6

Elements of Euclidean optics

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ABSTRACT Euclidean optics are models of the manifold of rays and wavefronts in terms of coset spaces of the Euclidean group. One realization of this construction is the geometric model of Hamilton's optical phase space. Helmholtz optics is a second Euclidean model examined here. A wavization procedure is given to map the former one on the latter. Non-euclidean transformations of the manifold of rays are provided by Lorentz boosts that produce a global "4π" comatic aberration.

6.1 Introduction

A plane is said to be similarly inclined to a plane as another is to another when the said angles of the inclinations are equal to one another... Equal and similar solid figures are those contained by similar planes equal in multitude and in magnitude.

Euclid, *Elements*
Book XI, *Definitions 7 and 9*

This monograph structures the results of several previous articles, some written more than a decade ago, in the light of discussions held during and after the second Lie Optics workshop. The considerable advances in the use of Lie algebraic methods for magnetic and light optical design suggest their application to other closely related areas, such as polarization and wave optics, and invites incorporating more distant fields such as signal analysis and tomography, in the directions of polychromatic and far-metaxial optics.

The motivation of the present work may be focused in two questions that can be posed, in the context of their current solution, in the following terms:

1. *How far off the optical axis can we go?* Paraxial optics with its linear transformations of phase space is, of course, the starting point. The art of aberration expansions into the metaxial regime has been refined by their classification, computation, and understanding the way they compound in propagating through a system. The validity of such expansions must stop somewhere, however; probably much before 90°, and certainly before 180°.

2. *How do we wavize geometric optics?* Again, the paraxial regime may be said to be domestic territory: the symplectic group of canonical integral transforms provides a well-established bridge to transit from geometric to wave optics, as it does between classical and quantum mechanics in systems with quadratic potentials. On the wave side, it couples naturally with Fourier and coherent-state optics. Hilbert spaces, Wigner distribution functions, and measurement theories, often translated from quantum mechanics, are available. The metaxial regime does not seem to have such a reliable wavization procedure. In particular, we would like to be able to design an optical system with the tools of Lie geometric optics, and thereby know its behavior as we turn on the wave nature of light. Lastly, there is a gulf in the *global* regime, *i.e.*, "4π"optics that extends over the *full* sphere of ray directions.

From the experience of quantum mechanics, it is evident that both questions are related. There, *global* properties of the potential (over the full real line or 3-space) are most important, and the 'far-away' regions can be seldom ignored.[1] Yet, the most striking *difference* between (nonrelativistic point-particle) mechanics and (geometric) optics is in their phase spaces: the former is flat and unbounded both in position and in momentum, while in the latter momentum ranges over a sphere projected flat on two disks in its equatorial plane. The Heisenberg-Weyl group of phase space motions underlies the symplectic geometry of the former, but not of the latter, unless we replace optics by its paraxial regime. It is our contention that the basic group of *global* optics is the three-dimensional *Euclidean* group $\mathcal{E}_3 = \mathrm{ISO}(3)$ of rigid motions of three-space.

In Section 2 we examine the structure of the manifold of rays of geometric optics, a vector bundle $\wp$, introducing the local and standard screen coordinates, and the *Descartes* sphere of ray directions. Section 3 reviews the composition rule for the Euclidean group of Lie operators that take a standard frame to any position and orientation. Thus we have a Euclidean theory of *frames*. The infinitesimal *generators* yield the Euclidean algebra on that group manifold given in Section 4.

When the objects that we regard as elementary have a *symmetry* group $\mathcal{H} \subset \mathcal{E}_3$, a corresponding model of optics follows. For example, in geometric optics rays are pictured as straight lines filling space, with a $\mathcal{T}_1$ symmetry under translations along the line times $\mathcal{R}_2$ under rotations around it. Sections 5, 6, and 7 show that the manifold of rays $\wp$ is the space of *cosets* $\mathcal{H}\backslash\mathcal{E}_3$. The Euclidean group and algebra are realized on that space, and shown to be canonical in the usual symplectic sense of Hamilton's theory, here derived from the conservation of the Haar measure. Indeed, also po-

[1] On the other hand, in quantum mechanics, kinetic energy is mostly of the fixed standard form $p^2/2m$.

larization and signal optics may be identified with coset spaces, $\mathcal{R}_2\backslash\mathcal{E}_3$ and $\mathcal{T}_1\backslash\mathcal{E}_3$, respectively.

Our second main model of interest is *wave* optics. When the elementary object is a plane, it defines *wavefront* optics.[2] This we introduce in Section 8 as the manifold $\mathcal{E}_2\backslash\mathcal{E}_3$, and note that it can carry *signals* in a train amenable to Fourier analysis. In Section 9 we show that the Fourier decomposition is irreducible under Euclidean transformations, each wavenumber component satisfying a corresponding *Helmholtz* equation that is the Casimir invariant of the algebra. Following the *modus operandi* of quantum mechanics, Section 10 builds the Hilbert space of oscillatory solutions over the standard screen whose inner product, uniquely invariant under Euclidean motion and endowed with a non-local measure, was previously found with Stanly Steinberg (Albuquerque). This, rather than $\mathcal{L}_2(\Re^2)$, seems to be the Hilbert space appropriate for wavized optics because, as shown in Section 11, it involves by necessity and on the same footing, both the wave function and its *normal derivative* at the screen.[3] With these elements we propose a definite "4π wavization" process that leads from geometric to wave optics on the level of the Euclidean group.

Up to here, we deal only with a group theory of rigid motions. Section 12 introduces the *Lorentz* transformation responsible for stellar aberration. Although this phenomenon has been known for centuries, its implications for Hamiltonian optics in image formation seem to have been overlooked. Sections 13 and 14 contain the results of two recent papers with Natig Atakishiyev (Baku) and Wolfgang Lassner (Leipzig) that predict a global *comatic aberration* of geometric and wave images on boosted screens, stemming from the nonlinear action of the Lorentz group on the corresponding coset space models. The proposed wavization process is applied to these transformations, and seen to hold.

Field theories on groups in empty space have been abundant —and some have been very important; yet optics visibly needs the dynamics of inhomogeneous media. We have felt obliged thus to add some preliminary reflections on refraction among the concluding remarks in Section 15. This process appears as a *coupling* between representations of the Euclidean group through the conservation law due to Willebrord Snell (experimentally) and René Descartes (theoretically). We are inspired by the Cartesian *Mèthode* in regarding Optics as Nature observing Symmetry, because in that way it pleases the mind.

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[2]When only $\mathcal{T}_2$ symmetry is present, it describes *polarized* polichromatic wave optics.

[3]Quantum mechanics uses only the first one because the Schrödinger equation has a first degree derivative only in the evolution variable.

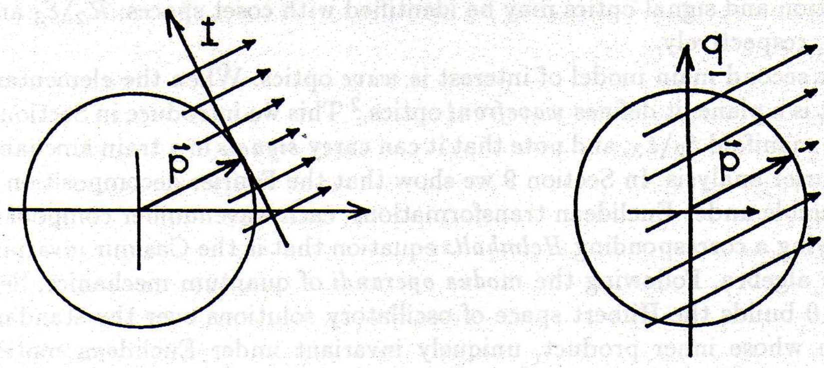

FIGURE 1. (a) Local screen parametrization of a bundle of parallel rays. (b) Parametrization by a fixed reference screen.

6.2 The bundle of rays in geometric optics

> ***We ought to give the whole of our attention to the most insignificant and most easily mastered facts, and remain a long time in contemplation of them until we are accustomed to behold the truth very clearly and distinctly.***
>
> René Descartes, *Rules for the Direction of the Mind*
> Rule IX

In geometric optics light rays are modeled as lines in 3-space oriented in all directions. In a homogeneous medium the lines are straight, and usual Euclidean geometry applies. We examine here this manifold $\wp$ of oriented lines to show its structure, and find good sets of coordinates.

With the tools of thought we can sort out all those lines that are oriented in a chosen direction and identify them by a point $\vec{n}$ on a sphere $\mathcal{S}_2$. This is a projection $\pi : \wp \mapsto \mathcal{S}_2$. The inverse image $\pi^{-1}(\vec{n})$ of such a point is a set of paralell rays that can be brought one onto the other by translations within their perpendicular plane. The set of translations constitute a group and a vector space $\mathcal{T}_2$. These properties of structure are those of a *vector bundle* [1]. The manifold of light rays in geometric optics is thus a (non-trivial) vector bundle $\wp$, with an $\mathcal{S}_2$ *base space* of ray directions, and a $\mathcal{T}_2$ *local screen*, the typical *fiber* of the bundle. The screens are local because each is associated to its set of parallel rays and perpendicular to them. Two rays that are not paralell will have their screens oriented differently. Let us now introduce coordinates.

The set of directions of the rays —the $\mathcal{S}_2$ base space of the bundle— is the Descartes sphere. This can be parametrized by a pair of usual Euler angles $\{\theta, \phi\}$, with the well-known but treatable difficulties of coordinates on spheres. Alternatively, we may choose a Cartesian three-vector $\vec{p} = (p_x, p_y, p_z)$ on a sphere of fixed radius $n = |\vec{n}| = \sqrt{p_x^2 + p_y^2 + p_z^2}$. (Of course,

n will be the refractive index of the medium!) Two of the three coordinates, say $\mathbf{p} = (p_x, p_y)$, supplemented by the sign σ of the otherwise redundant third component $p_z = \sigma\sqrt{n^2 - p_x^2 - p_y^2}$, will also serve to indicate a point of the Descartes sphere uniquely.[4] Depending on the need we shall use one or the other parametrization for $\mathcal{S}_2$.

The local screen of each paralell pencil of rays is an $\Re^2$ Cartesian manifold of translation two-vectors $\{t_{x'}, t_{y'}\}$ with an origin or *optical center* and an orientation of its local x' and y' axes. We may choose these in the direction of the p_x and p_y axes of the Descartes sphere, but we will have orientation trouble when effecting the paralell transport of the screens around the sphere.[5] Such features are normal in bundles that are not direct products, but we may always work on local charts in a sizable neighborhood of some standard ray. See Figure 1(a).

A local parametrization that is preferred in Hamiltonian optics is that of a *standard screen*. This screen is referred to an optical axis $\vec{p}_0 = (0, n)$, placed at $z = 0$. Rays within paralell pencils are parametrized by their observable of *position*, *i.e.*, their intersection $\mathbf{q} = (q_x, q_y)$ with that standard screen. See Figure 1(b). This works well except for rays that are paralell to the screen, all of which will map on a point at infinity. Again, the coordinates parametrize well sizable neighborhoods of rays, but cannot be global. Since it *does* happen that $\mathbf{q}$ and $\mathbf{p}$ are *canonically conjugate* in every neighborhood, they do deserve particular attention.

We may approach similarly other models, such as the manifold of frames (oriented point-objects), ribbons or of screws for polarization optics, and planes (for wavefront optics). We see naturally such manifolds as coset spaces of the group of Euclidean motions, modulo the symmetry subgroup of the object.

6.3 Lie operators on the Euclidean group

The action of the elements g of a Lie group $\mathcal{G}$ on the space of functions f of its own manifold variables γ [2], is carried by Lie transformations $\mathcal{L}_g$ [3] that may be realized in at least two ways: through right or left action, *viz.*,

$$\mathcal{L}_g^{\mathrm{R}} f(\gamma) = f(\mathcal{L}_g^{\mathrm{R}}\gamma) = f(\gamma g), \tag{3.1a}$$

$$\mathcal{L}_g^{\mathrm{L}} f(\gamma) = f(\mathcal{L}_g^{\mathrm{L}}\gamma) = f(g^{-1}\gamma). \tag{3.1b}$$

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[4] When $p_z = 0$ we may take $\sigma = 0$.

[5] The rays of geometric optics may be easily parametrized in Cartesian solid geometry by the line $\vec{r}(s) = s\vec{p} + \vec{t}$, where $s \in \Re$ measures length along the ray in units of $1/n$. The vector $\vec{t}$ may be chosen orthogonal to $\vec{p}$, namely $\vec{t}\cdot\vec{p} = 0$, so it contains only two independent local screen parameters.

Still another one is the conjugation $\mathcal{L}_g^{\mathrm{C}} f(\gamma) = f(g^{-1}\gamma g)$, and in fact any subgroup $\mathcal{G}$ of $\mathcal{G} \times \mathcal{G}$ may be used with bilateral action [4].

Lie transformations may be generally represented as exponentials of differential operators of first degree in the manifold coordinates of γ. They have the above property of "jumping into" the function's arguments. Let us verify step by step that the right action (3.1*a*) is consistent with the requirement that the group composition property be preserved, *i.e.*, that $\mathcal{L}_{g_1}^{\mathrm{R}} \mathcal{L}_{g_2}^{\mathrm{R}} = \mathcal{L}_{g_1 g_2}^{\mathrm{R}}$ holds when acting on any function f. We use (3.1*a*) for the rightmost factor to write

$$\mathcal{L}_{g_1}^{\mathrm{R}} \mathcal{L}_{g_2}^{\mathrm{R}} f(\gamma) = \mathcal{L}_{g_1}^{\mathrm{R}} f(\mathcal{L}_{g_2}^{\mathrm{R}} \gamma) = \mathcal{L}_{g_1}^{\mathrm{R}} f(\gamma g_2), \tag{3.2a}$$

and call

$$f_2(\kappa) = f(\mathcal{L}_{g_2}^{\mathrm{R}} \kappa) = f(\kappa g_2), \tag{3.2b}$$

for any κ such as γ. Thus we continue (3.2*a*) writing

$$\mathcal{L}_{g_1}^{\mathrm{R}} f_2(\gamma) = f_2(\mathcal{L}_{g_1}^{\mathrm{R}} \gamma) = f_2(\gamma g_1). \tag{3.2c}$$

Now we can use (3.2*b*) again, with $\kappa = \gamma g_1$, and finish (3.2*a*) with

$$\mathcal{L}_{g_1}^{\mathrm{R}} \mathcal{L}_{g_2}^{\mathrm{R}} f(\gamma) = f(\gamma g_1 g_2) = \mathcal{L}_{g_1 g_2}^{\mathrm{R}} f(\gamma). \tag{3.2d}$$

Observe carefully that $\mathcal{L}_{g_1}^{\mathrm{R}}$ acts *first* and $\mathcal{L}_{g_2}^{\mathrm{R}}$ *second* on γ to yield $\gamma g_1 g_2$, as read in the usual direction.[6]

The Euclidean group $\mathcal{E}_3 = \mathrm{ISO}(3)$ contains the group of rotations $\mathcal{R}_3 = \mathrm{SO}(3)$ and of translations $\mathcal{T}_3$. It has the following well-known structure that we display in coordinates. Let $\mathcal{R}_3 \ni \mathbf{R}$ be an orthogonal 3×3 matrix, of unit determinant, and $\mathcal{T}_3 \ni \vec{v} = (v_x, v_y, v_z)$ a Cartesian three-dimensional row-vector.[7] We may denote the elements of the Euclidean group $\mathcal{E}_3$ as

$$\mathsf{E}(\mathbf{R}, \vec{v}) = \mathcal{L}_{\mathbf{R}}^{\mathrm{R}} \mathcal{L}_{\vec{v}}^{\mathrm{R}} = \mathsf{E}(\mathbf{R}, \vec{0})\mathsf{E}(\mathbf{1}, \vec{v}). \tag{3.3a}$$

The two subgroup products are denoted by matrix multiplication and (commutative) sum as

$$\mathsf{E}(\mathbf{R}_1, \vec{0})\mathsf{E}(\mathbf{R}_2, \vec{0}) = \mathsf{E}(\mathbf{R}_1\mathbf{R}_2, \vec{0}), \tag{3.3b}$$

$$\mathsf{E}(\mathbf{1}, \vec{v}_1)\mathsf{E}(\mathbf{1}, \vec{v}_2) = \mathsf{E}(\mathbf{1}, \vec{v}_1 + \vec{v}_2). \tag{3.3c}$$

The yuxtaposition of the two subgroups is that of *semidirect product*, $\mathcal{E}_3 = \mathcal{R}_3 \triangleright \mathcal{T}_3$, specified by the action of $\mathcal{R}_3$ on $\mathcal{T}_3$, in row-vector and matrix notation

$$\mathsf{E}(\mathbf{R}, \vec{0})\mathsf{E}(\mathbf{1}, \vec{v})\mathsf{E}(\mathbf{R}^{-1}, \vec{0}) = \mathsf{E}(\mathbf{1}, \vec{v}\mathbf{R}^{-1}). \tag{3.3d}$$

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[6] A similar argument verifies that (3.1*b*), *i.e.*, action from the left by $g^{-1}\gamma$ is also consistent with the group property. *Not* consistent with it would be $g\gamma$, γg^{-1}, $g\gamma g$, etc.

