaches to LCTs, and the context in which our local research continued to develop. Thus, Sect. 1.3 reviews the salient properties of LCTs as integral transform realizations of the *double cover* of the group $\mathsf{Sp}(2,\mathsf{R})$ of 2×2 real matrices of unit determinant, and as generated by an algebra of second-order differential operators in Sect. 1.4. Section 1.5 recapitulates the radial and the (lesser-known) hyperbolic LCTs, geared to answer the question "*what are LCTs?*" In that section we propose what seems to be the proper context to accommodate all realizations ("faces") of integral and (infinite) matrix LCTs. Section 1.6 recalls complex extensions of LCTs that can be made unitary, such as heat diffusion, and a hint of applications to special function theory. Realizations of LCTs as finite matrices are addressed in Sect. 1.7 because there is a growing interest in fast algorithms to digitally treat LCTs for finite signals or pixelated images, where several tactics have been proposed to handle them, and on which I add a few words in the concluding Sect. 1.8. Here too it seems that at least two schools of thought contend, one strives for æsthetics and the other for efficacy.

1.2 Diffraction Integrals, Uncertainty Relations

Geometric and wave optics, as well as classical and quantum mechanics, agree with each other in the linear approximation—except for complex phases. It should be evident therefore that the paraxial régime of optics and quadratic systems in mechanics are closely related in their mathematical structure. They are both Hamiltonian systems whose waveforms, or states in any number of dimensions, can be displayed on a flat phase space. There, evolution is canonical (keeping the symplectic structure invariant) and linear (consisting only of translations, rotations, and shears). In paraxial wave optics, shears of phase space result from thin lenses and empty spaces, which, respectively, multiply the input functions by quadratic phases, and subject them to an isotropic Fresnel integral transform. In quantum mechanics on the other hand, beside the shear of free propagation, the harmonic oscillator is the most privileged actor; it generates a fractional Fourier transform on the initial state—times a phase.

1.2.1 Matrix Representation of Paraxial Optical Systems

The evolution in linear systems can be represented mathematically in three ways: by linear operators, by integral kernels, and by finite or infinite matrices. These will act on the states of the system, which in turn are realized, respectively, as differentiable and/or integrable functions of position (or momentum, or other observables), and as finite- or infinite-dimensional vectors. Since LCTs form a *group*, there will be locally a 1:1 correspondence between the three realizations, so one can use the algebraically simpler finite matrix realization to compute products and actions. Many authors point to the books by Willem Brouwer [16] and by Gerrard and Burch [17] for introducing the use of matrix algebra to paraxial optical design for resonators and the evolution of Gaussian beams therein. In two-dimensional ($2D$) optics, free propagation by z is represented by the 2×2 matrix $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, and a thin lens of focal distance f by $\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$; these act on rays represented by a two-vector $\begin{pmatrix} x \\ p \end{pmatrix}$, where x is the *position* of the ray on the $z = 0$ *screen*, and $p = n \sin\theta \approx n\theta$ is the *momentum* of the ray that crosses the screen with the "small" angle θ to its normal, in a transparent optical medium of refractive index n. In the paraxial régime one lets the phase space coordinates (x, p) roam over the full plane R^2. Products of these matrices correspond with the concatenation of the optical elements, and every paraxial $2D$ optical system is thus represented by a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1$ because the two generator matrices have unit determinant.

The paper by Stuart A. Collins, Jr. [1] considered the generic 3D paraxial, generally nonsymmetric but centered and aligned system.[1] These systems are represented by a 4×4 matrix $\mathbf{M} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$,

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{p}' \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix}, \quad \text{i.e.,} \quad \mathbf{w}' = \mathbf{M}\mathbf{w}, \tag{1.1}$$

where $\mathbf{w} := \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix}$, with $\mathbf{x}, \mathbf{x}'$, $\mathbf{p}, \mathbf{p}'$ being 2-vectors, and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are the 2×2 submatrices of $\mathbf{M}$. Since free propagation of an input function $f(\mathbf{x})$ is described by the Fresnel transform, whose integral kernel has a quadratic phase, and thin lenses multiply the function by a quadratic phase also, one should guess that the output $f_{\mathrm{M}}(\mathbf{x})$ of an $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$-transform should be an integral transform which, for the generic N-dimensional case is

$$f_{\mathrm{M}}(\mathbf{x}) \equiv (\mathcal{C}_{\mathrm{M}} f)(\mathbf{x}) := \int_{\mathsf{R}^N} \mathrm{d}^N\mathbf{x}'\, C_{\mathrm{M}}(\mathbf{x}, \mathbf{x}') f(\mathbf{x}'), \tag{1.2}$$

with a quadratic phase kernel $C_{\mathrm{M}}(\mathbf{x}, \mathbf{x}')$ in the components of $\mathbf{x}$ and $\mathbf{x}'$, and the matrix parameters of $\mathbf{M}$. The Collins paper considers transverse scalar fields $E_i = A_i \exp(\mathrm{i}kL_i)$ in each element of the optical setup, using the Fermat principle to show how the eikonal (optical distance) can be expressed in terms of the initial and final ray positions and slopes.[2] The resulting linear relations between these two 4-vectors with the parameters of the optical system turn out to be equivalent to the definition of *symplectic* matrices, whose generic form is

$$\mathbf{M}\,\boldsymbol{\Omega}\,\mathbf{M}^\top = \boldsymbol{\Omega}, \qquad \boldsymbol{\Omega}^\top = -\boldsymbol{\Omega}, \quad \boldsymbol{\Omega}^2 = -\mathbf{1}, \tag{1.3}$$

where the skew-symmetric *metric* matrix $\boldsymbol{\Omega}$ is usually written as $\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$. In the 2×2 submatrix form (1.1), this is

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{a}^\top & \mathbf{c}^\top \\ \mathbf{b}^\top & \mathbf{d}^\top \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{1.4}$$

which implies that the following submatrix products are symmetric,

$$\mathbf{a}\,\mathbf{b}^\top = (\mathbf{a}\,\mathbf{b}^\top)^\top, \quad \mathbf{c}\,\mathbf{d}^\top = (\mathbf{c}\,\mathbf{d}^\top)^\top, \quad \mathbf{a}\,\mathbf{d}^\top - \mathbf{b}\,\mathbf{c}^\top = \mathbf{1}. \tag{1.5}$$

[1]All lens centers are assumed to be on a common straight optical axis with their planes orthogonal to it; the "center" of cylindrical lenses is a line that should also intersect this axis. The consideration of displacement and (paraxial) tilt can be made using 2 + 2 more parameters for *inhomogeneous* LCTs, which are not explicitly considered here. See [18].

[2]The paper by Collins uses momenta in the form $n_i\mathbf{p}_i$ with $|\mathbf{p}_i| = \sin\theta_i$, and orders the 4-vector components as $(x_1, p_1, x_2, p_2)^\top$.

These conditions (for $N = 2$) were found [1, Appendix B] and thereby the optical distance between initial and final ray positions $L := L_0 + L_{\rm M}$, consisting of the distance L_0 along the axis plus that gained for rays between positions off this axis, $L_{\rm M}(\mathbf{x}, \mathbf{x}')$, which is a quadratic function of its arguments and contains the parameters of the transfer matrix $\mathbf{M}$. The integral kernel (1.2) is thus determined to be of the form $A_{\rm M} \exp(\mathrm{i}kL_{\rm M})$. The normalization factor $A_{\rm M}$ is computed by demanding the conservation of energy, and its phase is taken from the Fresnel diffraction kernel [1, Eq. (28)]. The paper by Collins applies this result for the analysis of Hermite–Gaussian beams in resonators and for the reconstruction of holographic images.

1.2.2 Evolution in Quadratic Quantum Systems

Marcos Moshinsky had been studying the harmonic motion of Gaussian wavepackets that represent alpha bondings in various oscillator models of the nucleus. This is the context in which he seems to have been motivated to touch upon canonical transformations in quantum mechanics. His paper was presented at the XV Solvay Conference in Physics of 1970 [2], whose Proceedings were delayed 4 years. Upon returning to Mexico with the Belgian postdoctoral associate Dr. Christiane Quesne, they stated the problem in the following terms [3, 4]: What are the transformations of phase space that leave the structure of quantum mechanics invariant? This included the important uncertainty relation $\Delta_f \, \Delta_{\tilde{f}} \geq \frac{1}{4}$ ($\hbar \equiv 1$) that is a mathematical property of the Fourier integral transform. The question remitted them to the basic Heisenberg commutators

$$[\hat{x}_i, \hat{p}_j] := \hat{x}_i \hat{p}_j - \hat{p}_j \hat{x}_i = \mathrm{i}\, \delta_{i,j}, \tag{1.6}$$

between the Schrödinger position operators $\hat{x}_i = x_i \cdot$ and the momentum operators $\hat{p}_j = -\mathrm{i}\partial_j$ (where $\partial_j \equiv \partial/\partial x_j$), for $i, j = 1, 2, \ldots, N$ in N-dimensional systems. Such transformations can be linear or nonlinear; some of the latter were examined a few years later, but the more immediate ones were the linear, for N-vector operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ forming a $2N$-vector $\hat{\mathbf{w}} \equiv \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}} \end{pmatrix}$ as before, acted upon by a transformation $\mathcal{C}_{\rm M}$ depending on the elements of a $2N \times 2N$ matrix $\mathbf{M}$. For operators, these are written somewhat differently from (1.1),

$$\mathcal{C}_{\rm M}\, \hat{\mathbf{w}}\, \mathcal{C}_{\rm M}^{-1} = \mathbf{M}^{-1} \hat{\mathbf{w}}. \tag{1.7}$$