[7] We use the top arrow $\vec{\ }$ to denote three-dimensional row-vectors.

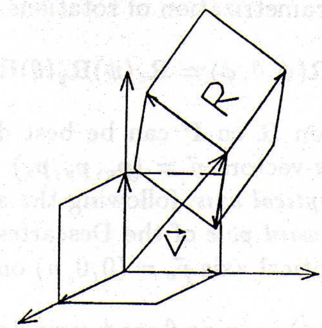

FIGURE 2. The standard and a Euclidean-displaced frame.

The translation subgroup is thus the invariant or *normal* subgroup of the Euclidean group. From here, the group multiplication law is

$$\mathsf{E}(\mathbf{R}_1, \vec{v}_1)\mathsf{E}(\mathbf{R}_2, \vec{v}_2) = \mathsf{E}(\mathbf{R}_1\mathbf{R}_2, \vec{v}_1\mathbf{R}_2 + \vec{v}_2). \tag{3.3e}$$

The group unit is $\mathsf{E}(\mathbf{1}, \vec{0})$, and the inverse is $\mathsf{E}(\mathbf{R}, \vec{v})^{-1} = \mathsf{E}(\mathbf{R}^{-1}, -\vec{v}\mathbf{R}^{-1})$.

Functions on the six-dimensional $\mathcal{E}_3$ manifold will be denoted by $f(\mathbf{P}, \vec{r})$, and we shall refer generically to $\mathbf{P}$ as *direction* and to $\vec{r}$ as *position*, because of ulterior motives. Direction is subject to rotation, and position to both rotation and translation. We act on these functions first with the right Lie transformation of the rotation $\mathcal{L}^{\mathrm{R}}_{\mathbf{R}}$ and second with that of the translation $\mathcal{L}^{\mathrm{R}}_{\vec{v}}$, to get

$$\mathsf{E}(\mathbf{R}, \vec{v})f(\mathbf{P}, \vec{r}) = \mathcal{L}^{\mathrm{R}}_{\mathbf{R}}\mathcal{L}^{\mathrm{R}}_{\vec{v}}f(\mathbf{P}, \vec{r}) = \mathcal{L}^{\mathrm{R}}_{\mathbf{R}}f(\mathbf{P}, \vec{r} + \vec{v}) = f(\mathbf{P}\mathbf{R}, \vec{r}\mathbf{R} + \vec{v}), \tag{3.4}$$

by (3.1*a*) and (3.3*e*). Upon this, the group unit $\mathsf{E}(\mathbf{1}, \vec{0})$ is rotated by $\mathbf{R}$ and translated by $\vec{v}$ to $\mathsf{E}(\mathbf{R}, \vec{v})$. The group unit may be seen as a *standard frame* of three orthogonal axes, defining the origin both of direction and position. See Figure 2. Through rotations and translations, we may bring this frame to any other one, mounted on the generic $\vec{r}$ and with the orientation obtained by the action of the rotation matrix $\mathbf{P}$ on the standard frame. At this stage we may say that we are defining the model of frames as the set of $\mathcal{R}_3$–orientable objects in $\mathcal{T}_3$–space. Note that the translation by $\vec{v}$ of the frames' origin is performed with reference to the *standard* frame, *i.e.*, $\vec{r}\mathbf{R} + \vec{v}$, and not some $(\vec{r} + \vec{v})\mathbf{R}$ appears in (3.4). This corresponds to the preferred parametrization of rays in optics by a *fixed* standard screen[8]. Lie operators act moving the underlying space referred to a fixed frame.

[8] On the other hand, the natural bundle parametrization by *local* screens discussed in the previous Section proceeds in accordance with the *left* action of Lie operators in (3.1*b*). For these, the parametrization of $\mathcal{E}_3$ by (3.3*a*) may be

The Euler angle parametrization of rotations is

$$\mathbf{R}(\psi,\theta,\phi) = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_z(\phi). \tag{3.5a}$$

The effect of a rotation $\mathbf{R}$ on $\mathbf{P}$ can be best des by letting $\mathbf{R}$ act on at least two distinct row-vectors $\vec{p} = (p_x, p_y, p_z)$ of length n, and referring to the z–axis as the *optical axis*, following the ancient convention used by opticists. It is the *forward pole* of the Descartes sphere. Thus $\mathbf{R}$ will map the direction of the optical axis $\vec{p}_o = (0,0,n)$ onto

$$\vec{p} = \vec{p}_o\mathbf{R}(\psi,\theta,\phi) = (n\sin\theta\cos\phi, n\sin\theta\sin\phi, n\cos\theta). \tag{3.5b}$$

We note that the angle ψ is absent from the right-hand side of the last equation. Similar formulae can be produced for the other directions, say $(n,0,0)$ (that *will* contain ψ). The action of $\mathbf{R}$ on the whole $\vec{p}$-sphere is *transitive* (a right frame may be mapped to any other right frame) and *effective* (no frame is left invariant except by the group identity).

6.4 Generators of the Euclidean group

As a standard assumption in Lie theory, our group $\mathcal{G}$ has its manifold parametrized by a set of coordinates, $g(x)$, $x = \{x_i\}_{i=1}^N$, $g(0) =$ identity, and for vanishing ϵ we can write $\mathcal{L}_{g(\epsilon)} f(\gamma) = f(\gamma) + \epsilon_i \hat{\Gamma}_i f(\gamma) + \cdots$. The $\hat{\Gamma}_i$ are the *generators* of the group and are *realized* as first-order differential operators in the coordinates of γ [2]. The set of these generators $\{\hat{\Gamma}_i\}$ closes under commutation into a Lie *algebra* $[\hat{\Gamma}_i, \hat{\Gamma}_j] = c_{ijk}\hat{\Gamma}_k$, and the Lie transformations may be written as $\mathcal{L}_{g(x)} = \exp x_i \hat{\Gamma}_i$.

We may find the generators of the Euclidean group on functions of its own six-dimensional manifold $f(\mathsf{E}(\mathbf{P},\vec{r})) = f(\mathbf{P},\vec{r})$ from (3.4); these will constitute the *Euclidean* Lie algebra in the *frame* realization. The right

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conveniently replaced by

$$\mathsf{E}^{\mathrm{l}}(\vec{v},\mathrm{R}) = \mathcal{L}^{\mathrm{L}}_{\vec{v}}\mathcal{L}^{\mathrm{L}}_{\mathrm{R}} = \mathsf{E}(\mathrm{R}, \vec{v}\mathrm{R}).$$

Accordingly, the action on the space of frames now parametrized accordingly is

$$\mathsf{E}^{\mathrm{l}}(\vec{v},\mathrm{R}) f(\vec{t},\mathrm{P}) = f([\vec{t}-\vec{v}]\mathrm{R}^{-1}, \mathrm{R}^{-1}\mathrm{P}).$$

Here, the translation $\vec{v}$ is in the frame of $\vec{t}$ as it is rotated by R^{-1}. Parallel developments can be made for left group action $g^{-1}\gamma$ and their Lie operators. The third possibility, $g^{-1}\gamma g$, favors parametrization by *conjugation classes.* For the $\mathcal{R}_3$ subgroup, this is through specifying the rotation axis $\hat{n}(\theta,\phi)$ and the rotation angle χ around that axis. The latter labels the classes. Under the adjoint action of $\mathcal{R}_3$, $\hat{n}$ transforms as a vector and χ is invariant.

translations $\mathcal{L}^{\mathrm{R}}_{\vec{v}} = \exp \vec{v} \cdot \hat{\vec{T}}$ act as

$$\exp \vec{v} \cdot \hat{\vec{T}} f(\mathbf{P}, \vec{r}) = f(\mathbf{P}, \vec{r} + \vec{v}). \tag{4.1a}$$

We thus find

$$\hat{T}_x = \frac{\partial}{\partial r_x}, \qquad \hat{T}_y = \frac{\partial}{\partial r_y}, \qquad \hat{T}_z = \frac{\partial}{\partial r_z}. \tag{4.1b}$$

Translations do not affect ray direction, so no p_i–derivatives appear.

For rotations, we may use the explicit expressions of the 3×3 matrices that act in (3.4) both the $\vec{p} = (p_x, p_y, p_z)$ and $\vec{r} = (r_x, r_y, r_z)$ row-vectors, that transform in unison. We call $\hat{R}_i$ the generators of the finite rotation matrices $\mathbf{R}_i$, in the following way:

$$\exp \alpha_x \hat{R}_x \quad \mapsto \quad \mathbf{R}_x(\alpha_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_x & \sin\alpha_x \\ 0 & -\sin\alpha_x & \cos\alpha_x \end{pmatrix}, \tag{4.2a}$$

$$\exp \alpha_y \hat{R}_y \quad \mapsto \quad \mathbf{R}_y(\alpha_y) = \begin{pmatrix} \cos\alpha_y & 0 & -\sin\alpha_y \\ 0 & 1 & 0 \\ \sin\alpha_y & 0 & \cos\alpha_y \end{pmatrix}, \tag{4.2b}$$

$$\exp \alpha_z \hat{R}_z \quad \mapsto \quad \mathbf{R}_z(\alpha_z) = \begin{pmatrix} \cos\alpha_z & \sin\alpha_z & 0 \\ -\sin\alpha_z & \cos\alpha_z & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.2c}$$

The rotation generators take their very familiar form

$$\hat{R}_x = p_y \frac{\partial}{\partial p_z} - p_z \frac{\partial}{\partial p_y} + r_y \frac{\partial}{\partial r_z} - r_z \frac{\partial}{\partial r_y}, \tag{4.3a}$$

$$\hat{R}_y = p_z \frac{\partial}{\partial p_x} - p_x \frac{\partial}{\partial p_z} + r_z \frac{\partial}{\partial r_x} - r_x \frac{\partial}{\partial r_z}, \tag{4.3b}$$

$$\hat{R}_z = p_x \frac{\partial}{\partial p_y} - p_y \frac{\partial}{\partial p_x} + r_x \frac{\partial}{\partial r_y} - r_y \frac{\partial}{\partial r_x}. \tag{4.3c}$$

The length n of the vector $\vec{p}$, the radius of the Descartes sphere, is naturally a Euclidean invariant in homogeneous media —only.

The hatted operators are a vector basis for the Lie algebra. As we can check, their *commutators* $[\hat{X}, \hat{Y}] = \hat{X}\hat{Y} - \hat{Y}\hat{X}$ close:

$$[\hat{T}_i, \hat{T}_j] = 0, \qquad i, j = x, y, z \tag{4.4a}$$

$$[\hat{R}_i, \hat{T}_j] = -\varepsilon_{ijk} \hat{T}_k, \qquad \varepsilon_{xyz} = 1, \tag{4.4b}$$

$$[\hat{R}_i, \hat{R}_j] = -\varepsilon_{ijk} \hat{R}_k, \tag{4.4c}$$

and all other independent commutators are zero. This clearly displays the Euclidean algebra as the *semi-direct sum* of the translation and rotation subalgebras [2].[9]

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[9] At this point we could introduce rotations that act exclusively on the frame

6.5 Coset spaces and rays

The realization of frames seen in the last Section will serve now to define other objects subject to Euclidean group action. In scalar geometric optics, light rays are modeled by straight lines, with no particular origin nor polarization plane. If such an object is rotated around its axis or translated in its direction, it is still the same elementary object. These are *symmetry transformations* of the object, and they always form a *group* [2]. In the geometric optics case, the group is $\mathcal{H}^{\rm geom} = \mathcal{R}_2 \times \mathcal{T}_1$, the direct product of the rotation group in two dimensions with the translation group in one dimension. The *rays* of a model will be identified thus as the *equivalence classes* of the Euclidean group modulo the symmetry subgroup of the object. We will now formalize this construction presenting some standard material on the equivalence classes in a group called *cosets*, before we apply it to the Euclidean group.

If $\mathcal{G}$ is a group and $\mathcal{H}$ a subgroup, we may divide the manifold of $\mathcal{G}$ into disjoint subsets by $\mathcal{H}$ in the following way: let $g \in \mathcal{G}$ and consider the set $\{hg\}_{h\in\mathcal{H}}$, called the (left) *coset* of g by $\mathcal{H}$. We thereby introduce the relation $g_1 \equiv g_2$ between two elements of $\mathcal{G}$ when $\mathcal{H}g_1 = \mathcal{H}g_2$. The coset of the identity $e \in \mathcal{G}$ is $\mathcal{H}e = \mathcal{H}$. Clearly, the cosets of g and of hg are equal, and from here it is easily shown that $\equiv$ is an *equivalence* relation, *i.e.*, it divides the manifold of $\mathcal{G}$ into disjoint subsets; the *set* of left cosets is denoted by $\mathcal{H}\backslash\mathcal{G}$. Within every coset we may choose a *representative* element $\gamma \in \mathcal{G}$. We can thus display the structure of $\mathcal{G}$ to be that of a *fiber* bundle whose base space is the set of these representatives γ, namely $\mathcal{H}\backslash\mathcal{G}$, and whose typical fiber is $\mathcal{H}$. The projection operator $\pi{:}\mathcal{G} \mapsto \mathcal{H}\backslash\mathcal{G}$ may be used to introduce subgroup-adapted coordinates on the manifold of $\mathcal{G}$ by writing g as $g(\rho, v) = h(\rho)\gamma(v)$, with h parametrized by coordinates ρ for $\mathcal{H}$, and coordinates v for $\gamma \in \mathcal{H}\backslash\mathcal{G}$ as representative. It may be that in a badly chosen set of representatives, some elements of will not admit such a decomposition; in approaching these elements, some of its coordinates in v will escape to infinity.

Left cosets transform under *right* group action. If under $k \in \mathcal{G}$ the group manifold transforms as $g \mapsto g' = \mathcal{L}_k^{\rm R}{:}g = gk$, and the subgroup–coset coordinates read $g(\rho, v) = h(\rho)\gamma(v)$ and $g'(\rho, v) = g(\rho', v') = h'(\rho)\gamma'(v) = h(\rho')\gamma(v')$, then we may subduce the mapping $f(v) \mapsto f'(v) = f(v') = \mathcal{L}_k^{\rm R,\mathcal{H}}{:}f(v)$, where $\mathcal{L}_k^{\rm R,\mathcal{H}}$ is now a Lie transformation acting on the functions f of the coset space coordinates v.

For the Euclidean group $\mathcal{E}_3 = \mathcal{R}_3 \triangleright \mathcal{T}_3$, we have that $\mathcal{R}_3\backslash\mathcal{E}_3 = \mathcal{T}_3$ and

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orientation $\vec{p}$–space, as $\hat{S}_x = p_y\partial_{p_z} - p_z\partial_{p_y}$, etc., with commutators $[\hat{S}_i, \hat{T}_j] = 0$, $[\hat{S}_i, \hat{R}_j] = -\hat{S}_k$, $[\hat{S}_i, \hat{S}_j] = -\hat{S}_k$. The structure of this group would be $\mathcal{R}_3^{\rm S} \times (\mathcal{R}_3^{\rm R-S} \triangleright \mathcal{T}_3)$.

$\mathcal{E}_3/\mathcal{T}_3 = \mathcal{R}_3$ are groups themselves and have been used for the global parametrization of $\mathbf{E}(\mathbf{R}, \vec{v})$.[10] We will now show that geometric optics is a model based on the $\wp = (\mathcal{R}_2 \times \mathcal{T}_1)\backslash\mathcal{E}_3$ manifold. Let us now see the way the coordinates of $\mathcal{R}_2\backslash\mathcal{E}_3$ and $\mathcal{T}_1\backslash\mathcal{E}_3$ transform under Euclidean action.

For the rotation group $\mathcal{R}_3 \subset \mathcal{E}_3$ the Euler angle parametrization (3.5) is the appropriate one when the symmetry group $\mathcal{R}_2$ is $\mathbf{R}_z(\psi)$ with $\psi \in \mathcal{S}_1$. Each coset is thus a one-dimensional sphere, *i.e.*, a circle. The cosets $\mathcal{R}_2\mathbf{R}(\psi, \theta, \phi) = \mathcal{R}_2\mathbf{R}(0, \theta, \phi)$ are the points of the space $\mathcal{R}_2\backslash\mathcal{R}_3$, and the appropriate coset *representatives* are plainly $\gamma(\theta, \phi) = \mathbf{R}(0, \theta, \phi)$. Their manifold is the two-sphere: $\mathcal{R}_2\backslash\mathcal{R}_3 = \mathcal{S}_2$. Transformations of the points of this sphere by $\mathcal{L}^{\mathrm{R}}_{\mathbf{S}}$ under a rotation $\mathbf{S}(\alpha, \beta, \gamma)$ are found, as usual, by simply applying the matrix $\mathbf{S}$ to the row vector in (3.5*b*). The result will yield $f'(\theta, \phi) = f(\theta', \phi')$, $\theta'(\theta, \phi; \alpha, \beta, \gamma)$ and $\phi'(\theta, \phi; \alpha, \beta, \gamma)$. The manifold of cosets $\mathcal{R}_2\backslash\mathcal{E}_3$ is thus parametrized by $\{\gamma(\theta, \phi), (r_x, r_y, r_z)\} = \{\vec{p}, \vec{r}\}$. The model $\mathcal{R}_2\backslash\mathcal{E}_3$ describes objects that are points in space $\vec{r}$ with a direction vector $\vec{p}$ of fixed length on each point, *i.e.*, a special kind of *vector field*. It has one dimension less than the original six-dimensional full group manifold.

We examine now the cosets by the *translation* symmetry subgroup $\mathcal{T}_1$ along the z–axis. These are built in an analogous way, but with the difference that while $\mathcal{T}_3$ and its subgroups splits off easily to the *right* of $\mathcal{E}_3 = \mathcal{R}_3 \triangleright \mathcal{T}_3$, we need it to the *left*. The standard frame $e = \mathbf{E}(\mathbf{1}, (0, 0, 0))$ and the line of z-translated frames $\mathbf{E}(\mathbf{1}, (0, 0, s))$, $s \in \Re$, constituting the $\mathcal{T}_1$ coset of the identity, parametrize the same standard ray in the geometric optics model. Similarly, an arbitrary frame $\mathbf{E}(\mathbf{P}, \vec{r})$ and the coset of frames $\mathbf{E}(\mathbf{1}, (0, 0, s))\mathbf{E}(\mathbf{P}, \vec{r}) = \mathbf{E}(\mathbf{P}, \vec{r} + (0, 0, s)\mathbf{P})$, $s \in \Re$ describe the same ray, whose position vector is, in coordinates

$$(r_x, r_y, r_z) + (0, 0, s)\mathbf{P} = (r_x + s \sin\theta \cos\phi, r_y + s \sin\theta \sin\phi, r_z + s \cos\theta), \tag{5.1}$$

and carries the orientation $\mathbf{P}(\psi, \theta, \phi)$. Further cosetting by $\mathcal{R}_2$ will elliminate the plane polarization angle ψ.

The space of cosets $\mathcal{H}^{\mathrm{geom}}\backslash\mathcal{E}_3$, $\mathcal{H}^{\mathrm{geom}} = \mathcal{R}_2 \times \mathcal{T}_1$ we shall show now, is the manifold of geometric optics rays $\wp$ described in Section 2. The elements of $\mathcal{H}^{\mathrm{geom}}$ are $\mathbf{E}(\mathbf{R}_z(\psi), (0, 0, s))$, $\psi \in \mathcal{S}_1$, $s \in \Re$, a cylindrical submanifold of $\mathcal{E}_3$. Every other coset is a right-translated version of this submanifold by $\mathbf{E}(\mathbf{P}, \vec{r})$. The coset *representatives* may now be chosen in writing the decomposition

$$\mathbf{E}(\mathbf{P}(\psi, \theta, \phi), (r_x, r_y, r_z)) = \mathbf{E}(\mathbf{R}(\psi, 0, 0), (0, 0, s))\ \mathbf{E}(\mathbf{P}(0, \theta, \phi), (q_x, q_y, 0)), \tag{5.2}$$

where the vector $(q_x, q_y, 0) = \vec{q}$ indicates a point *on the screen*, whose

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[10] We may use $\mathcal{T}_3\backslash\mathcal{E}_3$ and $\mathcal{E}_3/\mathcal{R}_3$ for the *left* parametrization of $\mathbf{E}^l(\vec{t}, \mathbf{R})$, corresponding to the coordinates of $\wp$ by local screens.

z–component will be henceforth assumed to be always *zero*. Thus we write

$$\vec{r} = (r_x, r_y, r_z) = (0,0,s)\mathbf{P}(0,\theta,\phi) + (q_x, q_y, 0) = s\vec{p}/n + \vec{q}. \tag{5.3}$$

The coordinates of each $\mathcal{H}^{\text{geom}}$ coset are $\{\psi, s\}$ and those of the space of coset representatives are the four independent parameters in $\{\vec{p},\vec{q}\}$, where $\vec{p}$ lies on the Descartes sphere of radius n, and $\vec{q}$ indicates the ray intersection at the standard screen. Any nonzero value of ψ and r_z will fall into the $\mathcal{H}^{\text{geom}}$ factor to the left. Both are (by definition) unobservable in scalar geometric optics, but may be retained in the less stringent models of geometric optics with a polarization orientation or signals along the line.

We relate the group and coset parameters through

$$\begin{aligned} r_x &= q_x + s\sin\theta\cos\phi, & q_x &= r_x - r_z p_x/p_z, & (5.4a)\\ r_y &= q_y + s\sin\theta\sin\phi, \quad i.e. & q_y &= r_y - r_z p_y/p_z, & (5.4b)\\ r_z &= 0 + s\not{n}\cos\theta, & s &= r_z\sec\theta = \not{n}r_z/p_z. & (5.4c)\end{aligned}$$

There is a submanifold where these coordinates fail, however: rays parallel to the standard screen —as may have been expected. For $\theta = \pi/2, p_z = 0$, and both s and $|\vec{q}|$ go to infinity because the decomposition (5.2) is impossible there. Figure 3 shows the geometrical situation in two dimensions. Comparison with Figure 1(b) justifies the identification $\wp = \mathcal{H}^{\text{geom}}\backslash\mathcal{E}_3$.

6.6 Euclidean group action on rays in geometric optics

The Euclidean transformations $\mathsf{E}(\mathbf{R}(\alpha,\beta,\gamma),\vec{v})$ of the rays in $\wp$ can now be found by acting from the right on $\mathsf{E}(\mathbf{P},\vec{r}) = h(\psi,s)\gamma(\vec{p}(\theta,\phi),(q_x,q_y))$[11] and then decomposing the product. See:

$$\mathsf{E}(\mathbf{P},\vec{r})\mathsf{E}(\mathbf{R},\vec{v}) = \mathsf{E}(\mathbf{PR},\vec{r}\mathbf{R}+\vec{v}) = h(\psi',s')\gamma(\vec{p}(\theta',\phi'),\vec{q}'). \tag{6.1}$$

We use (5.2) for $\mathbf{P}' = \mathbf{PR}$, with $\mathbf{R}$–rotated angles $\{\psi',\theta',\phi'\}$ in place of $\{\psi,\theta,\phi\}$,

$$\vec{p}'(\theta,\phi) = \vec{p}(\theta',\phi') = \vec{p}(\theta,\phi)\mathbf{R}(\alpha,\beta,\gamma). \tag{6.2}$$

This takes care of the direction part. Next, we look at the ray length parameter s' and the position coordinates $\vec{q}'$ on the screen in (5.4) with $\vec{r}' = \vec{r}\mathbf{R}(\alpha,\beta,\gamma) + \vec{v}$ in place of $\vec{r}$. This allows us to determine $s' = nr'_z/p'_z$ and

$$\vec{q}' = \vec{r}' - s'\vec{p}'/n = \vec{r}\mathbf{R} + \vec{v} - s'\vec{p}\mathbf{R}/n \tag{6.3a}$$

[11] We recall that $\vec{q} = (q_x, q_y, 0) = (\mathbf{q}, 0)$ is a vector *on the standard screen* whose z–component is always zero. We shall use boldface $\mathbf{q}$ to indicate the two-vector $\mathbf{q} = (q_x, q_y)$.

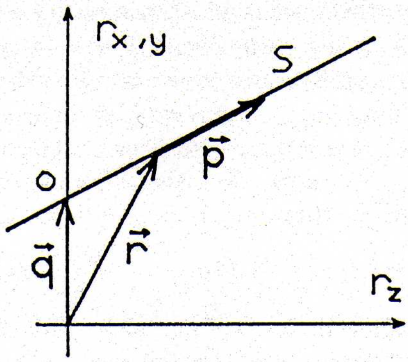

FIGURE 3. The vector $\vec{r}$ on a ray, its length parameter s from the screen, and its position $\vec{q} = (q_x, q_y, 0)$ on the standard screen $r_z = 0$.

$$= \quad (\vec{r} - s'\vec{p}/n)\mathbf{R} + \vec{v} = (\vec{q} + [s - s']\vec{p}/n)\mathbf{R} + \vec{v} \qquad (6.3b)$$

$$= \quad \left(\vec{q} + \left[\frac{r_z}{p_z} - \frac{r'_z}{p'_z}\right]\vec{p}\right)\mathbf{R} + \vec{v}, \qquad (6.3c)$$

that is independent of s, with the third component zero.

As a particular case we may obtain from (6.3) explicitly the transformation of the rays in $\wp$ under translations along the optical axis by $\mathsf{E}(\mathbf{1}, (0, 0, v_z))$:

$$\vec{p} \mapsto \vec{p}' = \vec{p}, \qquad \mathbf{q} \mapsto \mathbf{q}' = \mathbf{q} - v_z/p_z\,\mathbf{p}, \qquad (6.4)$$

where $\mathbf{q} = (q_x, q_y)$ and $\mathbf{p} = (p_x, p_y)$. This is geometrically obvious in Figure 3, but underscores the fact that while $r_z \mapsto q'_z = r_z + v_z$ for any point $\vec{r}$ of the bearing space, the *screen*-coordinate $\mathbf{q}$ of the *ray* slides down; thence the minus sign. If we want to *advance the screen* along the optical axis, we should translate the space by $-z$.

The detailed formulas for the general transformation of the ray and coset coordinates under elements g of the Euclidean group may be found from ordinary vector analysis and are, by themselves, not particularly succint. What is important is that the ray coordinates $\{\vec{p}(\theta, \phi), \vec{q}\}$ map only amongst themselves while the coordinates in the coset $\{\psi, s\}$ transform according to the coset to which they belong, *i.e.*, $\psi'(\psi, \vec{p}(\theta, \phi); g)$ and $s'(s, \vec{p}, \vec{q}; g)$. This is a general consequence of the fact that the spaces of cosets are base spaces for the group bundle. Another consequence of this pertains the factorization of the Haar measure as we shall now show for the $\wp$ manifold of cosets in $\mathcal{E}_3$.

We may ask from (6.1)–(6.3) if there is a volume element in $\wp$, the space of geometric optics rays, that is invariant under the Euclidean group; if so, this will provide a good integration measure for purposes of harmonic analysis and Wigner distribution theory. In fact, such exists and appears to be the origin of the *symplectic* structure of $\wp$. The Haar measure dg of a group manifold is a volume element such that under $g \mapsto g' = gg_0$ or $g \mapsto g'' = g_0^{-1}g$, the Jacobians at the group identity, are unity: $\partial g/\partial g'|_{g=e} = 1$ or $\partial g/\partial g''|_{g=e} = 1$ [2]. A semidirect-product group such as the Euclidean groups in any dimension does have a right- *and* left-invariant Haar measure that is the simple product of the Haar measures of the rotation and translation subgroups, because the latter is invariant under the former. They are, for $\mathcal{E}_3$, $\mathcal{R}_3$, and $\mathcal{T}_3$, respectively,

$$dg = d^3\mathbf{P}\, d^3\vec{r}, \qquad d^3\mathbf{P} = d\psi\, \sin\theta\, d\theta\, d\phi, \qquad d^3\vec{r} = dr_x\, dr_y\, dr_z. \tag{6.5}$$

where $\{\psi, \theta, \phi\}$ are the Euler angles. We can also write the invariant measure in terms of the coset-decomposed coordinates $\{\psi, s; \vec{p}, \vec{q}\}$. For the rotation subgroup, recalling (3.5*b*),

$$d^3\mathbf{R} = d\psi\, \frac{dp_x\, dp_y}{np_z}. \tag{6.6}$$

The expression for $d^3\vec{r}$ in terms of $ds\, dq_x\, dq_y$ can be found from (5.4),

$$d^3\vec{r} = (dq_x + p_x/n\, ds + s\, dp_x/n) \wedge (dq_y + p_y/n\, ds + s\, dp_y/n) \wedge (p_z\, ds + s\, dp_z)/n, \tag{6.7}$$

where $dp_z = -(p_x\, dp_x + p_y\, dp_y)/p_z$. Using differential form calculus [5], where only unlike-differential factors remain, the outer product of (6.6) and (6.7) is then the Euclidean-invariant Haar measure

$$dg = d\psi\, ds\, n^{-2}\, dp_x\, dp_y\, dq_x\, dq_y. \tag{6.8}$$

The factor $d\psi\, ds$ is the volume element over the space of each coset, while the second factor is the Euclidean-invariant volume element over the space of rays, $\wp$.

6.7 The Euclidean algebra generators on rays

In Section 4 we displayed the generators of the Euclidean algebra on the $\mathcal{E}_3$ manifold $\{\mathbf{P}, \vec{r}\}$. Now that we have found how the rays of geometric optics live on the submanifold $\wp \subset \mathcal{E}_3$, we shall restrict the generators of $\mathcal{E}_3$ to this, by *elliminating* p_z and r_z. We will arrive thus at those previously found for geometric optics [6].[12]

The direction manifold, the Descartes sphere, is a two-dimensional manifold. We have written on occasion $\vec{p}(\theta, \phi)$, but for a large neighborhood $(0 \le \theta < \pi/2)$ of the optical axis $(\theta = 0)$ we can work equivalently with $\mathbf{p} = (p_x, p_y)$, since the third component is $p_z = \sqrt{n^2 - p^2}$ (we denote $p = |\mathbf{p}| \le n$). If we go beyond this *forward hemisphere*, into the

[12] We note a difference of a $\pi/2$ rotation around the z axis with respect to this reference.

backward hemisphere, we must supplement the coordinates in $\wp$ by a *sign* $\sigma \in \{+,0,-\}$, $p_z = \sigma\,|p_z|$, as done in Section 2. All operators on $\wp$ involving $\partial/\partial\mathbf{p}$ are thus in principle *two-chart* operators, with forms perhaps differing by a sign on the two hemispheres of the Descartes sphere, and matching requirements for functions $f(\mathbf{p},\mathbf{q},\sigma)$ near the equator. In this Section we shall not build a Hilbert space of such functions, however, so these precautions are not as indispensable as they will be for Helmholtz optics in Section 9 *et seq.*

On functions $f(p_x,p_y,q_x,q_y;\sigma)$ of $\wp$, $\partial f/\partial p_z$ will be zero. Hence, in subducing operators from $\mathcal{E}_3$ to $\wp$, the restrictions on $\vec{p}$ and $\partial/\partial\vec{p}$ are:

$$p_z \mapsto h = \pm\sqrt{n^2-p^2}, \qquad \frac{\partial}{\partial p_z} \mapsto 0. \tag{7.1}$$

For the Euclidean position manifold $\vec{r} \in \Re^3$ we similarly reduce the independent position variables $\vec{r} = (r_x,r_y,r_z)$ to the two coordinates on the standard screen $\mathbf{q} = (q_x,q_y)$ and length along the ray s through (5.4), and thereafter require the independence of our function space on s. The action of $\partial/\partial r_z$ on functions f of $\mathbf{q}$ is given by the chain rule for (5.4*a*) and (5.4*b*), namely

$$\frac{\partial f(\mathbf{q})}{\partial r_z} = \frac{\partial \mathbf{q}}{\partial r_z}\cdot\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}} = -\frac{\mathbf{p}}{h}\cdot\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}}. \tag{7.2}$$

The proper replacement of $\vec{r}$ by $\mathbf{q}$ thus entails[13]

$$r_z \mapsto 0, \qquad \frac{\partial}{\partial r_z} \mapsto -\frac{\mathbf{p}}{h}\cdot\frac{\partial}{\partial \mathbf{q}}. \tag{7.3}$$

The generators of the Euclidean translations, Eqs. (4.1*b*) on the *screen* variables of $\wp$, are[14]

$$\hat{T}^{\wp}_x = \frac{\partial}{\partial q_x} = -\{p_x,\circ\}, \tag{7.4a}$$

$$\hat{T}^{\wp}_y = \frac{\partial}{\partial q_y} = -\{p_y,\circ\}, \tag{7.4b}$$

$$\hat{T}^{\wp}_z = -\frac{\mathbf{p}}{h}\cdot\frac{\partial}{\partial \mathbf{q}} = \{h,\circ\}. \tag{7.4c}$$

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[13] Notice *very* carefully that, as announced, the operator exhibiting h is actually a *two-chart* operator, having two different signs when acting on the forward and the backward hemispheres of the Descartes sphere of ray directions.