The reason for having the inverse matrix on the right-hand side is that this alone ensures that the composition of transforms follows that of the matrices: $\mathcal{C}_{\rm M_1} \mathcal{C}_{\rm M_2} = \varphi\, \mathcal{C}_{\rm M_1 M_2}$, with φ a constant undetectable in (1.7). Next, direct replacement into (1.6) yields the symplectic conditions (1.3)–(1.5) for $\mathbf{M}$. Symplectic matrices are invertible,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}^{-1} = \mathbf{\Omega}\mathbf{M}^\top\mathbf{\Omega}^\top = \begin{pmatrix} \mathbf{d}^\top & -\mathbf{b}^\top \\ -\mathbf{c}^\top & \mathbf{a}^\top \end{pmatrix}, \tag{1.8}$$

the unit $\mathbf{1}$ is symplectic and associativity holds. Hence symplectic matrices that are real form the *real symplectic group* $\mathsf{Sp}(2N, \mathsf{R})$ with $N(2N+1)$ independent parameters. When $N = 1$, $\mathsf{Sp}(2, \mathsf{R})$ is identical with the group of all 2×2 real matrices of unit determinant. (The complex case will be considered in Sect. 1.6.)

The action of the linear operators $\mathcal{C}_{\mathrm{M}}$ on the usual Hilbert space $\mathcal{L}^2(\mathsf{R}^N)$ of quantum mechanical Lebesgue square-integrable functions, $f \mapsto f_{\mathrm{M}} \equiv \mathcal{C}_{\mathrm{M}} f$, is expected to be integral in R^N as (1.2), and unitary, because such is quantum evolution. The integral kernel can be found applying $\mathcal{C}_{\mathrm{M}}$ to $\hat{x}_i f$ and to $\hat{p}_j f$ using (1.7) and (1.8),

$$\mathcal{C}_{\mathrm{M}}\,(\hat{x}_i f) = (\mathcal{C}_{\mathrm{M}}\hat{x}_i\mathcal{C}_{\mathrm{M}}^{-1}) f_{\mathrm{M}} = \textstyle\sum_j (d_{j,i}\hat{x}_j - b_{j,i}\hat{p}_j) f_{\mathrm{M}}, \tag{1.9}$$

$$\mathcal{C}_{\mathrm{M}}\,(\hat{p}_i f) = (\mathcal{C}_{\mathrm{M}}\hat{p}_i\mathcal{C}_{\mathrm{M}}^{-1}) f_{\mathrm{M}} = \textstyle\sum_j (-c_{j,i}\hat{x}_j + a_{j,i}\hat{p}_j) f_{\mathrm{M}}. \tag{1.10}$$

On the right, $\hat{x}_i$ and $\hat{p}_i$ act outside of the integral, on the $\mathbf{x}$ argument of the kernel $C_{\mathrm{M}}(\mathbf{x}, \mathbf{x}')$, while those on the left act inside, on $f(\mathbf{x}')$; the derivatives of the latter can be integrated by parts to act on the $\mathbf{x}'$ argument of the kernel. Since f is arbitrary, one obtains the $2N$ simultaneous linear differential equations satisfied by the LCT kernel,

$$x_i' C_{\mathrm{M}}(\mathbf{x}, \mathbf{x}') = \textstyle\sum_j \Big(d_{j,i}x_i + \mathrm{i}b_{j,i}\partial_j\Big) C_{\mathrm{M}}(\mathbf{x}, \mathbf{x}'), \tag{1.11}$$

$$\partial_i' C_{\mathrm{M}}(\mathbf{x}, \mathbf{x}') = \textstyle\sum_j \Big(\mathrm{i}c_{j,i}x_i - a_{j,i}\partial_j\Big) C_{\mathrm{M}}(\mathbf{x}, \mathbf{x}'). \tag{1.12}$$

The solution, up to a multiplicative constant K_{M}, is

$$C_{\mathrm{M}}(\mathbf{x}, \mathbf{x}') := K_{\mathrm{M}} \exp \mathrm{i}\Big(\tfrac{1}{2}\mathbf{x}^\top\mathbf{b}^{-1}\mathbf{d}\mathbf{x} - \mathbf{x}^\top\mathbf{b}^{-1}\mathbf{x}' + \tfrac{1}{2}\mathbf{x}'^\top\mathbf{a}\mathbf{b}^{-1}\mathbf{x}'\Big). \tag{1.13}$$

The constant K_{M} is found from the limit to the $2N \times 2N$ unit matrix, $\mathbf{M} \to \mathbf{1}$ (with $\det \mathbf{b}$ in the lower-half complex plane), so that $C_{\mathrm{M}}(\mathbf{x}, \mathbf{x}') \to \delta^N(\mathbf{x} - \mathbf{x}')$, regaining the unit transform $\mathcal{C}_{\mathbf{1}} = \mathit{1}$. The result is

$$K_{\mathrm{M}} = \frac{1}{\sqrt{(2\pi\mathrm{i})^N \det \mathbf{b}}} \equiv \frac{e^{-\mathrm{i}\pi N/4} \exp \mathrm{i}(-\frac{1}{2}\arg\det\mathbf{b})}{\sqrt{(2\pi)^N |\det \mathbf{b}|}}. \tag{1.14}$$

Finally, when only $\mathbf{b} \to \mathbf{0}$ from the lower complex half-plane, the matrix is $\mathbf{M}_0 := \begin{pmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{c} & \mathbf{a}^{\top -1} \end{pmatrix}$, the Gaussian kernel converges weakly to a Dirac δ, and the integral operator action becomes a change of scale of the function multiplied by a quadratic phase,

$$(\mathcal{C}_{\mathrm{M}_0} f)(\mathbf{x}) = \frac{\exp \mathrm{i}(\frac{1}{2}\mathbf{x}^\top \mathbf{c}\mathbf{a}^{-1}\mathbf{x})}{\sqrt{\det \mathbf{a}}} f(\mathbf{a}^{-1}\mathbf{x}). \tag{1.15}$$

In the case of $N = 1$-dimensions, Eqs. (1.2) and (1.13)–(1.14) simplify to the best-known form of LCTs,

$$f_{\mathrm{M}}(x) \equiv (\mathcal{C}_{\mathrm{M}} f)(x) = \int_{\mathsf{R}} \mathrm{d}x'\, C_{\mathrm{M}}(x,x') f(x'), \tag{1.16}$$

$$C_{\mathrm{M}}(x,x') := \frac{1}{\sqrt{2\pi \mathrm{i}\, b}} \exp\left(\frac{\mathrm{i}}{2b}(dx^2 - 2xx' + ax'^2)\right), \tag{1.17}$$

where it should be understood that $1/\sqrt{\mathrm{i}b} = \exp(-\mathrm{i}\frac{1}{2}\pi(\operatorname{sign} b + \frac{1}{2}))/\sqrt{|b|}$. The generalization of the Fourier–Heisenberg uncertainty relation to LCTs is of the form $\Delta_f \Delta_{f_{\mathrm{M}}} \geq \frac{1}{4}|b|$. The last two chapters of [6] were written based on the works of Marcos Moshinsky and his associates on LCTs, complemented with results by the author on translations of phase space, complex extensions, and applications to the evolution of Gaussians and other wavefunctions of quantum quadratic systems (oscillator wavefunctions, parabolic cylinder and Airy functions) under diffusion.

1.2.3 LCTs in a Broader Context

Optical models are richer than mechanical ones because they provide a wider view of canonical transformations beyond the linear regime. Mechanical Hamiltonians are mostly of the form $\frac{1}{2}p^2 + V(x)$, where the potential $V(x)$ with a smooth minimum may be expanded using perturbation series in powers of x around the harmonic oscillator; in geometric and magnetic metaxial optics on the other hand, the presence of aberrations generally requires evolution Hamiltonians expressible in series of terms $p^n x^m$. As Alex J. Dragt applied for accelerators [12–15, 19], Hamiltonian and Lie-theoretic tools served to calculate carefully one turn in the accelerator, and then one raises that transformation to the power of any number of turns, while canonicity ensures the conservation of the beam area in phase space. The usefulness of these techniques for optical design was facilitated by a neat theorem on the canonical transformations produced by refraction between two media separated by a surface of smooth but arbitrary shape [20]: they can be factored into the product of two canonical transformations, each depending on the surface and only one of the media. This allowed the computation of the aberration coefficients for polynomial surfaces of revolution, and the group structure translated the concatenation of optical elements along the optical axis into matrix multiplication. Interest in these lines led us to convene two gatherings on *Lie optics* (a convenient neologism), in 1985 and 1988 [21, 22]. In fact, LCTs were taken for granted and served as basis for chapters on Fourier optics, coherent states, holography, computational aspects for aberrations, and optical models that take into account that the optical momentum vector ranges over a sphere, and not over a plane as the paraxial theory assumes. Yet, it is the linear regime (paraxial optics or quadratic mechanics) that displays naturally the cleanest symmetries.