[14] The eye catches an apparent *sign* difference between the last expressions in (7.4*a*, *b*) and in (7.4*c*). To set intuition straight, look at Figure 3, displacing space with the embedded rays in x–y and z directions from a fixed reference screen. In the former cases the intersection moves *with* the space, while in the latter it *slides* down along $\mathbf{p}$ with a factor $-|\mathbf{p}|/h = -\tan\theta$.

The generators of rotations are on $\wp$ given by Eqs. (4.3) with the replacements (7.3). They are

$$\hat{R}_x^{\wp} = -h\frac{\partial}{\partial p_y} - q_y\frac{\mathbf{p}}{h}\cdot\frac{\partial}{\partial \mathbf{q}} = \{-q_y h, \circ\}, \quad (7.5a)$$

$$\hat{R}_y^{\wp} = h\frac{\partial}{\partial p_x} + q_x\frac{\mathbf{p}}{h}\cdot\frac{\partial}{\partial \mathbf{q}} = \{q_x h, \circ\}, \quad (7.5b)$$

$$\hat{R}_z^{\wp} = p_x\frac{\partial}{\partial p_y} - p_y\frac{\partial}{\partial p_x} + q_x\frac{\partial}{\partial q_y} - q_y\frac{\partial}{\partial q_x} = \{\mathbf{p}\times\mathbf{q}, \circ\}. \quad (7.5c)$$

The last term in each line of the above six expressions, writes the Euclidean group generators as *Poisson* operators of functions $f(\mathbf{p}, \mathbf{q})$, with the symbol

$$\{f, \circ\} = \frac{\partial f}{\partial \mathbf{q}}\cdot\frac{\partial}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{q}}. \quad (7.6)$$

That such can be done is in principle quite remarkable, although our acquaintance with Hamiltonian optics makes us expect this to happen, and rightfully call the manifold $(\mathbf{p}, \mathbf{q})$ the *phase space* of geometrical optics. What it means is that Euclidean motions, in particular screen motion along the z–axis (6.4), is a *canonical* evolution of the system governed by a Hamiltonian h and conserving the phase space volume element $d\mathbf{p}\,d\mathbf{q}$, as seen at the end of last Section. The Euclidean Lie algebra is thus the natural *dynamical* algebra of the manifold of rays $\wp$ in a homogeneous medium, with z taking the role of time. Indeed, *Hamilton*'s equations hold on $\wp$ with canonically conjugate coordinates $\mathbf{p}$ and $\mathbf{q}$. The first of these equations, on $d\mathbf{q}/dz$, is found from (5.4a,b). Since r_x, r_y and $\vec{p}$ are independent of $r_z = z$, the total and partial derivatives with respect to this z are the same. It follows through (7.1) that

$$\frac{d\mathbf{q}}{dz} = -\frac{\mathbf{p}}{h} = \frac{\partial h}{\partial \mathbf{p}}. \quad (7.7)$$

It is known that the origin of this *first* equation is geometrical [7]. The second, *dynamical*, Hamilton equation is of the form $d\mathbf{p}/dz = -\partial h/\partial \mathbf{q}$. It is trivially satisfied in a homogeneous medium since $\vec{p}$ is constant, so $d\mathbf{p}/dr_z = 0$, and also $\partial h/\partial \mathbf{q}$ is zero.[15]

Being familiar with the Hamiltonian formalism, we know that the invariance of this measure extends to *all* transformations generated by operators $\{f, \circ\}$. This is not sufficient, however, to guarantee that $\mathbf{p}'(\mathbf{p}, \mathbf{q}) = \exp\{f, \circ\}\mathbf{p}$, for arbitrary f will remain within a disk of radius n, *i.e.*, that

[15] In *inhomogeneous* media, translating from place to place will change the size of the Descartes sphere, $n(\vec{r})$; the vector $\vec{p}$ will accomodate according to Snell's law, conserving the components of $\vec{p}$ perpendicular to $\vec{\nabla} n$, and leading to Hamilton's second equation in nontrivial form [7].

it will be an *optical* transformation mapping $\wp$ onto itself. That f's beyond the Euclidean algebra are permissible will be seen below, in Sections 12 *et seq.*, regarding the generators of a Lorentz group. The Poisson bracket formalism is rooted in the Heisenberg-Weyl algebra, that is royal road to quantization. It will become clear in Section 11, when we draw the way to wavization provided by the *Euclidean* algebra, that screen coordinates are *not* fit to follow it because they are not operators within this algebra.

As a concrete example, we produce a finite translation by z along the optical axis of the space that bears the rays by exponentiating its infinitesimal generator (7.4c) on the screen coordinates. We have:

$$\mathbf{p} \quad \mapsto \quad \mathbf{p}' = \exp(z\hat{T}_z^{\wp})\,\mathbf{p} = \sum_{m=0}^{\infty} \frac{z^m}{m!}\left(-\frac{\mathbf{p}}{h}\cdot\frac{\partial}{\partial \mathbf{q}}\right)^m \mathbf{p} = \mathbf{p}, \qquad (7.8a)$$

$$\mathbf{q} \quad \mapsto \quad \mathbf{q}' = \exp(z\hat{T}_z^{\wp})\,\mathbf{q} = \left(1 - z\frac{\mathbf{p}}{h}\cdot\frac{\partial}{\partial \mathbf{q}} + \cdots\right)\mathbf{q} = \mathbf{q} - z\frac{\mathbf{p}}{h}. \quad (7.8b)$$

This result is the same as (6.4), and is geometrically obvious.

6.8 The coset space of wavefront optics

We shall now follow the cosetting strategy seen above to describe other kinds of optics where the elementary objects are not lines, but *planes*, *i.e.*, *wavefronts*. The symmetry group $\mathcal{H}^{\rm wf}$ of a plane wavefront is the two-dimensional Euclidean group $\mathcal{E}_2$.

Consider the two-dimensional Euclidean subgroup $\mathcal{H}^{\rm wf} = \mathcal{E}_2 \subset \mathcal{E}_3$ given by the elements $\mathsf{E}(\mathbf{R}_z(\psi), (t_x, t_y, 0))$, $\psi \in \mathcal{S}_1$, $(t_x, t_y) \in \Re^2$ and a generic decomposition of the elements of $\mathsf{E}(\mathbf{P}, \vec{r}) \in \mathcal{E}_3$ that is paralell to (5.2). The symmetry subgroup factor and a representative of the coset space $\mathcal{W} = \mathcal{E}_2 \backslash \mathcal{E}_3$, are

$$\mathsf{E}(\mathbf{P}(\psi,\theta,\phi),(r_x,r_y,r_z)) = \mathsf{E}(\mathbf{R}(\psi,0,0),(t_x,t_y,0))\mathsf{E}(\mathbf{P}(0,\theta,\phi),(0,0,u)). \qquad (8.1)$$

The factorization of the rotation subgroup into a polarization angle and a coset representative of the two-sphere, $\vec{p}(\theta,\phi) \in \mathcal{R}_2\backslash\mathcal{R}_3 = \mathcal{S}_2$, proceeds as in Section 5.

Regarding the position parameters $\vec{r}$, the analogue of Eqs. (5.4) is

$$(r_x, r_y, r_z) = (t_x, t_y, 0)\mathbf{P}(0,\theta,\phi) + (0,0,u). \qquad (8.2a)$$

Directly replacing the matrix $\mathbf{P}(0,\theta,\phi) = \mathbf{R}_y(\theta)\mathbf{R}_z(\phi)$ found through (4.2), we obtain

$$(r_x, r_y) = (t_x\cos\theta, t_y)\begin{pmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{pmatrix}, \quad r_z = -t_x\sin\theta + u. \qquad (8.2b)$$

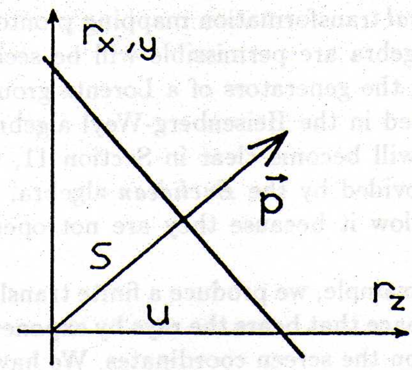

FIGURE 4. Plane wavefronts in space showing the coset space parameters $\vec{p}$ and u, as well as $s = up_z/n$.

These can be inverted for the new parameters in terms of $\mathbf{p} = (p_x, p_y)$, $\mathbf{r} = (r_x, r_y)$, dot and cross products, as

$$t_x = \frac{n}{p_z}\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\mathbf{r}, \qquad t_y = \frac{\mathbf{p}}{|\mathbf{p}|}\times\mathbf{r}, \qquad u = r_z + \frac{\mathbf{p}\cdot\mathbf{r}}{p_z} = \frac{\vec{p}\cdot\vec{r}}{p_z}. \tag{8.2c}$$

The coordinates $\{\psi, t_x, t_y\}$ are in the $\mathcal{H}^{\text{wf}}$ coset and $\{\vec{p}(\theta,\phi), u\}$ in the space of cosets $\mathcal{W} = \mathcal{E}_2\backslash\mathcal{E}_3$. The decomposition fails when $p_z = 0$, *i.e.*, for planes parallel to the z–axis. This was also a feature of geometric optics.

The points in $\mathcal{W}$, $\mathcal{H}^{\text{wf}}\mathsf{E}(\mathbf{P}(0,\theta,\phi),(0,0,u))$, as in (8.1), are each a *wavefront*, *i.e.*, a plane $\mathcal{H}^{\text{wf}}$ in space, given by $\vec{r}\cdot\vec{p} = up_z$, orthogonal to $\vec{p}$, and with intercept u on the z–axis. See Figure 4. The quantity $up_z = \vec{r}\cdot\vec{p} = ns$ is n times the distance s of the plane to the origin, *i.e.*, ns is the usual *optical distance*. We may picture a function $S_{\vec{p}}(s) = f(\vec{p}, u = ns/p_z)$ as representing a *signal* through a train of paralell planes in the direction $\vec{p}$. The coordinates (t_x, t_y) and polarization angle ψ on the plane are absent in $\mathcal{W}$, of course. On the $r_z = 0$ screen, where the r_x and r_y coordinates may be called q_x and q_x as in the last Section, this plane $\{\vec{p}, u\}$ cuts the line $p_x q_x + p_y q_y = \mathbf{p}\cdot\mathbf{q} = up_z$.

The translation subgroup $\mathcal{T}_3 \subset \mathcal{E}_3$ acts in the following way as a Lie transformation through the decomposition (8.1):

$$\mathsf{E}(\mathbf{1},\vec{v})f(\vec{p},u) = f[\mathcal{H}^{\text{wf}}\mathsf{E}(\vec{p},(0,0,u))\mathsf{E}(\mathbf{1},\vec{v})] = f(\vec{p}, u + \vec{v}\cdot\vec{p}/p_z). \tag{8.3}$$

The rotations that transform $\vec{p}\mapsto\vec{p}\,' = \vec{p}\mathbf{R}$ and $\vec{r}\mapsto\vec{r}\,' = \vec{r}\mathbf{R}$, will turn $u = \vec{r}\cdot\vec{p}/p_z$ into $u' = \vec{r}\,'\cdot\vec{p}\,'/p_z' = \vec{r}\cdot\vec{p}/p_z' = up_z/p_z'$ and $s = s'$. Hence,

$$\mathsf{E}(\mathbf{R},\vec{0})f(\vec{p},u) = f[\mathcal{H}^{\text{wf}}\mathsf{E}(\vec{p},(0,0,u))\mathsf{E}(\mathbf{R},\vec{0})] = f(\vec{p}\mathbf{R}, up_z/p_z'). \tag{8.4}$$

We now realize the Lie algebra of the Euclidean group $\mathcal{E}_3$ on the space of wavefronts $\mathcal{W}$. We may find the Lie generators as in last Section, or *ab initio*, since the group action in (8.3) and (8.4) is explicit. For the translations, as in (4.1) and (7.4),

$$\hat{T}_x^{\mathcal{W}} = \frac{p_x}{p_z}\frac{\partial}{\partial u}, \qquad \hat{T}_y^{\mathcal{W}} = \frac{p_y}{p_z}\frac{\partial}{\partial u}, \qquad \hat{T}_z^{\mathcal{W}} = \frac{\partial}{\partial u}. \tag{8.5a, b, c}$$

For the rotations (8.4) we can compute explicitly the scale factor p_z/p_z' for $\mathbf{R}_x(\alpha_x)$, $\mathbf{R}_y(\alpha_y)$, and $\mathbf{R}_z(\alpha_z)$ from the matrices (4.2) acting on the row vector $\vec{p}$, and then let $\alpha \to 0$. The results are:

$$\hat{R}_x^{\mathcal{W}} = p_y\frac{\partial}{\partial p_z} - p_z\frac{\partial}{\partial p_y} - \frac{p_y}{p_z}u\frac{\partial}{\partial u}, \tag{8.6a}$$

$$\hat{R}_y^{\mathcal{W}} = p_z\frac{\partial}{\partial p_x} - p_x\frac{\partial}{\partial p_z} + \frac{p_x}{p_z}u\frac{\partial}{\partial u}, \tag{8.6b}$$

$$\hat{R}_z^{\mathcal{W}} = p_x\frac{\partial}{\partial p_y} - p_y\frac{\partial}{\partial p_x}. \tag{8.6c}$$

We may check that the Lie brackets (4.4) hold.

6.9 Helmholtz optics

The Euclidean generators (8.5)–(8.6) form a Lie algebra, and within its *enveloping* algebra we can find group *invariants*. The three-dimensional Euclidean algebra has two quadratic invariants, $\hat{T}^2 = \hat{\vec{T}}\cdot\hat{\vec{T}}$ and $\hat{\vec{T}}\cdot\hat{\vec{R}}$, that may be used to label the irreducible representations of the algebra and group. The latter invariant is identically zero on $\mathcal{W}$. The former, on the other hand, is

$$\hat{T}^2 = (\hat{T}_x^{\mathcal{W}})^2 + (\hat{T}_y^{\mathcal{W}})^2 + (\hat{T}_z^{\mathcal{W}})^2 = \frac{n^2}{p_z^2}\frac{\partial^2}{\partial u^2}. \tag{9.1}$$

The action of the Euclidean group on $\mathcal{W}$ will thus map functions $f(\vec{p}, u)$ amongst themselves, but respecting the linear *eigenspaces* of the operator in (9.1), that will remain invariant under that process. Their u-dependent factors will be linear combinations of $\sin kp_z u/n$ and $\cos kp_z u/n$, for k in principle complex, with eigenvalues $-k^2$. We may thus apply Fourier analysis in $u \in \Re$ to the functions f on $\mathcal{W}$. Each of its partial wave components $\mathcal{W}_k$, is the form

$$f(\vec{p}, u) = \Phi(\vec{p})\exp iks, \qquad s = up_z/n = \vec{p}\cdot\vec{r}/n, \tag{9.2}$$

and they will transform irreducibly under the Euclidean group. The Euclidean invariant k is the *wavenumber* in the medium of refractive index n. When this number k is real, the functions (9.2) exhibit a *translational* *invariance* under $s \mapsto s + \lambda$, $\lambda = 2\pi/k$, as true plane *waves* do.[16] We thus have all and only plane waves of a given wavenumber, in all directions of the Descartes sphere.

In the same way as in geometric optics, the direction sphere $\vec{p} \in \mathcal{S}_2$ may be projected (twice) on its equatorial screen plane, the disk δ_n, where $|\mathbf{p}| < n$ and the boundary $|\mathbf{p}| = n$ sews the two disks. We will write the function $\Phi(\vec{p})$ as $\Phi_\pm(\mathbf{p})$ independent of p_z through for $p_z \mapsto \sigma\sqrt{n^2 - p^2}$ (with $p^2 = \mathbf{p}\cdot\mathbf{p}$); the *sign* of p_z, $\sigma \in \{+, 0, -\}$ distinguishes the two open hemispheres and the common boundary circle. (We shall usually disregard the zero.) The functions will be understood to be continuous between the two charts, *i.e.*, matching as $\lim_{|\mathbf{p}|\to n} \Phi_+(\mathbf{p}) = \lim_{|\mathbf{p}|\to n} \Phi_-(\mathbf{p}) = \Phi_0(\mathbf{p})$.

As in ordinary Fourier analysis, the operator $\partial/\partial u$ acts only on the e^{iks} factor (recall that $s = up_z/n$), so in the translation generators (8.5), it is replaced by the factor ikp_z/n, with one sign in each chart. The generators of Euclidean translations in $\mathcal{W}_k$ are thus

$$\hat{T}_x^{k\pm} = \frac{ikp_x}{n}, \qquad \hat{T}_y^{k\pm} = \frac{ikp_y}{n}, \qquad \hat{T}_z^{k\pm} = \frac{\pm ik\sqrt{n^2 - p^2}}{n}. \tag{9.3}$$

The restriction from $\mathcal{W}$ to $\mathcal{W}_k$ of the generators of rotation (8.6) proceeds through noting that neither $\partial/\partial p_z$ nor $\partial/\partial u$ act on $\Phi_\pm(\mathbf{p})$, and that $p_z\partial s/\partial p_z = u\partial s/\partial u$. Hence, their form on functions of this space is:

$$\hat{R}_x^{k\pm} = \mp\sqrt{n^2 - p^2}\frac{\partial}{\partial p_y}, \tag{9.4a}$$

$$\hat{R}_y^{k\pm} = \pm\sqrt{n^2 - p^2}\frac{\partial}{\partial p_x}, \tag{9.4b}$$

$$\hat{R}_z^{k\pm} = p_x\frac{\partial}{\partial p_y} - p_y\frac{\partial}{\partial p_x}. \tag{9.4c}$$

This realization of the Euclidean algebra, by construction, belongs to a definite irreducible representation, determined by the values of the Casimir invariants. Plane waves moreover, are a representation *basis* further reduced and classified by the translation subalgebra $\hat{T}_j^{k\pm}$, $j = x, y$, and the sign of $\hat{T}_z^{k\pm}$. Another subgroup basis, where the diagonal generators are $\hat{T}_z^{k\pm}$ and $\hat{R}_z^{k\pm}$, are functions with support on a ring $\theta = \theta_0$, with a definite rotation covariance. These are the *nondiffracting J_m-beams* [8]. Finally, *multipole fields* are obtained as the rotation subgroup eigenbasis of the subalgebra

[16] We note that there is a fundamental wavenumber (light *color*) k_0 associated with the vacuum $n = 1$, and for any other medium $k = n\,k_0$. Thus even though k and n will be written jointly in most of the text, they always appear as the ratio $k/n = k_0$.

Casimir operator

$$\sum_{j=x,y,z} (\hat{R}_j^{k\pm})^2 = n^2 \left(\frac{\partial^2}{\partial p_x^2} + \frac{\partial^2}{\partial p_x^2} \right) - \left(p_x \frac{\partial}{\partial p_x} + p_y \frac{\partial}{\partial p_y} \right) \left(p_x \frac{\partial}{\partial p_x} + p_y \frac{\partial}{\partial p_y} + 1 \right) \quad (9.5)$$

and $\hat{R}_z^{k\pm}$, *i.e.*, the projection of the spherical harmonics of the sphere on the disk. Each of these bases will yield a different realization of the Euclidean algebra through differential or difference operators in the row labels of the representation.

The effect of the *exponentials* of these operators on the two-chart functions are simple when the direction hemispheres do not mix, and rather complicated otherwise. In any case, we may conveniently revert to the description by functions $\Phi(\vec{p})$ over the sphere. Translations $\exp(\sum_j v_j \hat{T}_j)$ multiply it with the phase $\exp(ik \sum_j v_j p_j / n) = \exp(ik_0 \sum_j v_j p_j)$, and rotations act on the argument row vector $\vec{p}$ through right matrix multiplication as usual.

6.10 The Hilbert space for Helmholtz optics

Quantum mechanics works with $\mathcal{L}^2(\Re^n)$ Hilbert spaces of wavefunctions where real observables are eigenvalues of self-adjoint operators, and symmetry transformations are unitary. We shall now proceed to build a Hilbert space for the oscillatory solutions of the Helmholtz equation that is unitarily equivalent to $\mathcal{L}^2(\mathcal{S}_2)$, the well-known space of square-integrable functions on the Descartes direction sphere. We call the structure *Helmholtz optics.*

Let us return to the Haar measure over $\mathcal{E}_3$ given in (6.5)–(6.6). On the direction sphere $|\vec{p}| = n$, the invariant *surface* element is

$$d^2S(\vec{p}) = n^2 \sin\theta \, d\theta \, d\phi = \frac{n}{p_z} \, dp_x \, dp_y. \quad (10.1)$$

In the second form, the Descartes sphere surface element has been projected over the screen plane as before. We must specify that $p_z > 0$ in the 'forward' hemisphere $0 \leq \theta < \pi/2$, and $p_z < 0$ in the 'backward' hemisphere $\pi/2 < \theta \leq \pi$, taking account of the change in the surface element orientation.

A continuous linear superposition $\Phi(\vec{p})$ of plane waves (9.2) over all directions is in $\Re^3$,[17]

$$F(\vec{r}) = \frac{k}{2\pi n} \int_{\mathcal{S}_2} d^2S(\vec{p}) \Phi(\vec{p}) \exp(ik\vec{p} \cdot \vec{r}/n), \quad (10.2)$$

[17]We choose the normalization factor for the purposes of symmetry in the Fourier analysis formulas. Position $\vec{r}$ has units of $k^{-1} = \lambda/2\pi$, the reduced wavelength. We may ascribe to $\vec{p}$ units of n, although physically dimensionless. Inte-

and satisfies the Helmholtz equation in this space:

$$\left(\frac{\partial^2}{\partial r_x^2}+\frac{\partial^2}{\partial r_y^2}+\frac{\partial^2}{\partial r_z^2}\right)F(\vec{r})=-k^2F(\vec{r}). \tag{10.3}$$

This may be also written in *evolution* form on a space of two-component functions

$$\begin{pmatrix}0 & 1\\ -\Delta_k & 0\end{pmatrix}\begin{pmatrix}F(\vec{r})\\ F'(\vec{r})\end{pmatrix}=\frac{\partial}{\partial r_z}\begin{pmatrix}F(\vec{r})\\ F'(\vec{r})\end{pmatrix}, \quad \Delta_k=k^2+\frac{\partial^2}{\partial r_x^2}+\frac{\partial^2}{\partial r_x^2}. \tag{10.4}$$

The first component of this equation defines $F'(\vec{r})$ to be the r_z–derivative of $F(\vec{r})$, while the second reproduces the Helmholtz equation (10.3).

Previously, the geometric optics model reduced the description of rays in position 3-space to a standard 2-dimensional *screen* at $r_z=0$. This suggests that we perform the Helmholtz analogue of this *regression to screen values* $\vec{r}\mapsto(\mathbf{q},0)$ —with some extra care due to the *two-chart* structure pointed out before. Oscillating solutions to the Helmholtz equation are determined thoughout $\Re^3$ by specifying their initial value and normal derivative at a plane. From (10.1) and (10.2) these are, *on the screen* $r_z=0$ and expressed as integrals over the $\mathbf{p}$–*disk* δ_n of radius n,[18]

$$\begin{aligned}F(\mathbf{q}) &= F(\vec{r})\big|_{r_z=0}\\ &= \frac{k}{2\pi}\int_{\delta_n}\frac{d^2\mathbf{p}}{\sqrt{n^2-p^2}}\left[\Phi_+(\mathbf{p})+\Phi_-(\mathbf{p})\right]e^{ik\mathbf{p}\cdot\mathbf{q}/n}, \qquad (10.5a)\\ F'(\mathbf{q}) &= \frac{\partial F(\vec{r})}{\partial r_z}\bigg|_{r_z=0}\\ &= \frac{ik^2}{2\pi n}\int_{\delta_n}d^2\mathbf{p}\left[\Phi_+(\mathbf{p})-\Phi_-(\mathbf{p})\right]e^{ik\mathbf{p}\cdot\mathbf{q}/n}. \qquad (10.5b)\end{aligned}$$

Both the function $F(\mathbf{q})$ at the screen *and* its normal derivative $F'(\mathbf{q})$ are needed to encode the information contained in the two functions $\Phi_\pm(\mathbf{p})$ on the disk δ_n. The inversion of (10.5) to solve for the function on the sphere is found through Fourier transformation in the plane:

$$\Phi_\pm(\mathbf{p})=\frac{k}{4\pi n}\int_{\Re^2}d^2\mathbf{q}\left[\frac{\sqrt{n^2-p^2}}{n}F(\mathbf{q})\pm\frac{1}{ik}F'(\mathbf{q})\right]e^{-ik\mathbf{p}\cdot\mathbf{q}/n}. \tag{10.6}$$

---

gration over the sphere with the measure (10.1) endows it with units of n^2. Hence, if we regard $\Phi(\vec{p})$ as having units n^{-1}, $F(\vec{r})$ will have units of k. We recall that $k/n=k_0$, the fixed wavenumber of vacuum in the Fourier signal decomposition of last Section.

[18]Note the factor $k/2\pi n=k_0/2\pi=1/\lambda_0=1/n\lambda$, with λ_0 the wavelength in vacuum $n=1$.

For example, a single plane wave $\Omega_{\vec{p}_0}$ directed by some $\vec{p}_0$ will have its coset representative function given by a Dirac delta on the Descartes sphere, *i.e.*, under the measure (10.1) and appropriate range,

$$\Omega_{\vec{p}_0}(\vec{p}) = \delta_{S_2}(\vec{p}_0, \vec{p}) = (n^2 \sin\theta)^{-1}\delta(\theta_0 - \theta)\delta(\phi_0 - \phi), \quad (10.7a)$$

or,

$$\Omega_{\mathbf{p}_0,\sigma}(\mathbf{p}) = \frac{\sqrt{n^2 - p^2}}{n}\,\delta(p_{0x} - p_x)\,\delta(p_{0y} - p_y)\delta_{\sigma,\mathrm{sign}\, p_{0z}}. \quad (10.7b)$$

The corresponding Helmholtz plane-wave solution function and its normal derivative at the screen form then the two-function

$$\mathbf{W}_{\vec{p}_0}(\mathbf{q}) = \begin{pmatrix} W_{\vec{p}_0}(\mathbf{q}) \\ W'_{\vec{p}_0}(\mathbf{q}) \end{pmatrix} = \frac{k}{2\pi n}\begin{pmatrix} 1 \\ ikp_{0z}/n \end{pmatrix} e^{ik\mathbf{p}_0\cdot\mathbf{q}/n}. \quad (10.8)$$

The normal derivative distinguishes the two distinct plane waves $\vec{p}^{+} = (\mathbf{p}, p_z)$ and $\vec{p}^{-} = (\mathbf{p}, -p_z)$ that are *reflected* versions of each other by a mirror in the screen. A Helmholtz function whose normal derivative on the screen is *zero* contains, for every constituent plane wave, its reflection. The last Section will elaborate further on this.

Square-integrable functions on the sphere with the measure (10.1) are well known to constitute a Hilbert space $\mathcal{L}^2(\mathcal{S}_2)$, for which a definite value of k is implied. The sesquilinear inner product of two functions in that space is

$$(\Phi_1, \Phi_2)_{\mathcal{S}_2} = \int_{\mathcal{S}_2} d^2S(\vec{p})\,\Phi_1(\vec{p})^*\,\Phi_2(\vec{p}) \quad (10.9a)$$

$$= \int_{\delta_n} \frac{n\,d^2\mathbf{p}}{\sqrt{n^2 - p^2}} \times [\Phi_{1,+}(\mathbf{p})^*\,\Phi_{2,+}(\mathbf{p}) + \Phi_{1,-}(\mathbf{p})^*\,\Phi_{2,-}(\mathbf{p})], \quad (10.9b)$$

where the asterisk * indicates complex conjugation. This inner product is manifestly invariant under rotations of the Descartes sphere, as well as under translations, since the latter only multiply the functions by a phase that cancels on account of the sesquilinearity of the inner product. It is an invariant under Euclidean transformations: $(\mathcal{L}_g\Phi_1, \mathcal{L}_g\Phi_2)_{\mathcal{S}_2} = (\Phi_1, \Phi_2)_{\mathcal{S}_2}$. The last expression shows the form of the inner product as an integral over the disk δ_n of radius n, for both 'forward' and 'backward' waves. The Euclidean generators in (9.3) and (9.4) will be skew-adjoint under this inner product.

Let us now write the inner product (10.9) in terms of the initial value and normal derivative on the screen of solutions to the Helmholtz equation, as given by equations (10.5), replacing the $\Phi_\pm(\mathbf{p})$'s from (10.6) into (10.9).