Closely related with LCTs, a line of research on the Wigner distribution function applied to optical waveforms and their transformation in first-order optical systems was opened by Martin J. Bastiaans by the end of the 1970s [23, 24]. Both papers are highly cited (>330 and >400 citations), indicating that many authors have followed the analysis of non-imaging linear systems in phase space [25–27]. More recent work with Tatiana Alieva, María Luisa Calvo, and several coworkers addressed LCTs to obtain phase information out of intensity measurements [28–30], and the processing of two-dimensional images [31, 32] by means of optical setups of cylindrical lenses that can be rotated in fixed positions to synthesize any LCT transformation [33], in particular fractional Fourier transforms [34, 35] and gyrators [36, 37]. Both the Wigner function and the two-dimensional LCTs that form the group $\mathsf{Sp}(4, \mathsf{R})$ cannot be surveyed in this chapter for reasons of space even though they are now widely used for many applications in quantum optics. See, for example, [38] (>1120 citations).

Linear canonical transformations include fractional Fourier transforms in the subgroup $\mathcal{F}^\nu \equiv \mathcal{F}_\theta \in \mathsf{SO}(2) \subset \mathsf{Sp}(2, \mathsf{R})$ of matrices $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, of power $\nu \in \mathsf{R}$ or angle $\theta = \frac{1}{2}\pi\nu$, times the metaplectic phase (to be seen below). This development also has a story behind: in 1937, Edward Condon thanks Profs. Bochner, von Neumann, and Bohnenblust for conversations leading to the article [39], where he clearly defines the fractional Fourier transform and finds its kernel following the reasoning in (1.11)–(1.12), recognizing the metaplectic problem. Condon's result seems to have been in suspended animation for decades, unnoticed by Victor Namias [40] who in 1980 rediscovered $\mathcal{F}^\nu$ proposing that it self-reproduces the harmonic oscillator wavefunctions with a phase $(-\mathrm{i})^\nu$ (to be taken as $e^{-\mathrm{i}\pi\nu/2}$), and the kernel found from the bilinear generating function of Hermite polynomials (inexplicably, [6] disregarded this specialization of LCTs). Interest of the optical community in fractional Fourier transforms grew in the early nineties around their optical implementation through the slicing of graded-index media and non-imaging lens systems, by Mendlovic and Ozaktas [41–44] (>780, >437, >250 and >254 citations). Their work was formalized in the 2001 book [45] by Ozaktas, Zalevsky, and Kutay, which spread the use of the fractional Fourier transform and LCTs in general. This book contains a bibliography of >500 references which hardly any of us can read entirely, and which I certainly cannot reproduce.

1.3 LCTs, Matrices, Signs and Covers

An important property of the LCTs (1.13)–(1.15) is that they conserve the norms [1] and overlaps [3], i.e., the transformations are unitary under the $\mathcal{L}^2(\mathsf{R}^N)$ inner product,

$$(f, g)_{\mathcal{L}^2(\mathsf{R}^N)} := \int_{\mathsf{R}^N} \mathrm{d}\mathbf{x} f(\mathbf{x})^* \, g(\mathbf{x}) = (f_{\mathrm{M}}, g_{\mathrm{M}})_{\mathcal{L}^2(\mathsf{R}^N)}, \tag{1.18}$$

because

$$C_{\mathrm{M}}(\mathbf{x}, \mathbf{x}') = C_{\mathrm{M}^{-1}}(\mathbf{x}', \mathbf{x})^*. \tag{1.19}$$

However, the group composition property of LCTs is satisfied by the integral kernels only as

$$\int_{\mathsf{R}^N} \mathrm{d}\mathbf{x}' \, C_{\mathrm{M}_1}(\mathbf{x}, \mathbf{x}') \, C_{\mathrm{M}_2}(\mathbf{x}', \mathbf{x}'') = \sigma \, C_{\mathrm{M}_1 \mathrm{M}_2}(\mathbf{x}, \mathbf{x}''), \tag{1.20}$$

where σ is a phase—the *metaplectic* phase (actually a sign). This problem is announced by the square root in the denominator of (1.14) and (1.15); it can be seen most clearly in the Fourier integral transform $\mathcal{F}$ for $N = 1$, which for dimensionless matrix elements corresponds to $\mathbf{F} = \mathbf{\Omega}$[3]; the integral kernel is then $C_{\mathrm{F}}(x, x') = e^{-\mathrm{i}\pi/4} e^{-\mathrm{i}xx'}/\sqrt{2\pi}$, so

$$\mathcal{C}_{\mathrm{F}} = e^{-\mathrm{i}\pi/4} \mathcal{F}, \qquad \mathbf{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.21}$$

Thus, while $\mathcal{F}^4 = \mathit{1}$ we have $\mathcal{C}_{\mathrm{F}}^4 = -\mathit{1}$; this is reminiscent of the behavior of spin under 2π rotations.

The metaplectic sign has bedeviled many papers, and it can be said that it was not really understood until the group theory behind brought to the fore the fact that the correspondence between integral LCTs and matrices is not 1:1, but 2:1. The problem is not crucial in optical setups because overall phases are commonly not registered, but in mathematics signs cannot be just ignored. Indeed, the structure of the symplectic groups (even that of 2×2 matrices) is unexpectedly imbricate [46]. The problem for $N = 1$ was clarified early by Valentin Bargmann in 1947 [47, Sects. 3, 4] using the *polar* decomposition of matrices. This is a generalization of the factorization of complex numbers $z = e^{\mathrm{i}\phi}|z|$ into a phase $e^{\mathrm{i}\phi}$ times a positive number $|z|$; multiple Riemann sheets of a function around its branch points need the phase ϕ to range beyond its basic interval modulo 2π. A real 2×2 matrix can be similarly decomposed into the product of a unitary and a symmetric positive definite matrix,

[3] I thank Dr. George Nemeş for the remark that when dimensions are respected, $\mathbf{F} \neq \mathbf{\Omega}$ because the parameters b and $1/c$ have units of momentum/position, while a and d have no units. In our presentation of the kernel (1.17) we assume that momentum p bears no units (as in optics), and that a unit of distance has been agreed for position so that x is its numerical multiple.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \lambda + \mathrm{Re}\,\mu & \mathrm{Im}\,\mu \\ \mathrm{Im}\,\mu & \lambda - \mathrm{Re}\,\mu \end{pmatrix}, \tag{1.22}$$

where μ is complex, and $\lambda := +\sqrt{|\mu|^2 + 1} \geq 1$. Under multiplication of two matrices, their Bargmann parameters (with subindices 1 and 2) compose through

$$\phi = \phi_1 + \phi_2 + \arg\nu, \qquad \mu = e^{-\mathrm{i}\arg\nu}(\lambda_1\mu_2 + e^{-2\mathrm{i}\phi_2}\mu_1\lambda_2), \tag{1.23}$$

where $\nu := 1 + e^{-2\mathrm{i}\phi_2}\mu_1\mu_2/\lambda_1\lambda_2$ is an auxiliary complex quantity whose phase is determined to range in $\arg\nu \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, and $\lambda = \lambda_1|\nu|\lambda_2 \geq 1$. The composite ϕ can thus take values on the full real line R and hence parametrize all elements of $\overline{\mathsf{Sp}(2,\mathsf{R})}$, the *infinite cover* of the group $\mathsf{Sp}(2,\mathsf{R})$. Thus, while the unitary spin group $\mathsf{SU}(2)$ covers twice the orthogonal rotation group $\mathsf{SO}(3)$, the symplectic group is infinitely covered; the realization by LCTs is then a *two*fold cover of the group of 2×2 real matrices of unit determinant. Below we shall comment on this feature of the group of integral transforms, called the *metaplectic group* $\mathsf{Mp}(2,\mathsf{R})$. (See also [48, Sect. 9.4].)

The generic case of $\mathsf{Sp}(2N,\mathsf{R})$ follows suit, as proved by Bargmann some years later [49]. The polar decomposition is then into a real $2N \times 2N$ orthosymplectic matrix that represents the group $\mathsf{U}(N)$ of $N \times N$ unitary matrices, and again a symmetric positive definite matrix [48, p. 173]. This $\mathsf{U}(N)$ group is the maximal *compact* (i.e., of finite volume) subgroup of $\mathsf{Sp}(2N,\mathsf{R})$, and has been called the *Fourier* group [50]. In the $N = 2$-dimensional case, $\mathsf{U}(2)$ contains the isotropic and anisotropic fractional Fourier and gyration integral transforms [34, 36], as well as joint rotations of position and momentum around the optical center and axis. In turn, this $\mathsf{U}(2)$ is the direct product of a $\mathsf{U}(1)$ subgroup of isotropic fractional Fourier transforms (a circle), times the group $\mathsf{SU}(2)$ of 2×2 matrices of unit determinant; the latter is simply connected, so the onus of multivaluation falls on the former. For $N = 2$ and the 4×4 Fourier matrix $\mathbf{F} = \mathbf{\Omega} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$ the relation between the LCT and the $2D$ Fourier integral transform is thus $\mathcal{C}_{\mathrm{F}} = e^{-\mathrm{i}\pi/2}\mathcal{F}$.

1.4 LCTs Are Generated by Second-Order Differential Operators

In retrospect it is obvious that unitary LCTs $\mathcal{C}_{\mathrm{M}}$ and self-adjoint second-order differential operators $\hat{J} = \alpha\hat{p}_i\hat{p}_j + \beta\frac{1}{2}(\hat{x}_i\hat{p}_j + \hat{p}_j\hat{x}_i) + \gamma\hat{x}_i\hat{x}_j$ should be closely related, the latter generating the former through $\mathcal{C}_{\mathrm{M}(\tau)} = \exp(\mathrm{i}\tau\hat{J})$. The LCT integral kernels $C_{\mathrm{M}(\tau)}(\mathbf{x}, \mathbf{x}')$ are Green functions of quadratic Hamiltonians that can be found through

$$\hat{J}f(\mathbf{x}) = -\mathrm{i}\frac{\partial}{\partial\tau}\int_{\mathrm{R}^N} \mathrm{d}\mathbf{x}'\, C_{\mathrm{M}(\tau)}(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')\Big|_{\tau=0}, \tag{1.24}$$

and $C_{\mathrm{M}(0)}(\mathbf{x},\mathbf{x}') = \delta^N(\mathbf{x}-\mathbf{x}')$, as was done in [51]. Probably the reason for not having recognized this relation earlier was that since the time of Sophus Lie only first-order differential operators, $f(x)\partial_x + g(x)$, were used to generate Lie groups.

Writing $\mathcal{C}_{\mathrm{M}} \equiv \mathcal{C}(\mathbf{M})$, we have the following $N=1$ paraxial optical elements generated by operators and their LCTs,

$$\text{thin lens:} \qquad \exp\left(\mathrm{i}\tfrac{1}{2}\tau\hat{x}^2\right) = \mathcal{C}\begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}, \tag{1.25}$$

$$\text{free flight:} \qquad \exp\left(\mathrm{i}\tfrac{1}{2}\tau\hat{p}^2\right) = \mathcal{C}\begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix}, \tag{1.26}$$

$$\text{magnifier:} \quad \exp\left(\mathrm{i}\tfrac{1}{2}\tau(\hat{p}\hat{x}+\hat{x}\hat{p})\right) = \mathcal{C}\begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix}, \tag{1.27}$$

$$\text{repulsive guide:} \quad \exp\left(\mathrm{i}\tfrac{1}{2}\tau(\hat{p}^2-\hat{x}^2)\right) = \mathcal{C}\begin{pmatrix} \cosh\tau & -\sinh\tau \\ -\sinh\tau & \cosh\tau \end{pmatrix}, \tag{1.28}$$

$$e^{\mathrm{i}\pi\tau/4}\times\text{Fourier}^{-\tau}\text{:} \quad \exp\left(\mathrm{i}\tfrac{1}{2}\tau(\hat{p}^2+\hat{x}^2)\right) = \mathcal{C}\begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}. \tag{1.29}$$

For vanishing τ, $\mathbf{M}(\tau) \approx \mathbf{1} + \tau\mathbf{m}$, we can associate the generator operators with traceless 2×2 matrices $\mathbf{m}$: thin lens, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; free flight, $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$; magnifier, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; repulsive guide, $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$; and harmonic guide, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This infinitesimal "portion" of $\mathsf{Sp}(2,\mathsf{R})$ constitutes a linear space called its *Lie algebra*, denoted by the lowercase name $\mathsf{sp}(2,\mathsf{R})$, and whose structure is determined by the commutators of its elements. Under $\mathcal{C}_{\mathrm{M}}$ the "infinitesimal" matrices $\mathbf{m}\in\mathsf{sp}(2,\mathsf{R})$ will transform by similarity as $\mathbf{m}\mapsto\mathbf{m}' = \mathbf{M}\,\mathbf{m}\,\mathbf{M}^{-1}$, and with all $\mathbf{M}\in\mathsf{Sp}(2,\mathsf{R})$ we build the *orbit* of $\mathbf{m}$. Thus the generators of lenses and of free flights are in the same orbit related by the Fourier matrix $\mathbf{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the generators of magnifiers are the same with those of repulsive guides, related by the square root $\mathbf{F}^{1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Analysis shows that $\mathsf{sp}(2,\mathsf{R})$ has three orbits (excluding the orbit of $\mathbf{0}$): *elliptic* containing (1.29); *hyperbolic* (1.27)–(1.28); and *parabolic* (1.25)–(1.26). The last forms a cone in R^3, the first and second fill the inside and outside of that cone. This division into disjoint orbits in the R^3 linear space of the algebra extends to the group, but the group $\mathsf{Sp}(2,\mathsf{R})$ of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an extra *non-exponential* region identified by the range of the trace, $a+d\in(-\infty,-2)$, where the matrices have no real logarithm. For $N=2$ dimensions, the identification of generating Hamiltonians in $\mathsf{sp}(4,\mathsf{R})$ with optical elements can be found in [48, Chap. 12]; there are 4 continua of orbits and 12 isolated points, few of which have been exploited.

The relations (1.25)–(1.29) also determine that the eigenfunctions of an operator $\hat{J}\,\psi_\nu = E_\nu\,\psi_\nu$ (whose eigenvalues E_ν are common to all elements in its orbit), will self-reproduce under the generated LCT as $\mathcal{C}_{\mathrm{M}(\tau)}\psi_\nu = e^{\mathrm{i}\tau E_\nu}\psi_\nu$. In particular, the harmonic oscillator Hermite–Gauss eigenfunctions $\Psi_n(x)$ correspond to energies $E_n = n+\frac{1}{2}$, $n\in\{0,1,2,\ldots\}$. Thus, the $\mathcal{C}_{\mathrm{F}}$ LCT of the eigenfunctions $\psi_n^{\mathrm{M}} = \mathcal{C}_{\mathrm{M}}\psi_n$ of all operators in the elliptic orbit is

$$\mathcal{C}_{\mathrm{F}}\psi_n^{\mathrm{M}} = \exp\left[-\mathrm{i}\tfrac{1}{2}\pi(n+\tfrac{1}{2})\right]\psi_n^{\mathrm{M}} = e^{-\mathrm{i}\pi/4}(-\mathrm{i})^n\psi_n^{\mathrm{M}}, \tag{1.30}$$

having set $\tau = -\frac{1}{2}\pi$ in (1.29) and in agreement with (1.21). Here again the phase evinces the double cover of $\mathcal{C}_{\mathrm{F}^\alpha} \in \mathsf{Mp}(2,\mathsf{R})$ over the circle of fractional Fourier matrices $\mathbf{F}^\alpha \in \mathsf{Sp}(2,\mathsf{R})$. We may also see the metaplectic phase as the energy of the vacuum, $E_0 = \frac{1}{2}$.

We have thus associated three classes of mathematical actors in the $\mathsf{Sp}(2,\mathsf{R})$ troupe: LCT integral transforms, hyperdifferential (exponentials of second order) operators, and matrices (modulo a sign). Product operations in one class correspond with products in the other two. Hence, we can easily write Baker–Campbell–Hausdorff relations between quadratic operators [6, Sect. 9.3.2], and the LCTs of the eigenfunction set of one under LCTs generated by another, including phase space translations [6, Chap. 10]. Certainly, other authors have considered various aspects of the above constructions (see, e.g., [52]), so it is as grievous not to mention one as it is to mention all.

1.5 Radial, Hyperbolic, and Other LCTs

Isotropic LCTs in $N = 2$ or more dimensions that are represented by matrices $\mathbf{M} = \begin{pmatrix} a\mathbf{1} & b\mathbf{1} \\ c\mathbf{1} & d\mathbf{1} \end{pmatrix}$ with diagonal submatrices can be reduced to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *radial* LCTs acting on eigenspaces of functions of the radius and with definite angular momentum. One may also ask for separation of variables in other sets of coordinates and select eigenspaces under other operators, to find, e.g., *hyperbolic* LCTs. Not surprisingly, it turns out that for $N = 1$ the theory of $\mathsf{Sp}(2,\mathsf{R})$ representations studied by Bargmann [47], and Gel'fand and Naĭmark [53]—also in the same year 1947, provides an appropriate framework to phrase these and other derivate LCTs.