There is a triple integration where we can move to the right the integral over the compact domain,

$$(\Phi_1, \Phi_2)_{S_2} = \left(\frac{k}{2\pi n}\right)^2 \int_{\delta_n} \frac{n\, d^2\mathbf{p}}{\sqrt{n^2-p^2}} \int_{\Re^2} d^2\mathbf{q} \int_{\Re^2} d^2\mathbf{q}'\, e^{-ik\mathbf{p}\cdot(\mathbf{q}-\mathbf{q}')/n}$$
$$\times \left[\frac{n^2-p^2}{n^2} F_1(\mathbf{q})^* F_2(\mathbf{q}') + \frac{1}{k^2} F_1'(\mathbf{q})^* F_2'(\mathbf{q}')\right] \qquad (10.10a)$$

$$= \left(\frac{k}{2\pi n}\right)^2 \int_{\Re^2} d^2\mathbf{q} \int_{\Re^2} d^2\mathbf{q}'$$
$$\times \left[\omega(|\mathbf{q}-\mathbf{q}'|) F_1(\mathbf{q})^* F_2(\mathbf{q}')\right.$$
$$\left. + \varpi(|\mathbf{q}-\mathbf{q}'|) F_1'(\mathbf{q})^* F_2'(\mathbf{q}')\right]. \qquad (10.10b)$$

We have assimilated the $\mathbf{p}$–integration[19] into two *nonlocal weight functions*, ω and ϖ,

$$\begin{aligned}\omega(|\mathbf{q}-\mathbf{q}'|) &= \frac{1}{2}\int_{\delta_n} d^2\mathbf{p}\, \frac{\sqrt{n^2-p^2}}{n} e^{-ik\mathbf{p}\cdot(\mathbf{q}-\mathbf{q}')/n} \\ &= \frac{1}{2}\int_0^n p\, dp \frac{\sqrt{n^2-p^2}}{n} \int_0^{2\pi} d\varphi\, e^{-ikp|\mathbf{q}-\mathbf{q}'|\cos\varphi/n} \\ &= \frac{\pi}{n}\int_0^n p\, dp \sqrt{n^2-p^2}\, J_0(kp|\mathbf{q}-\mathbf{q}'|/n) \qquad (10.11a) \\ &= \pi n^2 \frac{j_1(k|\mathbf{q}-\mathbf{q}'|)}{k|\mathbf{q}-\mathbf{q}'|},\end{aligned}$$

and

$$\begin{aligned}\varpi(|\mathbf{q}-\mathbf{q}'|) &= \frac{1}{2k^2}\int_{\delta_n} d^2\mathbf{p}\, \frac{n}{\sqrt{n^2-p^2}} e^{-ik\mathbf{p}\cdot(\mathbf{q}-\mathbf{q}')/n} \\ &= \frac{\pi n^2}{k^2} j_0(k|\mathbf{q}-\mathbf{q}'|), \qquad (10.11b)\end{aligned}$$

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[19] We note the useful integral [Gradshteyn & Ryzhik, Eqs. 6.567.1 and 6.554.2]:

$$\int_0^n p\, dp\, (n^2-p^2)^\mu J_0(xp/n) = 2^\mu \Gamma(\mu+1) n^{2(\mu+1)} \frac{J_{\mu+1}(x)}{x^{\mu+1}}.$$

where we have the spherical Bessel functions

$$j_0(z) = \sqrt{\frac{\pi}{2z}}\, J_{1/2}(z) = \frac{\sin z}{z}, \qquad (10.12a)$$

$$\frac{j_1(z)}{z} = \sqrt{\frac{\pi}{2z^3}}\, J_{3/2}(z) = \frac{\sin z - z\cos z}{z^3}. \qquad (10.12b)$$

The weight functions are solutions of the Helmholtz equation; $\varpi(\mathbf{q})$ integrates $\Phi(\vec{p})$ = constant over the whole direction sphere [*cf.* (10.5), thus with zero normal derivative] and $\omega(\mathbf{q})$ correspondingly integrates $\Phi(\vec{p}) =$ constant $\times\, p_z^2$. Since they are the widest, smoothest functions on the sphere, they may be seen as the *narrowest* functions on the screen that are still purely oscillatory solutions of the Helmholtz equation.[20]

We may write this inner product on the space of screen conditions for the Helmholtz equation in 2–matrix form as

$$(\mathbf{F}_1, \mathbf{F}_2)_{\mathcal{H}_k} = \int_{\Re^2} d^2\mathbf{q} \int_{\Re^2} d^2\mathbf{q}'\; \mathbf{F}_1(\mathbf{q})^\dagger \mathbf{H}_k(|\mathbf{q}-\mathbf{q}'|)\mathbf{F}_2(\mathbf{q}'), \qquad (10.13a)$$

$$\mathbf{F}_j(\mathbf{q}) = \begin{pmatrix} F_j(\mathbf{q}) \\ F_j'(\mathbf{q}) \end{pmatrix}, \qquad (10.13b)$$

$$\mathbf{H}_k(|\mathbf{q}-\mathbf{q}'|) = \frac{1}{4\pi}\begin{pmatrix} k^2\dfrac{j_1(k|\mathbf{q}-\mathbf{q}'|)}{k|\mathbf{q}-\mathbf{q}'|} & 0 \\ 0 & j_0(k|\mathbf{q}-\mathbf{q}'|) \end{pmatrix}. \qquad (10.13c)$$

This inner product is also *Euclidean invariant*: if $F_j(\vec{r})$, $j = 1, 2$ are two *solutions* of the Helmholtz equation, whose values and normal derivatives at the standard screen $r_z = 0$ are $F_j(\mathbf{q})$ and $F_j'(\mathbf{q})$, their inner product is unchanged if we move or rotate the screen to any other plane. Since it was built unitarily equivalent to $\mathcal{L}^2(\mathcal{S}_2)$, it thus serves to define a *Hilbert space* of oscillatory solutions to the Helmholtz equation that we shall call $\mathcal{H}_k$. Such an inner product for the *two*-dimensional Helmholtz solutions was found by Steinberg and Wolf [9] searching for Euclidean-invariant inner products with an in general nonlocal matrix measure $\mathbf{H}_k(|\mathbf{q}-\mathbf{q}'|)$; its matrix elements were obtained through boundary and differential conditions that hold in the subspace of oscillatory solutions of that equation, and shown to be *unique*.[21]

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[20] Dirac δ's are not allowed in $\mathbf{q}$ since their Fourier conjugate has support outside the p-disk δ_n. Also, evanescent waves are not allowed unless we go into the complex–k extension of our group. This we shall not do here. The issue of *localizability* is correspondingly different from what we are familiar with in quantum mechanics.

[21] A similar treatment was made in [9] for the inner product in the Klein-Gordon equation solution space. In the form (10.13), the measure is then shown to be local (by Dirac δ's) and, in matrix form, antidiagonal. This verifies the known result for the three-dimensional Poincaré-invariant inner product. We should note that the inner product is not *total illumination* —that will be examined in the next Section.

6.11 The Euclidean algebra generators in Helmholtz optics —wavization

Our last realization of the Euclidean algebra generators was given in equations (9.3)–(9.4) for two-chart functions on the two disks of the squashed Descartes sphere. We now want to display the form of the generators on the Helmholtz Hilbert space of two-functions on the *screen* $\mathbf{q}$, with the nonlocal inner product (10.13). Upon comparison with the Euclidean generators on the geometric optics phase space, we will arrive at what appears to be a good recipe for wavization.

When the function over the direction sphere $\Phi(\vec{p})$ in (10.2) or its equivalents $\Phi_\pm(\mathbf{p})$ in (10.5), are multiplied by p_x/n or p_y/n, this factor becomes $ik\partial/\partial q_x$ and $ik\partial/\partial q_y$ on the Helmholtz solution $F(\vec{r})$ in (10.2) or its equivalent two-function $\mathbf{F}(\mathbf{q})$ in (10.5). The z–translation generator, multiplication by p_z/n, is different on the two charts: $p_z\Phi(\vec{p}) \mapsto \pm\sqrt{n^2-p^2}\Phi_\pm(\mathbf{p})$. Such a multiplication turns the integral for $F(\mathbf{q})$ in (10.5a) into the integral for $F'(\mathbf{q})$ in (10.5b). It also turns the latter into the former with an integrand factor of n^2-p^2. This factor, in company with $\exp(ik\mathbf{p}\cdot\mathbf{q}/n)$, becomes (minus) the Helmholtz operator Δ_k in equation (10.4) acting on the same exponential. This operator is then extracted from the integral. Hence, the translation generators in (9.3) may be written as 2×2 *matrix* operators on the Helmholtz Hilbert space, thus:

$$\hat{T}_x^{\mathcal{H}_k} = \begin{pmatrix} \partial_{q_x} & 0 \\ 0 & \partial_{q_x} \end{pmatrix}, \quad \hat{T}_y^{\mathcal{H}_k} = \begin{pmatrix} \partial_{q_y} & 0 \\ 0 & \partial_{q_y} \end{pmatrix}, \quad \hat{T}_z^{\mathcal{H}_k} = \begin{pmatrix} 0 & 1 \\ -\Delta_k & 0 \end{pmatrix}. \tag{11.1a, b, c}$$

We may follow a similar procedure for the generators of rotations around the x and y axes in (9.4) through (10.5a,b), the sign difference on the two charts turning one into the other. The roots of n^2-p^2 cancel or combine with their measures; derivatives with respect to p_j's can be integrated by parts because the boundary terms between functions on the two disks cancel, and are thus thrown on the exponential factor. Finally, the exponent turns p_j's into $\partial/\partial q_j$'s and $\partial/\partial p_j$'s into q_j's that can be extracted from the integral. We thus arrive at the following matrix operator realization:

$$\hat{R}_x^{\mathcal{H}_k} = \begin{pmatrix} 0 & q_y \\ -q_y\Delta_k - \partial_{q_y} & 0 \end{pmatrix}, \tag{11.2a}$$

$$\hat{R}_y^{\mathcal{H}_k} = \begin{pmatrix} 0 & -q_x \\ q_x\Delta_k + \partial_{q_x} & 0 \end{pmatrix}, \tag{11.2b}$$

$$\hat{R}_z^{\mathcal{H}_k} = \begin{pmatrix} q_x\partial_{q_y} - q_y\partial_{q_x} & 0 \\ 0 & q_x\partial_{q_y} - q_y\partial_{q_x} \end{pmatrix}. \tag{11.2c}$$

The above generators were also found by Steinberg and Wolf [9], for the two-dimensional Helmholtz equation and were further studied and applied by Atakishiyev, Lassner and Wolf in reference [17]. They are skew-adjoint under the nonlocal inner product in the Helmholtz Hilbert space (10.13). Their commutation relations are of course the same as (4.4), and as a *irreducible representation* of the Euclidean algebra they are identified by their invariants $(\hat{\vec{T}}^{\mathcal{H}_k})^2 = -k^2\mathbf{1}$ and $\hat{\vec{T}}^{\mathcal{H}_k}\cdot\hat{\vec{R}}^{\mathcal{H}_k} = 0$. Regarding the pure rotation subalgebra (11.2), the representation that is spanned is not irreducible, but quite closely so:

$$(\hat{R}_x^{\mathcal{H}_k})^2 + (\hat{R}_y^{\mathcal{H}_k})^2 + (\hat{R}_z^{\mathcal{H}_k})^2 = \begin{pmatrix} \hat{D}(\hat{D}-1)+k^2q^2 & 0 \\ 0 & \hat{D}(\hat{D}+1)+k^2q^2 \end{pmatrix}, \tag{11.3a}$$

where

$$\hat{D} = \tfrac{1}{2}(\mathbf{q}\cdot\partial_{\mathbf{q}} + \partial_{\mathbf{q}}\cdot\mathbf{q}) = q_x\partial_{q_x} + q_y\partial_{q_y} + 1 \tag{11.3b}$$

is a 'dilatation' operator on the screen, that is self-adjoint on $\mathcal{L}^2(\Re^2)$, but not separately so in $\mathcal{H}_k$.

Let us exemplify the handling of the sphere and Helmholtz inner products $(\Phi_1,\Phi_2)_{\mathcal{S}_2} = (\mathbf{F}_1,\mathbf{F}_2)_{\mathcal{H}_k}$ by finding the matrix elements of the z–translation generator in $\mathcal{L}^2(\mathcal{S}_2)$, $\hat{T}_z^{k\pm}$ in Eqs. (9.3), and its Helmholtz version $\hat{T}_z^{\mathcal{H}_k}$ in Eqs. (11.1) through (10.2)–(10.5). This generator is the analogue of the quantum mechanical Hamiltonian, so we may asign $(\Phi,\hat{T}_z^k\Phi)_{\mathcal{S}_2} = (\mathbf{F},\hat{T}_z^{\mathcal{H}_k}\mathbf{F})_{\mathcal{H}_k}$ the interpretation of the energy —illumination— of the state described by $\Phi(\vec{p})$ as a function of direction, or by $\mathbf{F}(\mathbf{q})$ as the Helmholtz two-function on the screen.

To this end we calculate, following (10.9), the cross matrix elements

$$(\Phi_1,\hat{T}_z^k\Phi_2)_{\mathcal{S}_2} = \int_{\mathcal{S}_2} d^2S(\vec{p})\,\Phi_1(\vec{p})^*\,\frac{ikp_z}{n}\Phi_2(\vec{p}) \tag{11.4a}$$

$$= ik\int_{\delta_n} d^2\mathbf{p}\,[\Phi_{1,+}(\mathbf{p})^*\,\Phi_{2,+}(\mathbf{p})$$

$$-\,\Phi_{1,-}(\mathbf{p})^*\,\Phi_{2,-}(\mathbf{p})]\,. \tag{11.4b}$$

Next, we replace the $\Phi(\mathbf{p})$'s by $F(\mathbf{q})$'s and $F'(\mathbf{q})$'s through (10.6) with cancellation of summands, and exchange integrals. We thus obtain

$$\frac{2k^2}{(4\pi n)^2}\int_{\Re^2} d^2\mathbf{q}\int_{\Re^2} d^2\mathbf{q}'[F_1(\mathbf{q})^*F_2'(\mathbf{q}') - F_1(\mathbf{q})'^*F_2(\mathbf{q}')]$$

$$\times\int_{\delta_n} d^2\mathbf{p}\,\frac{\sqrt{n^2-p^2}}{n}e^{-ik\mathbf{p}\cdot(\mathbf{q}-\mathbf{q}')/n} \tag{11.4c}$$

$$= \left(\frac{k}{2\pi n}\right)^2\int_{\Re^2} d^2\mathbf{q}\int_{\Re^2} d^2\mathbf{q}'$$

$$\times\,(F_1(\mathbf{q})\;F_1'(\mathbf{q}))^*\begin{pmatrix} 0 & \omega(|\mathbf{q}-\mathbf{q}'|) \\ -\omega(|\mathbf{q}-\mathbf{q}'|) & 0 \end{pmatrix}\begin{pmatrix} F_2(\mathbf{q}') \\ F_2'(\mathbf{q}') \end{pmatrix} \tag{11.4d}$$

$$= \left(\frac{k}{2\pi n}\right)^2 \int_{\Re^2} d^2\mathbf{q} \int_{\Re^2} d^2\mathbf{q}' \, (F_1(\mathbf{q})\, F_1'(\mathbf{q}))^*$$
$$\times \begin{pmatrix} \omega(|\mathbf{q}-\mathbf{q}'|) & 0 \\ 0 & \varpi(|\mathbf{q}-\mathbf{q}'|) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\Delta_k & 0 \end{pmatrix} \begin{pmatrix} F_2(\mathbf{q}') \\ F_2'(\mathbf{q}') \end{pmatrix}. \quad (11.4e)$$
$$= (\mathbf{F}_1, \hat{T}_z^{\mathcal{H}_k} \mathbf{F}_2)_{\mathcal{H}_k}. \quad (11.4f)$$

The step (11.4c–d) recognizes the integral in (10.11a) while the equality (11.4d–e) proceeds through integration by parts on $\mathbf{q}'$, matrix multiplication, and the differential equality

$$\Delta_k \, \varpi = \omega \quad (11.5)$$

between the two weight functions of the Helmholtz Hilbert space measure.

In the form (11.4a) it is evident that we are integrating functions over the sphere of directions with an obliquity factor of $p_z/n = \cos\theta$, where θ is the angle between the plane of the waves and the $z =$ constant screen. Upon transformation to Helmholtz 'wavefunctions' over the screen in (11.4d), the same inner product takes a nonlocal Klein-Gordon-type of antidiagonal measure structure [9] that should merit further inquiry elsewhere.

Now that we have presented geometric and Helmholtz optics as two structures contained within the Euclidean group, let us define *wavization* "$\overset{\mathrm{W}}{\mapsto}$" heuristically as the passage from one to the other that is paralell to that from classical to quantum mechanics. Comparison of the translation generators on $\mathcal{W}_k$ given by (9.3) and recall of the relation $k/n = k_0$, suggest the replacement of the 'geometric' momentum components of $\vec{p}$ in the $\wp$ manifold by the self-adjoint matrix operators (11.1) in the Helmholtz Hilbert space $\mathcal{H}_k$, through the map

$$p_x \overset{\mathrm{W}}{\mapsto} -\frac{i}{k_0}\begin{pmatrix} \partial_{q_x} & 0 \\ 0 & \partial_{q_x} \end{pmatrix}, \quad (11.6a)$$
$$p_y \overset{\mathrm{W}}{\mapsto} -\frac{i}{k_0}\begin{pmatrix} \partial_{q_y} & 0 \\ 0 & \partial_{q_y} \end{pmatrix}, \quad (11.6b)$$
$$p_z \overset{\mathrm{W}}{\mapsto} -\frac{i}{k_0}\begin{pmatrix} 0 & 1 \\ -\Delta_k & 0 \end{pmatrix}. \quad (11.6c)$$

Note that the role of Planck's constant $\hbar$ in quantum mechanics is taken by vacuum wavenumber k_0 in Helmholtz screen optics.