1.5.1 Radial Canonical Transforms

Shortly after completing the initial two papers on LCTs based on the 2×2 $\mathsf{Sp}(2,\mathsf{R})$ matrices [3, 4], and Dr. Quesne having returned to Belgium, Marcos Moshinsky extended his inquiry to canonical transformations which he deemed to be nonlinear, but were closely related to the two-dimensional oscillator through the subgroup chain $\mathsf{Sp}(4,\mathsf{R}) \supset \mathsf{SO}(2) \otimes \mathsf{Sp}(2,\mathsf{R})$, where $\mathsf{SO}(2)$ is the group of rotations in the plane [54]. The representations of the two subgroups are *conjugate*, i.e., the representation $m \in \mathsf{Z}$ of $\mathsf{SO}(2)$ fixes the discrete-series representation $k = \frac{1}{2}(|m|+1)$ of $\mathsf{Sp}(2,\mathsf{R})$ (see below). This approach considered isotropic LCTs (1.2) in the *polar* coordinates of R^2,

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \qquad r \in \mathsf{R}_0^+ = [0,\infty), \quad \theta \in \mathsf{R} \bmod 2\pi. \tag{1.31}$$

Since angular momentum $\hat{L} = -\mathrm{i}(x_1\partial_2 - x_2\partial_1) = -\mathrm{i}\partial_\theta$ commutes with these LCTs, we can isolate an eigenspace of functions $f(\mathbf{x}) \sim f(r)\, e^{\mathrm{i}m\theta}/\sqrt{2\pi}$ with integer $m \in \mathsf{Z}$, to find the corresponding "radial" LCTs (RLCTs). There, $\nabla^2 = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$ where with $\partial_\theta^2 \mapsto -m^2$ is self-adjoint under the measure $r\,\mathrm{d}r$. In order to have "m-radial" spaces where ∂_r^2 be self-adjoint, we need the inner product

$$(f,g)_{\mathcal{L}^2(\mathsf{R}+)} := \int_0^\infty \mathrm{d}r\, f(r)^*\, g(r) \tag{1.32}$$

with measure $\mathrm{d}r$, so previous operators should be transformed through $\hat{J} \mapsto \sqrt{r}\,\hat{J}/\sqrt{r}$ to keep self-adjointness.

To find the RLCT integral kernel under (1.32), we project out the Fourier series coefficient of the $e^{\mathrm{i}m\theta}$ component of the $N = 2$ isotropic LCT kernel (1.13),

$$C_{\mathrm{M}}^{(m)}(r,r') = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, C_{\mathrm{M}}(\mathbf{x},\mathbf{x}')e^{-\mathrm{i}m\theta}. \tag{1.33}$$

Noting that only the factor $e^{-\mathrm{i}\mathbf{x}\cdot\mathbf{x}'/b}$ contains the mutual angle through $\mathbf{x}\cdot\mathbf{x}' = rr'\cos(\theta - \theta')$, we fix the reference axes by $\mathbf{x}$ to perform the integration. This is the angular momentum decomposition of the LCT, and defines the m-RLCT by

$$f_{\mathrm{M}}^{(m)}(r) \equiv (\mathcal{C}_{\mathrm{M}}^{(m)} f)(r) = \int_{\mathsf{R}+} \mathrm{d}r'\, C_{\mathrm{M}}^{(m)}(r,r')f(r'), \tag{1.34}$$

$$C_{\mathrm{M}}^{(m)}(r,r') = \frac{e^{\mathrm{i}\pi(m+1)/2}}{b}\exp\Big(\frac{\mathrm{i}}{2b}(dr^2 + ar'^2)\Big)J_m\Big(\frac{rr'}{b}\Big), \tag{1.35}$$

where $J_m(z)$ is the Bessel function of the first kind. An alternative derivation of this kernel can be found in [55].

1.5.2 Hyperbolic Canonical Transforms

Hyperbolic canonical transforms are obtained when instead of the polar coordinates (1.31), one introduces the two-chart *hyperbolic* coordinates [56],

$$\left.\begin{array}{ll} \sigma = + : x_1 = \rho\cosh\zeta, & x_2 = \rho\sinh\zeta, \\ \sigma = - : x_1 = \rho\sinh\zeta, & x_2 = \rho\cosh\zeta, \end{array}\right\} \quad \begin{array}{l} \rho,\ \zeta \in \mathsf{R}, \\ \sigma := \operatorname{sign}(x_1^2 - x_2^2). \end{array} \tag{1.36}$$

Here the subgroup chain to be used is $\mathsf{Sp}(4,\mathsf{R}) \supset \mathsf{O}(1,1) \otimes \mathsf{Sp}(2,\mathsf{R})$, where now $\mathsf{O}(1,1)$ consists of pseudo-orthogonal ("1+1 Lorentz") matrices, and inversions $\Pi : \mathbf{x} = -\mathbf{x}$ that also commute with $\mathsf{Sp}(2,\mathsf{R})$, reducing the range of the "hyperbolic radius" ρ to $[0,\infty)$. Instead of the isotropic LCTs used for RLCTs above, we now consider LCTs of the form $\mathbf{M} = \begin{pmatrix} a\bar{\mathbf{1}} & b\bar{\mathbf{1}} \\ c\bar{\mathbf{1}} & d\mathbf{1} \end{pmatrix}$ with $\bar{\mathbf{1}} := \begin{pmatrix} 1 & 0 \\ 0 & e^{-\mathrm{i}\pi} \end{pmatrix}$, where the phase $e^{-\mathrm{i}\pi}$ is important. Then in (1.13) the first exponential term is $\mathbf{x}^\top\mathbf{b}^{-1}\mathbf{dx} = \sigma d\rho^2/b$, and only the term $\mathbf{x}^\top\mathbf{b}^{-1}\mathbf{x}' = \sigma\rho\rho'\cosh(\zeta - \zeta')$ contains the boost "angle" $\zeta \in \mathsf{R}$ that will be subject to integration.

Fourier integral decomposition of the LCT kernel (1.13) into plane waves and parity yield the "hyperbolic" LCTs (HLCTs), characterized now by the Fourier conjugate variable $s \in \mathsf{R}$ and the parity eigenvalue $\varpi \in \{+1, -1\}$. But note that now there are also two charts $\sigma \in \{+, -\}$, so that functions should be represented by two σ-component functions with definite parity ϖ, as $\mathbf{f}^\varpi(\rho) = \begin{pmatrix} f^{+,\varpi}(\rho) \\ f^{-,\varpi}(\rho) \end{pmatrix}$, with $f^{\sigma,\varpi}(\rho) = \varpi f^{\sigma,\varpi}(-\rho)$, and the inner product

$$(\mathbf{f}, \mathbf{g})_{\mathcal{L}^2(\varpi,\mathsf{R}+)} := \sum_{\sigma\in\{+,-\}} \int_0^\infty \mathrm{d}\rho f^{\sigma,\varpi}(\rho)^*\, g^{\sigma,\varpi}(\rho). \tag{1.37}$$

The HLCT of a function $f(\rho)$ is then

$$\mathbf{f}^{\varpi,s}_{\mathrm{M}}(\rho) \equiv (\mathcal{C}^{\varpi,s}_{\mathrm{M}}\,\mathbf{f})(\rho) = \int_{\mathsf{R}+} \mathrm{d}\rho'\, \mathbf{C}^{(\varpi,s)}_{\mathrm{M}}(\rho, \rho')\, \mathbf{f}(\rho'), \tag{1.38}$$

where the matrix integral kernel is

$$\mathbf{C}^{(\varpi,s)}_{\mathrm{M}}(\rho, \rho') = \begin{pmatrix} G_{\mathrm{M},+,+}(\rho,\rho')\, H^{(\varpi,s)}_{+,+}(\rho\rho'/b) & G_{\mathrm{M},+,-}(\rho,\rho')\, H^{(\varpi,s)}_{+,-}(\rho\rho'/b) \\ G_{\mathrm{M},-,+}(\rho,\rho')\, H^{(\varpi,s)}_{-,+}(\rho\rho'/b) & G_{\mathrm{M},-,-}(\rho,\rho')\, H^{(\varpi,s)}_{-,-}(\rho\rho'/b) \end{pmatrix}, \tag{1.39}$$

$$G_{\mathrm{M},\sigma,\sigma'}(\rho, \rho') = \frac{\sqrt{\rho\rho'}}{2\pi\,|b|} \exp\left(\mathrm{i}\frac{\sigma d\rho^2 + \sigma' a\rho'^2}{2b}\right), \tag{1.40}$$

$$\begin{aligned} H^{(\varpi,s)}_{+,+}(\xi) &= \mathrm{i}\pi\,[\varpi e^{-\pi s} H^{(1)}_{2\mathrm{i}s}(\xi{+}\mathrm{i}0^+) - \varpi e^{\pi s} H^{(2)}_{2\mathrm{i}s}(\xi{-}\mathrm{i}0^+)] \qquad (1.41)\\ &= \varpi H^{(\varpi,s)}_{-,-}(\xi), \\ H^{(\varpi,s)}_{+,-}(\xi) &= 4c^{\varpi,s}_\xi K_{2\mathrm{i}s}(|\xi|) = \varpi H^{(\varpi,s)}_{-,+}(\xi), \qquad (1.42) \end{aligned}$$

and where $H^{(1)}_\nu$ and $H^{(2)}_\mu$ are Hankel functions of the first and second kind valued above and below the branch cut, K_ν is the MacDonald function, $c^{+1,s}_\xi := \cosh \pi s$ and $c^{-1,s}_\xi := -\mathrm{sign}\,\xi\, \sinh \pi s$.

1.5.3 LCTs as Representations of Sp(2, R)

In Sect. 1.3 I warned that the theory of 2×2 real matrices is more imbricate than expected.[4] Yet I believe that the natural context to understand the foundations and

[4]Once I said in front of a large student audience that I had devoted much work to understand 2×2 matrices, the giggles in the hall were sobering.

see the possible incarnations of linear canonical transformations is in the theory of unitary irreducible representations of the Lorentz group $\mathsf{SO}(2,1)$ of "2+1" special relativity [46]. Let me now place LCTs in this context.

After relating paraxial optical elements to LCTs and second-order differential operators in (1.25)–(1.29), we note further that the following operators

$$\hat{J}_1 := \frac{1}{4}\Big(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\gamma}{r^2} - r^2\Big), \tag{1.43}$$

$$\hat{J}_2 := \frac{-\mathrm{i}}{4}\Big(r\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r}r\Big), \tag{1.44}$$

$$\hat{J}_3 := \frac{1}{4}\Big(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\gamma}{r^2} + r^2\Big), \tag{1.45}$$

are essentially self-adjoint under the inner product (1.32) of $\mathcal{L}^2(\mathsf{R}^+)$, and that they close into an algebra with the commutation relations

$$[\hat{J}_1, \hat{J}_2] = -\mathrm{i}\hat{J}_3, \quad [\hat{J}_2, \hat{J}_3] = \mathrm{i}\hat{J}_1, \quad [\hat{J}_3, \hat{J}_1] = \mathrm{i}\hat{J}_2, \tag{1.46}$$

that characterize the isomorphic algebras $\mathsf{sp}(2,\mathsf{R}) = \mathsf{so}(2,1)$. Instead of starting with the preservation of the Heisenberg canonical commutation relations (1.6) between the Schrödinger quantum position and momentum operators, here we start from the preservation of the commutators (1.46) and their realization by the three operators (1.43)–(1.45). Their commutators are preserved under linear transformations with parameters taken from $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{Sp}(2,\mathsf{R})$,

$$\begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{pmatrix} \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} \frac{1}{2}(a^2-b^2-c^2+d^2) & bd-ac & \frac{1}{2}(a^2-b^2+c^2-d^2) \\ cd-ab & ad+bc & -cd-ab \\ \frac{1}{2}(a^2+b^2-c^2-d^2) & -bd-ac & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix} \begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{pmatrix}. \tag{1.47}$$

These 3×3 matrices form the "2+1" Lorentz group $\mathsf{SO}(2,1)$ with metric $(--+)$. Since both $\mathbf{M}$ and $-\mathbf{M}$ yield the same 3×3 matrix, this Lorentz group is covered 2:1 by $\mathsf{Sp}(2,\mathsf{R})$; however, their Lie *algebras*, defined by their commutation relations, are the same.

In Sect. 1.4 we came upon the three orbits of $\mathsf{sp}(2,\mathsf{R}) = \mathsf{so}(2,1)$, which can be also be characterized by the distinctive spectrum of the generator that we choose to be the operator of position, $\{\rho\}$, which can be discrete or continuous. We can use the realization (1.43)–(1.45) in $\mathcal{L}^2(\mathsf{R}^+)$ for $\gamma > 0$ to evince those spectra.[5] The are:

[5]For $\gamma < 0$ there is a doubling of the Hilbert space that requires some extra analytical finesse [57], which stems from a separation in hyperbolic coordinates such as that seen in the previous subsection.

- In the ω = **elliptic** orbit of the compact "harmonic oscillator $+\gamma/r^2$" operator, $\hat{J}_3$ in (1.45) has a discrete spectrum $\{\rho\}_3$ bounded from below, and equally spaced by 1.
- In the ω = **parabolic** orbit of the "thin lens" generator in (1.25), here $\hat{J}_- := \hat{J}_3 - \hat{J}_1 = \frac{1}{2}r^2 \geq 0$, the spectrum $\{\rho\}_-$ is continuous and non-negative. Its Fourier-Bessel transform is $\hat{J}_+ := \hat{J}_3 + \hat{J}_1$, which is the Hamiltonian of "free flight in a γ/r^2 potential," and has the same spectrum.
- In the ω = **hyperbolic** orbit of the "repulsive oscillator $+\gamma/r^2$" operator, $\hat{J}_1$ in (1.43), the spectrum $\{\rho\}_1$ is the real line.

Thus, while $\mathsf{su}(2) = \mathsf{so}(3)$ contains a single orbit and the spectrum $\{\mu\}$ of any generator J_z can provide the row and column labels—positions—for the spin j representation matrices and vectors, bound by integer-spaced $|\mu| \leq j$, in $\mathsf{sp}(2,\mathsf{R}) = \mathsf{so}(2,1)$ we have *three* orbits and *three* choices for the position $\{\rho\}$: discrete, continuous positive, or real. Moreover, while the representations of $\mathsf{so}(3)$ are simply labelled by the non-negative integers $j \in \mathsf{Z}_0^+$ in the eigenvalues $j(j+1)$ of the square angular momentum, the representation structure of $\mathsf{so}(2,1)$ and the bounds it imposes on $\{\rho\}$ are more complicated. The parameter γ in (1.43)–(1.45) is the strength of the centrifugal ($\gamma > 0$) or centripetal potential ($\gamma < 0$); the special case $\gamma = 0$ will remit us back to the original and best-known LCT face in (1.16)–(1.17). This parameter γ determines *almost* (see below) the representation of the algebra through the eigenvalues of the $\mathsf{so}(2,1)$ invariant *Casimir* operator,

$$\hat{C} := \hat{J}_1^2 + \hat{J}_2^2 - \hat{J}_3^2 = (-\tfrac{1}{4}\gamma + \tfrac{3}{16})\, 1 =: k(1-k)\, 1, \tag{1.48}$$

$$\gamma = (2k-1)^2 - \tfrac{1}{4}, \qquad k = \tfrac{1}{2}(1 \pm \sqrt{\tfrac{1}{4} + \gamma}). \tag{1.49}$$

Here, k is the all-important *Bargmann index*; it distinguishes the two main series of representations:

- Bargmann **discrete** $\mathcal{D}_k^\pm$ representations [47] (called *complementary* by Gel'fand and Naĭmark [53]). When the coefficient γ is of centrifugal origin in two dimensions, angular momentum $\mu \in \mathsf{Z}$ determines $\gamma = \mu^2 - \frac{1}{4} \geq -\frac{1}{4}$, which implies the range $k = \frac{1}{2}(|\mu| + 1) \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. This series can be extended to continuous $k \in \mathsf{R}^+$, representing multiple covers of $\mathsf{so}(2,1)$. In particular for k quarter-integers, they are faithful representations of $\mathsf{Mp}(2;\mathsf{R})$. The $\mathcal{D}_k^-$ representations are related to the $\mathcal{D}_k^+$ ones by an outer automorphism of the group that in geometric optics is reflection [48, Sect. 10.4].
- Bargmann **continuous** $\mathcal{C}_s^\varepsilon$ representations (called *principal* by Gel'fand and Naĭmark). When $\gamma < 0$, the potential is centripetal and we must further distinguish the *exceptional* range $-\frac{1}{4} \leq \gamma < 0$ where $\frac{1}{2} \leq k < 1$ is real, from the *principal* range $\gamma < -\frac{1}{4}$ where $k = \frac{1}{2} + \mathrm{i}s$, with $s = \pm\frac{1}{2}\sqrt{(|\gamma| - \frac{1}{4})} \in \mathsf{R}$, and $\varepsilon \in \{0, \frac{1}{2}\}$ is a multivaluation index. We shall exclude the exceptional range

$0 < k < 1$ from further detailed considerations.[6] We treat this interval as an extension of the $\mathcal{D}_k^{\pm}$ discrete series.

The best-known one-dimensional LCT in (1.16)–(1.17) occurs for $\gamma = 0$, namely the quarter-integers $k = \frac{1}{4}$ and $k = \frac{3}{4}$, for the subspaces of even and odd functions of ρ, respectively—recall that here we are on the "radial" half-line for the inner product (1.32) of $\mathcal{L}^2(\mathsf{R}^+)$.

Using Dirac's shorthand notation, let $|k, \rho\rangle^{\omega}$ be a basis vector for the unitary irreducible representation k (in $\mathcal{D}_k^+$ or $\mathcal{C}_s^{\varepsilon}$), with row ρ (discrete or continuous) determined by the orbit ω of the chosen position operator. We may then understand LCTs as the unitary irreducible representations of $\mathbf{M} \in \mathsf{Sp}(2, \mathsf{R})$ acting on those Hilbert space bases and functions,

$$f_{\mathrm{M}}^{k,\omega}(\rho) = \underset{\rho'(k,\omega)}{\mathsf{S}} D^{k,\omega}_{\rho,\rho'}(\mathbf{M}) f(\rho'), \tag{1.50}$$

$$D^{k,\omega}_{\rho,\rho'}(\mathbf{M}) := {}^{\omega}\langle k, \rho | \mathcal{C}_{\mathrm{M}} | k, \rho' \rangle^{\omega}, \qquad \text{i.e.,} \tag{1.51}$$

$${}^{\omega}\langle k, \rho | f_{\mathrm{M}} \rangle = {}^{\omega}\langle k, \rho | \mathcal{C}_{\mathrm{M}} | f \rangle = {}^{\omega}\langle k, \rho | \mathcal{C}_{\mathrm{M}} | k, \rho' \rangle^{\omega}\, {}^{\omega}\langle k, \rho' | f \rangle, \tag{1.52}$$

where S_{ρ} is a sum or integral over the range of eigenvalues of position $\rho(k, \omega)$ allowed in the representation k, where the chosen position operator is in the orbit ω. The ranges of its "position coordinate" are:

$$\begin{array}{lll} & \mathcal{D}_k^+ & \mathcal{C}_s^{\varepsilon} \\ \omega \text{ elliptic:} & \rho = k+n,\ n \in \mathsf{Z}_0^+ & \rho - \varepsilon \in \mathsf{Z} \\ \omega \text{ parabolic:} & \rho \in \mathsf{R}^+ & \rho \in \mathsf{R}^+ \oplus \mathsf{R}^+ \\ \omega \text{ hyperbolic:} & \rho \in \mathsf{R} & \rho \in \mathsf{R} \oplus \mathsf{R} \end{array} \tag{1.53}$$