Let us articulate first a naïve guess based on our Schrödinger "$\overset{\mathrm{Q}}{\mapsto}$" experience: classical functions satisfying certain Poisson brackets should quantize to operators satisfying analogous commutators. This will turn out to be somewhat off the mark, but may be instructive to point out pitfalls. If we map the space coordinates (r_x, r_y, r_z) to the multiplicative operators

$(\hat{q}_x, \hat{q}_y, 0)$ on functions in $\mathcal{H}_k$, and the symmetrization scheme is followed,[22] the classical components of angular momentum $\vec{R}^\times = \vec{r}\times\vec{p}$ that generate rotations, will map correctly as

$$R^\times_x = r_y p_z - r_z p_y \overset{Q}{\mapsto} \tfrac{1}{2}\{\hat{q}_y, \hat{p}_z\}_+ = -i/k_0\, \hat{R}^{\mathcal{H}_k}_x, \quad (11.7a)$$

$$R^\times_y = r_z p_x - r_x p_z \overset{Q}{\mapsto} -\tfrac{1}{2}\{\hat{q}_x, \hat{p}_z\}_+ = -i/k_0\, \hat{R}^{\mathcal{H}_k}_y, \quad (11.7b)$$

$$R^\times_z = r_x p_y - r_y p_x \overset{Q}{\mapsto} \hat{q}_x\hat{p}_y - \hat{q}_y\hat{p}_x = -i/k_0\, \hat{R}^{\mathcal{H}_k}_z. \quad (11.7c)$$

Compare with (11.2), noting that the 2–1 matrix element of $\hat{R}^{\mathcal{H}_k}_x$ is $\frac{1}{2}(q_y\Delta_k + \Delta_k q_y) = q_y\Delta_k + \partial_{q_y}$ and similarly for $\hat{R}^{\mathcal{H}_k}_y$. It may be somewhat surprising however that the 'wavized' factors $\hat{q}_y$ and $\hat{q}_y$ do *not* commute with $\hat{p}_z$.

Three reasons for **not** accepting this wavization recipe are: *(a)*, multiplicative operators $\hat{q}_x$ and $\hat{q}_y$ are by themselves not self-adjoint under the inner product (10.13) $\mathcal{H}_k$; *(b)* even classically, $\exp v_x\{q_x, \circ\}$ and $\exp v_y\{q_y, \circ\}$ map the momentum variables $\mathbf{p}$ outside their proper optical range $|\mathbf{p}| \le n$ —recall this is **not** the Heisenberg algebra and such operators are **not** within the Euclidean algebra; and finally, *(c)*, in Sections 13 and 14 we shall present the Lorentz group generators, properly constructed both in $\wp$ and $\mathcal{H}_k$, where the above recipe does not quite work. The difference will be small enough, however, to merit notice.

Our position here is that we should wavize only variables in $\wp$ that are within the Euclidean algebra into self-adjoint operators in $\mathcal{H}_k$. This means that the maps between (7.5)[23] and (11.2) that should complement the momentum wavization (11.6), are:[24]

$$R_x = q_y h \overset{W}{\mapsto} -\frac{i}{k_0}\hat{R}^{\mathcal{H}_k}_x = -\frac{i}{k_0}\begin{pmatrix} 0 & q_y \\ -q_y\Delta_k - \partial_{q_y} & 0 \end{pmatrix}, \quad (11.8a)$$

$$R_y = -q_x h \overset{W}{\mapsto} -\frac{i}{k_0}\hat{R}^{\mathcal{H}_k}_y = -\frac{i}{k_0}\begin{pmatrix} 0 & -q_x \\ q_x\Delta_k + \partial_{q_x} & 0 \end{pmatrix}, \quad (11.8b)$$

$$R_z = \mathbf{q}\times\mathbf{p} \overset{W}{\mapsto} -\frac{i}{k_0}\hat{R}^{\mathcal{H}_k}_z$$
$$= -\frac{i}{k_0}\begin{pmatrix} q_x\partial_{q_y} - q_y\partial_{q_x} & 0 \\ 0 & q_x\partial_{q_y} - q_y\partial_{q_x} \end{pmatrix}. \quad (11.8c)$$

In quantum mechanics, *position* is a very good observable: its eigenstates

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[22] In quantum mechanics, if two observables quantize as $a \overset{Q}{\mapsto} \hat{A}$ and $b \overset{Q}{\mapsto} \hat{B}$, the symmetrization scheme in entails that the product quantize through their *anticommutator*: $ab \overset{Q}{\mapsto} \frac{1}{2}\{\hat{A}, \hat{B}\}_+ = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})$. When ab is of the form $qf(p)$, $pf(p)$, or quadratic in q and p, this scheme is equivalent to any other quantization scheme [12].

[23] Note that the 'classical' functions to be wavized are *minus* the functions that appear in the Poisson operator.

[24] Although $h = p_z$, we write h in place of p_z here to emphasize that the components of $\mathbf{q}$ will not be wavized alone, but only *in company* with the h's.

are Dirac δ's that, while not quite in $\mathcal{L}^2(\Re)$, are nevertheless limit points of weak sequences of functions that are. One feature of Euclidean-based wavization is the absence of a good position operator in Helmholtz optics. Dirac δ's are nowhere near to functions in the space. The coordinates appear only as the arguments of functions in $\mathcal{H}_k$ and are not extractable from there as eigenvalues of a polynomial operator. Correspondingly, in $\mathcal{H}_k$ The closest we can come to "screen coordinates" are the *rotation* functions $q_x h$ and $q_y h$ above.[25] These functions do not have zero Poisson brackets nor do their operators commute, for they are generators of a rotation group $\mathcal{R}_3$. Educated intuition confirms that they *should not* be simultaneously observable, since $\omega(\mathbf{q})$ and $\varpi(\mathbf{q})$ are the 'sharpest' screen functions available. Indeed, we expect to find a form of the sampling theorem (valid for the *sphere*, rather than the circle or torus, as is usual in power spectrum and signal theory [10]). Such is a good program to be followed elsewhere.

6.12 The ray direction sphere under Lorentz boost transformations

It is natural to follow the strategies of classical and quantum mechanics in developing Euclidean optics. The formulation of mechanics may be derived from the nilpotent Heisenberg-Weyl [11], its enveloping algebra, group, and ring [12], from the Galilei group [13], or from the general symplectic group [14]. These groups are inappropriate for optics because here the *momentum* observable has a *bounded* range, whereas in mechanics it is infinite. The issue of *position* coordinates has arisen above and we have seen that they are outside the pale of good Euclidean operators. *Linear* canonical transformations of phase space, a well-studied terrain common to classical and quantum mechanics, are thus meaningless in global optics. Yet because they constitute the essence of the paraxial approximation and the tool of aberration expansions we shall continue to work with them in the near-metaxial regime –elsewhere. Here we now review a group of transformations that *is beyond the Euclidean group but whose action is well-defined and global* both in geometric and Helmholtz optics, and follows the wavization process proposed above.

In geometric optics the basic object, a light ray, is a coset $\{\mathbf{p}, \sigma; \mathbf{q}\}$ in the Euclidean group by the symmetry group of the ray. In Helmholtz optics, the basic object is a plane and its corresponding space of cosets has been divided into irreducible subspaces $\{\mathbf{p}; \sigma\}|_k$. They have in common the ray direction sphere $\vec{p} \in \mathcal{S}_2$. The sphere is a well-known subject of relativistic Lorentz SO(3,1) transformations because it is a space of cosets of that

[25]We are not allowed to divide by h, because the spectrum of $\hat{h} = -i/k_0\, \hat{T}_z^{\mathcal{H}_k}$ includes zero.

group by the noncompact factor. Physically, this comes about as follows.

Let $\ell = (\vec{\ell}, \ell_0) = (\ell_x, \ell_y, \ell_z, \ell_0)$ be a lightlike four-vector, $|\vec{\ell}| = |\ell_0|$, undergoing a Lorentz boost by $v = c \tanh \alpha$ in the z–direction,

$$\ell_{x,y} \mapsto \ell'_{x,y}, \tag{12.1a}$$

$$\ell_z \mapsto \ell'_z = \ell_z \cosh \alpha + \ell_0 \sinh \alpha, \tag{12.1b}$$

$$\ell_0 \mapsto \ell'_0 = \ell_z \sinh \alpha + \ell_0 \cosh \alpha. \tag{12.1c}$$

The *direction* of such a four-vector on a sphere $\mathcal{S}_2$ of radius n is given by the components of $\vec{\ell}$ normalized by division through ℓ_0/n, namely,

$$\vec{p} = (p_x, p_y, p_z) = \left(\frac{n\ell_x}{\ell_0}, \frac{n\ell_y}{\ell_0}, \frac{n\ell_z}{\ell_0}\right). \tag{12.2}$$

In these *homogeneous* coordinates, the boost (12.1) becomes the transformation

$$\mathbf{p} \mapsto \mathbf{p}' = \frac{\mathbf{p}}{\cosh \alpha + p_z/n \ \sinh \alpha}, \tag{12.3a}$$

$$p_z \mapsto p'_z = \frac{p_z \cosh \alpha + n \sinh \alpha}{\cosh \alpha + p_z/n \ \sinh \alpha}, \tag{12.3b}$$

where $\mathbf{p} = (p_x, p_y)$ as usual, and $p = |\mathbf{p}|$. In terms of angles $\{\theta, \phi\}$ over the sphere, [*cf.* (3.5*b*)], we find ϕ to be invariant while the colatitude θ follows the nonlinear transformation given by

$$\frac{p}{p_z + n} = \tan \tfrac{1}{2}\theta \mapsto \tan \tfrac{1}{2}\theta' = e^{-\alpha} \tan \tfrac{1}{2}\theta. \tag{12.4}$$

An observer in a spacecraft moving with respect to the stars will therefore see the directions of their rays concentrate towards his direction of motion by the amount (12.4). This effect was noticed in 1725 by Bradley, who termed it *stellar aberration*, recognized it to originate from the earth's orbital motion during the year, and provided the first estimate of the speed of light. It is a global *deformation*[26] transformation of $\mathcal{S}_2$ that has a group-theoretic origin.

To find the z–boost *generator* responsible for the nonlinear transformation (12.4), we may linearize it to a translation through the change of variables

$$\zeta = -\ln \tan \tfrac{1}{2}\theta \mapsto \zeta' = \zeta + \alpha = \exp\left(\alpha \frac{d}{d\zeta}\right)\zeta. \tag{12.5}$$

From $e^{-\zeta} = \tan \frac{1}{2}\theta = \dfrac{p}{n + \sqrt{n^2 - p^2}}$, we find

$$p_\pm = n\frac{2e^\zeta}{1 \pm e^{2\zeta}}, \qquad p_z = n\frac{e^{2\zeta} - 1}{1 + e^{2\zeta}}. \tag{12.6}$$

[26] *i.e.*, the measure over the sphere (10.1) is *not* preserved [15].

On the **p**–disk δ_n there are *two* values of p for each value of ζ, reflecting the map $\mathcal{S}_2 \overset{2:1}{\longmapsto} \delta_n$. The boost generator $\hat{B}_z$ effecting $\exp(\alpha\hat{B}_z)f(\zeta) = f(\zeta+\alpha)$, is thus

$$\hat{B}_z = \frac{d}{d\zeta} = -\frac{p_z\, p}{n}\frac{d}{dp} = \frac{\mp\sqrt{n^2-p^2}}{n}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{p}}. \tag{12.7}$$

6.13 Relativistic coma in geometric optics

When images are enlarged as in a slide projector, or reduced as in a camera, the angles of the rays that arrive at the screen to form the image are inversely reduced or enlarged. This well-known property of passive optical devices is succintly described by the statement that optical transformations must preserve the phase space volume element $d\mathbf{p}\,d\mathbf{q}$. That is, they are bound to produce only *canonical* transformations of the coset manifold seen in Sections 6 and 7. The relativistic Lorentz transformation (12.3) of ray directions, expanded in series of $|\mathbf{p}|$ to fifth order, is

$$\mathbf{p} \mapsto \mathbf{p}' = \frac{\mathbf{p}}{\cosh\alpha + h/n\,\sinh\alpha} \tag{13.1a}$$

$$= e^{-\alpha}\mathbf{p} + \tfrac{1}{2}n^{-2}\sinh\alpha\, e^{-2\alpha}p^2\mathbf{p} \tag{13.1b}$$

$$+ \tfrac{1}{4}n^{-4}\sinh\alpha\, e^{-2\alpha}(1-\tfrac{1}{2}e^{-2\alpha})(p^2)^2\mathbf{p} + \cdots. \tag{13.1c}$$

As before we abbreviate $h = \pm\sqrt{n^2-p^2}$, the sign indicating the hemisphere and $p^2 = p_x^2 + p_y^2$. To first order in p, we have a magnification by a factor of $e^{-\alpha}$ that is less than unity for $\alpha > 0$; after this we have the series of terms that tell us that the magnification is not linear, but a *distortion* of the ray direction sphere. Therefore, to first order in the ray *position* **q** (intersection with the standard screen), we expect a magnification with the inverse factor e^{α}. This will be followed by *aberration* of the nature of *coma*, as will be borne out below.

In order to *extend* the boost action (12.6) from the direction sphere $\vec{p}$ to the whole space $\wp$ of geometric optics $\{\mathbf{p},\sigma;\mathbf{q}\}$ *canonically*, we note that the boost generator in (12.7) may be written as

$$\hat{B}_z = -\frac{h}{n}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{p}} = \frac{\partial(-h\mathbf{p}\cdot\mathbf{q}/n)}{\partial\mathbf{q}}\cdot\frac{\partial}{\partial\mathbf{p}}. \tag{13.2a}$$

This is the $\partial/\partial\mathbf{p}$ part of a Poisson operator [*cf.* Eq. (7.6)], and suggests we *extend* it to $\wp$ as

$$\hat{B}_z^{\wp} = \{-h\mathbf{p}\cdot\mathbf{q}/n, \circ\} = -\frac{h}{n}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{p}} + \frac{h}{n}\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{q}} - \frac{\mathbf{p}\cdot\mathbf{q}}{nh}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{q}}. \tag{13.2b}$$

Now we can exponentiate this operator and show[27] that the boost action on ray position at the screen, conjugate to (13.1), is

$$\begin{aligned} \mathbf{q} &\mapsto \mathbf{q}' \\ &= \exp(\alpha \hat{B}_z^{\wp})\,\mathbf{q} & (13.3a) \\ &= (\cosh\alpha + h/n \,\sinh\alpha)\left(\mathbf{q} - \frac{\sinh\alpha}{n\sinh\alpha + h\cosh\alpha}\,\frac{\mathbf{p}\cdot\mathbf{q}}{n}\,\mathbf{p}\right) & (13.3b) \\ &= e^{\alpha}\mathbf{q} - n^{-1}\sinh\alpha\,\mathbf{p}\cdot\mathbf{q}\,\mathbf{p} - \tfrac{1}{2}n^{-2}\sinh\alpha\, p^2\mathbf{q} \\ &\quad - \tfrac{1}{2}n^{-3}\sinh\alpha\, e^{-2\alpha}p^2\mathbf{p}\cdot\mathbf{q}\,\mathbf{p} - \tfrac{1}{8}n^{-4}\sinh\alpha\,(p^2)^2\mathbf{q} \\ &\quad - \cdots & (13.3c) \end{aligned}$$

By construction it is guaranteed that the measure $d\mathbf{p}\,d\mathbf{q}$ will be preserved.[28] In the last expression we have developed the closed formula (13.3a) in series of powers of $|\mathbf{p}|$ and $|\mathbf{q}|$ to fifth order. The leading linear term shows indeed a magnification factor of e^{α} as required. The rest of the series contains terms in $(p^2)^m\mathbf{q}$ and $(p^2)^{m-1}\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}$, $m = 1, 2, \ldots$. The presence of only such terms determines the mapping to be circular *comatic*. This is the name of a class of aberrations that are *comet*-shaped, and that have the very important property of being 2:1 mappings of object rays to screen points: $(\mathbf{p},\mathbf{q})$ and $(-\mathbf{p},\mathbf{q})$ are mapped on the *same* image *point* $\mathbf{q}'(\mathbf{p},\mathbf{q})$.[29]

Lorentz boosts in directions other than the screen normal may be obtained transforming the generator $\hat{B}_z^{\wp}$ in (13.2b) by means of the generators of rotations given in (7.5) through Poisson operators. In this way we find

$$\hat{B}_x^{\wp} = \{nq_x - p_x\,\mathbf{p}\cdot\mathbf{q}/n, \circ\} = \{q_x h^2 + p_y\,\mathbf{q}\times\mathbf{p}, \circ\}, \qquad (13.4a)$$

$$\hat{B}_y^{\wp} = \{nq_y - p_x\mathbf{p}\cdot\mathbf{q}/n, \circ\} = \{q_y h^2 - p_x\,\mathbf{q}\times\mathbf{p}, \circ\}, \qquad (13.4b)$$

where $\times$ is the vector cross product. The three-vector of boosts is thus generated by the Poisson operator of the vector function $\vec{B} = \vec{p}\times\vec{R}$, where $\vec{R} = \vec{q}\times\vec{p}$ with $\vec{q} = (\mathbf{q}, 0)$ is the three-vector function generating rotations through Poisson operators given in (7.1)–(7.5).[30] Since these are three-vectors, it is natural to expect that

$$[\hat{B}_i^{\wp}, \hat{R}_j^{\wp}] = -\varepsilon_{ijk}\hat{B}_k^{\wp} \qquad (13.5a)$$

holds, as it does. Moreover, it is *also* true that

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[27]The way to derive this formula will be indicated below. *Prima facie*, this is a nontrivial task.