The orthogonality and completeness of the bases $|k, \rho'\rangle^{\omega}$ guarantees that the group composition property holds and that the transformation is unitary and hence invertible,

$$\underset{\rho'(k,\omega)}{\mathsf{S}} D^{k,\omega}_{\rho,\rho'}(\mathbf{M}_1)\, D^{k,\omega}_{\rho',\rho''}(\mathbf{M}_2) = D^{k,\omega}_{\rho,\rho''}(\mathbf{M}_1 \mathbf{M}_2), \tag{1.54}$$

$$D^{k,\omega}_{\rho,\rho'}(\mathbf{M}^{-1}) = D^{k,\omega}_{\rho',\rho}(\mathbf{M})^*. \tag{1.55}$$

The matrices and integral kernels $D^{k,\omega}_{\rho,\rho'}(\mathbf{M})$ are known in the literature. They were written out for ω = elliptic by Bargmann [47]; for ω = hyperbolic by Mukunda and Radhakrishnan [60]; and for ω = parabolic they are the radial and hyperbolic LCT kernels of this section. In [57] all ${}^{\omega}\langle k, \rho | \mathcal{C}_{\mathrm{M}} | k, \rho' \rangle^{\omega'}$ are listed, including the mixed cases $\omega \neq \omega'$; these were later used to find the $\mathsf{so}(2, 1)$

[6] The generators present a one-parameter family of self-adjoint extensions with non-equally spaced spectra [58] and also harbor the $\mathcal{E}_k$ exceptional (or *supplementary*) representation series [59].

Clebsch–Gordan coefficients between all representation series [61]. Finally, while writing this chapter, I completed the work in [62], giving explicitly (in the present notation) the six distinct *faces* of LCTs, $D^{k,\omega}_{\rho,\rho'}(\mathbf{M})$ for the three orbits in the two nonexceptional representation series. I close this section reminding the readers that there is a theorem stating that noncompact groups (i.e., of infinite volume) do *not* have faithful finite-dimensional unitary representations; thus, $\mathsf{Sp}(2,\mathsf{R})$ only has finite representations that are *not* unitary—such as the 2×2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the 3×3 matrix in (1.47), or others of "spin" k given in [48, Eq. (13.6)] that are used for Lie aberration optics.

1.6 Complex Extensions of LCTs

While LCTs allow a transparent formulation of the properties of resonators, where a paraxial wavefield is bounced repeatedly between two end-mirrors, it is natural to inquire about systems with loss or gain [10, 11]. On the other hand, applications to clustering in nuclei [63] required the description of Gaussian packets in terms of the raising and lowering operators of the harmonic oscillator [64], i.e.,

$$\begin{pmatrix} \hat{z}^{\uparrow} \\ -\mathrm{i}\hat{z}^{\downarrow} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{x} - \mathrm{i}\hat{p} \\ -\mathrm{i}(\hat{x} + \mathrm{i}\hat{p}) \end{pmatrix}, \quad -\mathrm{i} \equiv e^{-\mathrm{i}\pi/2}. \tag{1.56}$$

Issues related to the meshing between Bargmann and LCT transforms were discussed in a think-tank at the Centre de Recherches Mathématiques (Université de Montréal) during the closing months of 1973. It was also noted that the real heat diffusion kernel to time $t > 0$ is the $\begin{pmatrix} 1 & -2\mathrm{i}t \\ 0 & 1 \end{pmatrix}$ complex LCTs with kernel $\sim \exp(-(x - x')^2/4t)/\sqrt{t}$; for $t > 0$ these transforms form a *semi*-group (i.e., without inverses). Indeed, one can extend the $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ parameters as long as the LCT kernel (1.17) is a *decreasing* Gaussian in the argument x' subject to integration, namely $\operatorname{Re}(\mathrm{i}a/b) < 0$. If a is real, this means that the complex value of b must be in the lower complex half-plane, $-\pi < \arg b < 0$.

But unitarity is a cherished property among group theorists, so the question was posed to find appropriate Hilbert spaces to comply with this requirement. There was the precedent of Bargmann's space for analytic functions $f(z)^* = f(z^*)$ [49], so it was not difficult [51] to follow his construction in proposing a *measure* for the sesquilinear inner product that integrates over the whole complex plane $z \in \mathsf{C}$, of $\mathcal{L}^2(\mathsf{R})$ functions that have been transformed by a complex $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$(\mathbf{f}_{\mathrm{M}}, \mathbf{g}_{\mathrm{M}})_{\mathcal{B}_{\mathrm{M}}} := \int_{\mathsf{C}} \mathrm{d}^2\mu_{\mathrm{M}}(z, z^*) f_{\mathrm{M}}(z)^* g_{\mathrm{M}}(z) = (\mathbf{f}, \mathbf{g})_{\mathcal{L}^2(\mathsf{R})}, \tag{1.57}$$

$$\mathrm{d}^2\mu_{\mathrm{M}}(z, z^*) = \nu_{\mathrm{M}}(z, z^*)\, \mathrm{d}\operatorname{Re} z\, \mathrm{d}\operatorname{Im} z, \quad \text{where} \tag{1.58}$$

$$\nu_{\mathrm{M}}(z,z^*) \;=\; \sqrt{\frac{2}{\pi v}}\exp\Big(\frac{uz^2-2zz^*+u^*z^{*\,2}}{2v}\Big), \tag{1.59}$$

$$u := a^*d-b^*c, \quad v := 2\,\mathrm{Im}\,(a\,b^*) > 0. \tag{1.60}$$

This defines Bargmann-type Hilbert spaces $\mathcal{B}_{\mathrm{M}}$ such that the complex LCT between $\mathcal{L}^2(\mathsf{R}) = \mathcal{B}_1$ and $\mathcal{B}_{\mathrm{M}}$ is unitary, and can be inverted back to $\mathcal{L}^2(\mathsf{R})$ through

$$f(x) = \int_{\mathsf{C}} \mathrm{d}^2\mu_{\mathrm{M}}(z',z'^*)\,C_{\mathrm{M}^{-1}}(x,z')f_{\mathrm{M}}(z'). \tag{1.61}$$

In the limit when $\mathbf{M}$ becomes real, the measure weight function $\nu_{\mathrm{M}}(z,z^*)$ in (1.59) is a Gaussian that collapses to a Dirac δ on the $\mathrm{Re}\,z'$ axis [6, Sect. 9.2.2]. Of interest to mathematicians is the use of the hyperdifferential operator realization of complex LCTs to find an expression for Hermite polynomials, such as [51, App. A]

$$H_n(x) = \exp\Big(-\frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big)\,(2x)^n, \qquad x = \exp\Big(\frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big)\,H_n(x). \tag{1.62}$$

Similar relations could be found for parabolic cylinder and other special functions, but have not been investigated.

Radial LCTs can also be extended to the complex domain [55] when the radial kernel (1.35) is a decreasing Gaussian, $\mathrm{Re}\,(\mathrm{i}a/b) < 0$ as before. But now, noting in (1.32) that the argument of the functions is $r \in [0,\infty)$, it turns out that the complex-transformed functions will be analytic only in the right half-plane $\varrho \in \mathsf{C}^+$ where $\mathrm{Re}\,\varrho > 0$. The Bargmann-type inner products that preserve the unitarity of the complex RLCTs that map $f(r) \in \mathcal{L}^2(\mathsf{R}^+)$ to $f_{\mathrm{M}}^{(m)}(r) \in \mathcal{B}_{\mathrm{M}}^{(m)}$ are

$$(\mathbf{f}_{\mathrm{M}}^{(m)},\mathbf{g}_{\mathrm{M}}^{(m)})_{\mathcal{B}_{\mathrm{M}}^{(m)}} := \int_{\mathsf{C}+} \mathrm{d}^2\mu_{\mathrm{M}}^{(m)}(\varrho,\varrho^*)f_{\mathrm{M}}^{(m)}(\varrho)^*\,g_{\mathrm{M}}^{(m)}(\varrho) = (\mathbf{f},\mathbf{g})_{\mathcal{L}^2(\mathsf{R}+)}, \tag{1.63}$$

$$\mathrm{d}^2\mu_{\mathrm{M}}^{(m)}(\varrho,\varrho^*) \;=\; \nu_{\mathrm{M}}^{(m)}(\varrho,\varrho^*)\;\mathrm{d}\,\mathrm{Re}\,\varrho\;\mathrm{d}\,\mathrm{Im}\,\varrho, \quad \text{where} \tag{1.64}$$

$$\nu_{\mathrm{M}}^{(m)}(\varrho,\varrho^*) \;=\; \frac{2}{\pi v}\exp\Big(\frac{u\varrho^2+u^*\varrho^{*\,2}}{2v}\Big)\,K_m\Big(\frac{\varrho\varrho^*}{v}\Big), \tag{1.65}$$

where $K_m(z)$ is the MacDonald function, while u and v are given by (1.60). The inversion and real limit properties are similar to those of the complex LCTs seen above.

A specific case of interest is the treatment of the Barut–Girardello transform and coherent state [65]. Similar to (1.62), one obtains a hyperdifferential form for the Laguerre polynomials [55],

$$L_n^{(m)}(\tfrac{1}{2}r^2) = \frac{(-1)^n}{n!\,2^n}r^{-m}\exp\Big[\frac{-1}{2}\Big(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{m^2}{r^2}\Big)\Big]\,r^{2n+m}. \tag{1.66}$$

Finally, hyperbolic LCTs do not allow for any complex extension [56].

1.7 Finite Data Sets and LCTs

Most data sets in the real world are finite and can be represented as N-component vectors, $\mathbf{f} \equiv \{f_m\}_{m=1}^N$. If the numbers come from sensing a continuous wavefield $f(x)$ at N discrete points, how can we compute their propagation through an optical LCT setup? The most direct answer is sampling the assumed smooth wavefield $f_m := f(x_m)$ and the LCT kernel (1.17) at the same points $\{x_m\}_{m=1}^N$, and simply performing the product of the $N \times N$ matrix $\mathbf{C}_{\mathrm{M}}$, function of $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and the vector $\mathbf{f}$. Following the discretization adopted in [66] for $x_m = m\sqrt{(2\pi/N)}$, this is

$$(\mathbf{C}_{\mathrm{M}}\,\mathbf{f})_m := \sum_{m'=1}^{N} \frac{1}{\sqrt{N}} \exp\Big(\frac{\mathrm{i}\pi}{bN}(dm^2 - 2mm' + am'^2)\Big) f_{m'}, \tag{1.67}$$

$$(\mathbf{C}_{\mathrm{M}}\,\mathbf{f})_m := \exp(\mathrm{i}\pi\, cm^2/aN) f_m \qquad \text{when } b = 0, \tag{1.68}$$

where we leave out phases. Yet this transformation is only an approximation to the LCT, and it is generally not unitary.

Regarding the spacing of the sampling points, Ding [67] has given a sampling theorem that generalizes that of Shannon in terms of the desired extent of the LCT transform signal. The requirement of unitarity on the kernels (1.67) is that $\mathbf{C}_{\mathrm{M}}^{\dagger}\mathbf{C}_{\mathrm{M}} = \mathbf{1}$; this occurs only for values of the parameter b such that $1/b$ is an integer relatively prime to N [68]. Combining both results, in [69], the authors present sufficient conditions on the sampling rate of $\{x_m\}_{m=1}^N$ for any one LCT to ensure its unitarity. However, two such matrices will not concatenate as integral LCTs do because, as we mentioned at the end of Sect. 1.5, $\mathsf{Sp}(2, \mathsf{R})$ has no finite-dimensional unitary irreducible representations. Alternatively, if one discretizes the LCT kernel by using the LCT sampling theorem [67, 70], a unitary discrete LCT which provides a provably good approximation to the continuous LCT can be obtained [71, 72]. In principle, one would like to have a relation between the discrete and continuous LCTs that mirrors and generalizes the corresponding relation for ordinary Fourier transforms. Such a relation has been provided in [71], showing that the discrete LCT, as defined in [73], approximates the LCT in the same sense that the discrete Fourier transform approximates the continuous Fourier transform, provided that the number of samples and the sampling intervals are chosen according to the LCT sampling theorem [67, 70–72].

There are some other problems to define subsets of finite LCTs that form a group, which we can point out for finite analogues of fractional Fourier transforms. If we search for a one-parameter subgroup of unitary $N \times N$ matrices $\mathbf{F}^{\nu} \in \mathsf{U}(\mathsf{N})$ such that $\mathbf{F}^{\nu}\mathbf{F}^{\mu} = \mathbf{F}^{\nu+\mu}$, with $\mathsf{F}^1 = \mathbf{F}$ being the well-known finite Fourier transform matrix,

$$F_{m,m'} = \frac{1}{\sqrt{N}} \exp\Big(-\mathrm{i}\frac{2\pi\, m\, m'}{N}\Big), \tag{1.69}$$

which is unitary and idempotent, $\mathbf{F}^4 = \mathbf{1}$, what we find is a deluge of possibilities: we can draw the N^2 real parameters of unitary $N \times N$ matrices inside a sphere in R^{N^2} space; the four matrix powers of $\mathbf{F}$ are but four points—with $\mathbf{1}$ on the origin. Unitary matrices of unit determinant form the simply connected subgroup $\mathsf{SU}(N)$, whose $\mathsf{SO}(2)$ subgroups of possible fractional $\mathbf{F}^\nu$'s are closed lines (picture them as circles) that can be freely rotated keeping the origin fixed, and are only required to pass through $\mathbf{F}$ for $\nu = 1$, since automatically the circles will pass also through its integer powers. Clearly, for dimensions $N > 2$ there is a continuum of such circles that can be drawn through two points. From that perspective, we analyzed this freedom in [74] in terms of choosing "good" bases for the R^N manifold. Alternative approaches to define good bases to build finite fractional Fourier transform matrices have used sampled harmonic oscillator wavefunctions, as done by Pei et al. [73, 75, 76], or other candidates such as the Harper functions [77] by Ozaktas et al.

Additionally, there is a problem with the phase space interpretation of these finite fractional Fourier transforms, which we can see through the commonly used finite Wigner function [25]. The finite Fourier transform matrix $\mathbf{F}$ brings N-vectors of position to N-vectors of momentum. And, being cyclic, $F_{m,m'} = F_{m+N,m'} = F_{m,m'+N}$ leads to consider a phase space that is discrete and connected as a torus. The "front face" of this torus is the origin of phase space $m = 0 = m'$. So, while the *integral* fractional Fourier transform rotates the phase space plane around the origin, we *cannot* rotate the front face of a torus without tearing it. Yet to be applicable, finite LCTs must be computed efficiently for one- or two-dimensional signals and images in real time. This line of research has been developed by Sheridan et al. in [66, 78–81] with the strategy of separating the finite LCT in (1.67) into a Fourier transform factor, for which the FFT algorithm exists, and factors of (1.68). Another strategy for fast and accurate computation of LCTs has been developed by Ozaktas et al. using the Iwasawa decomposition [82, 83]. Alternatively, a chirp-Fourier-chirp factorization with a fast-convergent quadrature formula was proposed in [84].

Finally, we should mention another fast computation method that also involves chirp multiplication, fast Fourier transform, and a second chirp multiplication [71]: this method has the advantage of involving the least number of samples possible as determined from the LCT sampling theorem [67, 70]. This discrete LCT has a well-defined relation to the continuous LCT and can be made unitary by adding a factor in front [71, 72]. This approach is attractive because it combines a desirable analytic discrete LCT definition with a computational method that is nearly as fast and accurate as the fast Fourier transform algorithm to compute continuous Fourier transforms.

1.8 Conclusion

I was deeply honored by the invitation of the Editors to write some pages about the development of linear canonical transforms, in company with distinguished researchers who are applying them in encryption, metrology, holography, and optical implementations. Emeritus Professor Stuart Collins is still active and has registered six patents to his name between 1982 and 2008; his work in optoelectronics has been applied for space science. I was an apprentice of Marcos Moshinsky and developed his work on quantum mechanics as a fruitful model for LCTs and related transforms. Only later did I learn that LCTs were excellent tools for paraxial optics as used by a community with whom I could then establish dialogue.

Perhaps a similar bifurcation of viewpoints may occur concerning finite LCTs. The previous section contains problems which I regard as indicative that a different approach can be useful to understand finite signals on phase space and their canonical transformations. Based on the rotation algebra $\mathsf{so}(3)$, instead of the Heisenberg–Weyl algebra of quantum mechanics, we have proposed a model for discrete Hamiltonian systems where phase space is a sphere [85]. When the number of position points and their density increase, the model contracts to that of quantum mechanics, the sphere blowing up into the quantum phase space plane. Canonical transformations in $\mathsf{so}(3)$ are those that preserve the surface elements of the sphere. *Linear* transformations of N-point signals are the rigid rotations of that sphere. Among these, the fractional Fourier-*Kravchuk* transform [86] describes the time evolution of this finite harmonic oscillator. Moreover, *nonlinear* canonical transformations can be defined in correspondence with optical aberrations as matrices in the full $\mathsf{U}(N)$ group of linear transformations of N-vectors [87]. Based on the Euclidean and Lorentz algebras, other discrete models are available in one and two dimensions [88].

On the other hand, it is not clear that expansions in group-theoretic bases have any advantage over other bases for expansion [89], since they do not seem amenable to fast algorithms. Still, based on previous experiences, I harbor the hope that the mathematical landscape succinctly described here can be of use to broaden the perspective we have of canonical transformations of phase space. The founders of this field must have been quite unaware of the full panorama they opened for us to see.

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