[28]The Poisson bracket of the transformed variables (13.1a)–(13.3a) is also conserved: $\{q_i', p_j'\} = \delta_{i,j} = \{q_i, p_j\}$, etc.

[29]We must emphasize the word *point* because the two image *rays* are distinguished by their *direction* at the screen. Witness in (13.1) that $\mathbf{p}'(\mathbf{p},\mathbf{q}) = -\mathbf{p}'(-\mathbf{p},\mathbf{q})$, as required by the essential 1:1 bijection of all *canonical* mappings of phase space.

[30]Note that $\{\vec{q}, \vec{p}, \vec{R}\}$ form a right triad of vectors.

$$[\hat{B}_i^{\wp}, \hat{B}_j^{\wp}] \quad = \quad +\varepsilon_{ijk}\hat{R}_k^{\wp}. \tag{13.5b}$$

Hence the $\hat{R}^{\wp}$'s and $\hat{B}^{\wp}$'s close into the algebra of the Lorentz SO(3,1) group of special relativity.[31] It is left as an exercise to the reader to decide whether this fact is natural or remarkable. We also note the x–y vector identity $\mathbf{q} = \mathbf{b}/n - b_z\mathbf{p}/nh$, that allowed us to derive the rather formidable Lie exponential of $\hat{B}_z^{\wp}$ in (13.3) through knowing the transformation properties of the pieces in the right-hand side. Not so obvious is the boost exponential in the x–direction, that may be shown to be[32]

$$\begin{aligned} \exp(\alpha\hat{B}_x^{\wp})\, q_x \quad &= \quad q_x' = (\cosh\alpha + p_x/n\, \sinh\alpha) \\ &\qquad \times (q_x \, \cosh\alpha + \mathbf{p}\cdot\mathbf{q}/n\, \sinh\alpha), \qquad (13.6a) \\ \exp(\alpha\hat{B}_x^{\wp})\, q_y \quad &= \quad q_y' = (\cosh\alpha + p_x/n\, \sinh\alpha)\, q_y. \qquad (13.6b) \end{aligned}$$

The transformation undergone by p_x and p_y may be found from (12.3) with the rotated replacement $(p_x, p_y, p_z) \mapsto (p_z, p_x, p_y)$.

Take an image-forming device that is in focus when at rest, and then boost it to α. Figures 5, 6, and 7 show[33] respectively what our mathematics predicts should be the image formed by an array of luminous points on a screen moving towards $(+z)$, sideways $(+x)$, and away from $(-z)$ the optical axis, at the rather considerable speed of $\alpha = 0.3$ ($v = 0.29131\, c$). The images are supposed to be formed out of a 45°–cone of directions around the forward pole (optical axis). The paralells and meridians of this spherical cap constitute the *spot diagram of the original point images (marked by* crosses) magnified and aberrated by the Lorentz motion represented by equations (13.1)–(13.3) and (13.6).

Some detailed geometric properties of the figures have been explored in reference [16]. Here we only want to remark that the three figures are faces of the same aberration, and that they are global: they appear as circular comatic for small angles in $\pm z$–motion and astigmatic/curvature of field for

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[31] Posed as a group deformation procedure, we may add to $\vec{B}$ a multiple μ of $\vec{p}$ and still have the commutation relations (13.5) close. The Lorentz invariants are $\vec{B}^2 - \vec{R}^2 = n^2\mu^2$ and $\vec{R}\cdot\vec{B} = 0$.

[32] These expressions were found by Wolfgang Lassner by a back-and-forth process involving hand and symbolic REDUCE computation on the trusty old IIMAS/Cuernavaca PC, checking that two successive x–boosts will compose properly.

[33] I would like to thank Guillermo Correa (IIMAS–UNAM/DF) for the graphics program SPOT_D, that is capable of reading muSIMP output files through PASCAL and plotting the corresponding spot diagrams; it works not only with aberration expansions, but with exact *global* formulas that apply for Euclidean optics. It is reported in: G.J. Correa–Gómez and K.B. Wolf, SPOT_D, *Programa para Graficación de Diagramas de Manchas en Optica.* Comunicaciones Técnicas IIMAS, Serie Desarrollo, No. 97 (1989), 51 págs. The program is open and may be requested from the author.

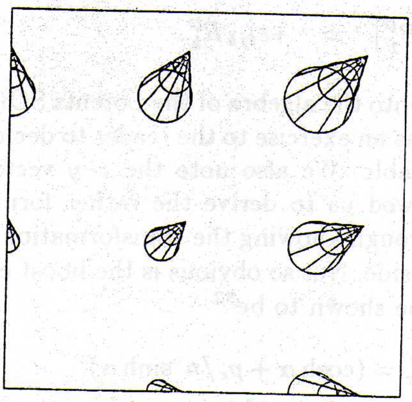

FIGURE 5. Relativistic coma in geometric optics. A screen receives the focused image of an array of object points through collecting rays from 45° cones. When the screen approaches the source at a velocity of $v = 0.29131\,c$ ($\alpha = 0.3$) the image amplifies and exhibits global coma.

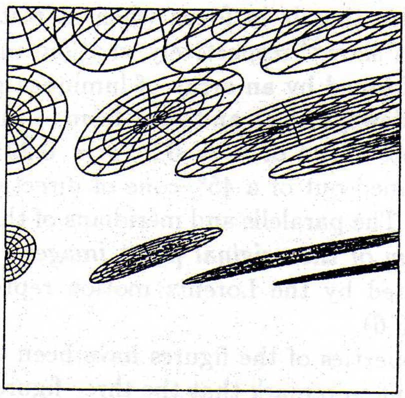

FIGURE 6. The screen moving at right angles (to the right) at the same velocity.

cross motion; they represent part of the 2:1 mapping of the *full* Descartes sphere on the screen. When the motion is but in the screen plane, there will be a circle of rays that become paralell to the moving screen; the position coordinate of these rays will then appear to escape to infinity, without implying any actual singularity in the ray manifold.

6.14 Relativistic coma in Helmholtz wave optics

Helmholtz optics was presented in Sections 9 to 11 as the Euclidean geometry of planes belonging to a definite irreducible representation k of that group. Here we want to explore the Helmholtz wave optics representation of the relativistic Lorentz transformation seen in Section 12. We shall do

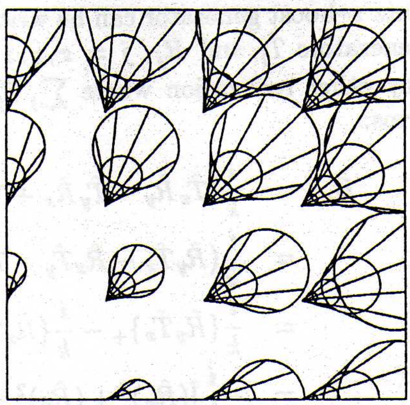

FIGURE 7. The screen moving away from the object. Figures 5 and 7, when superposed, show the $\theta < 45°$ and $\theta > 135°$ parts of global coma.

this first 'longhand', by conventional means of the group deformation arguments of Ref. [15], and then through application of the wavization process (11.6)–(11.8) on the results of last Section. Finally, we comment upon the results obtained in reference [17].

The Lorentz transformation of the sphere $\vec{p} \in \mathcal{S}_2$ was shown to have the formal generator (12.7). Taking into account the measure of the $\mathcal{L}^2(\mathcal{S}_2)$ inner product (10.9a)–(10.9b), and recalling the matching condition among the two functions of the latter, the *skew-adjoint* generator of z–boosts on $\mathcal{L}^2(\mathcal{S}_2)$ is its symmetrized version,

$$\hat{B}_z^{\mathcal{S}_2} = -\frac{p_z}{2n}\left(\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{p}} + \frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{p}\right), \tag{14.1a}$$

and the two-chart operator for $\mathcal{H}_k$ over the disk δ_n is

$$\hat{B}_z^{k\pm} = \frac{\mp\sqrt{n^2-p^2}}{n}\left(\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{p}} + 1\right). \tag{14.1b}$$

The upper sign applies on the 'forward' hemisphere ($p_z > 0$) and the lower sign in the 'backward' one ($p_z < 0$). Finally, through replacement in the integrand of the transform pair (10.5), integration by parts (with the appropriate cancellation of boundary terms for Φ_+ with Φ_-) and extraction from the integral, we obtain the z–boost generator skew-adjoint on the Helmholtz Hilbert space $\mathcal{H}_k$ on the screen:

$$\hat{B}_z^{\mathcal{H}_k} = \frac{i}{k}\begin{pmatrix} 0 & -\hat{D} \\ (\hat{D}+1)\Delta_k - k^2 & 0 \end{pmatrix}, \tag{14.2}$$

where $\hat{D} = \frac{1}{2}(\mathbf{q}\cdot\partial_\mathbf{q} + \partial_\mathbf{q}\cdot\mathbf{q})$ is, as in (11.3) the usual dilatation operator, and Δ_k the Laplacian on the screen plus k^2.

This Lorentz z-boost generator can be written abstractly in terms of the Euclidean generators $\hat{T}_j$ and $\hat{R}_j$, $j = x, y, z$ given in (9.3)–(9.4), (11.1)–(11.2), or any other realization where $\sum_j \hat{T}_j^2 = -k^2$, a constant, in the following forms:

$$\hat{B}_z = \frac{i}{k}(\hat{T}_x\hat{R}_y - \hat{T}_y\hat{R}_x + \hat{T}_z) \tag{14.3a}$$

$$= \frac{i}{k}(\hat{R}_y\hat{T}_x - \hat{R}_x\hat{T}_y - \hat{T}_z) \tag{14.3b}$$

$$= \frac{i}{k}\{\hat{R}_y\hat{T}_x\}_+ - \frac{i}{k}\{\hat{R}_x\hat{T}_y\}_+ \tag{14.3c}$$

$$= -\frac{i}{k}[(\hat{R}_x)^2 + (\hat{R}_y)^2 + (\hat{R}_z)^2, \hat{T}_z], \tag{14.3d}$$

where $\{A, B\}_+ = AB + BA$ is the anticommutator, $[A, B] = AB - BA$ the commutator. All operators are here skew-adjoint. The last form (14.3d) allows us to write the vector boost generator as

$$\hat{\vec{B}} = \frac{i}{k}[\hat{R}^2, \hat{\vec{T}}]. \tag{14.4}$$

This is the usual algebra deformation formula for ISO(3) $\Rightarrow$ SO(3,1) [15], and insures that the three components of $\hat{B}_j$ close, together with the rotation generators $\hat{R}_j$, into the Lorentz algebra:[34]

$$[\hat{B}_i, \hat{R}_j] = -\varepsilon_{ijk}\hat{B}_k, \tag{15.4a}$$

$$[\hat{B}_i, \hat{B}_j] = +\varepsilon_{ijk}\hat{R}_k. \tag{15.4b}$$

In $\mathcal{L}^2(\mathcal{S}_2)$ the boost generators are first-order skew-adjoint differential operators in the components of $\vec{p}$, whose expression includes the symmetrized version of the p–operator part of the corresponding geometric-optics operators in (13.2) and (13.4). In the Helmholtz Hilbert space $\mathcal{H}_k$ of two-functions, they are 2×2–matrices with $\partial/\partial\mathbf{q}$–operator entries that may be found from (11.1)–(11.2) and (14.3)–(14.4). The results so obtained were reported in reference [17]. They are given by (14.2) for $\hat{B}_z^{\mathcal{H}_k}$ and, in screen two-vector form,

$$\hat{\mathbf{B}}^{\mathcal{H}_k} = -\frac{i}{k}\begin{pmatrix} \hat{D}\partial_\mathbf{q} + k^2\mathbf{q} & 0 \\ 0 & (\hat{D}+1)\partial_\mathbf{q} + k^2\mathbf{q} \end{pmatrix}. \tag{14.6}$$

The same results for the boost generators may be obtained through applying the *wavization* rules given in (11.6)–(11.8) on the function compo-

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[34] We are again allowed to add any real multiple μ of $\hat{\vec{T}}$ to (14.4), leading to all nonexceptional continuous series of SO(3,1) representations according to the deformation algorithm. The values of the Casimir operators are $\hat{R}^2 - \hat{B}^2 = 1$ and $\hat{R} \cdot \hat{B} = 0$:

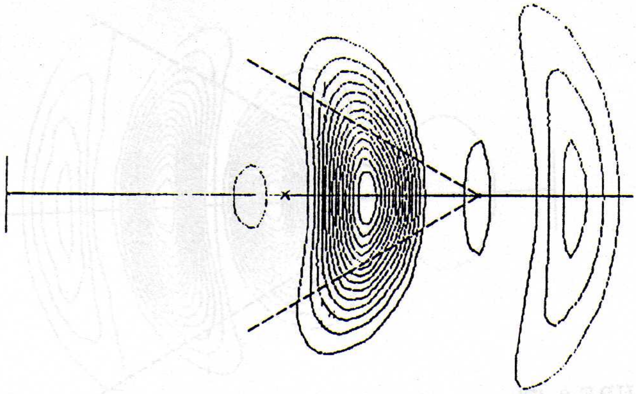

FIGURE 8. Relativistic coma in Helmholtz optics. A Gaussian beam displaced from the optical center by $2/k_0$ units (marked by the $\times$) and of width $4/k_0^2$ units is focused on a screen. When the screen approaches the source at $v = 0.29131\,c$ ($\alpha = 0.3$), as in Figure 5, the image exhibits the indicated 'isophote' lines given by the square of the first component of the Helmholtz two-function.

nents of the three-vector

$$\vec{B} = \vec{p}\times\vec{R} = \begin{pmatrix} p_z\, q_x h + p_y\, \mathbf{q}\times\mathbf{p} \\ -p_x\, \mathbf{q}\times\mathbf{p} + p_z\, q_y h \\ -p_x\, q_x h - p_y\, q_y h \end{pmatrix}, \tag{14.7}$$

where we write the components as a column to save space. They contain the same *functions* that appear in the Poisson operators (13.2b)–(13.4) for geometric optics. We emphasize that for the purpose of wavization, we must consider $q_x h$ and $q_y h$ as *single* observable subject to the wavization mapping. If we were to replace hp_z by $p_z^2 = n^2 - p_x^2 - p_y^2$ in the first two components of $\vec{B}$, we would obtain $n^2\mathbf{q} - \mathbf{p}\cdot\mathbf{q}\,\mathbf{p}$ —tempting us to wavize $\mathbf{q}$ alone. The wavization of *this* form of the function would be a *diagonal* matrix with equal elements $(\hat{D} + \frac{1}{2})\partial_{\mathbf{q}} + k^2\mathbf{q}$.[35] The z-component of $\vec{B}$, on the other hand, involves only the combinations $h\mathbf{q}$, and its wavization yields the correct result anyway.

Let us now report on the essentials of the exponentiation of the z-boost generator (14.2), $\exp(i\alpha\hat{B}_z^{\mathcal{H}_k})$, carried out in reference [17]. Let us prominently note that the 1–2 matrix element contains $\hat{D}$, by itself the generator of magnifications $\exp(\alpha\hat{D}) : f(\mathbf{q}) \mapsto e^{\alpha/2} f(e^{\alpha}\mathbf{q})$. The 2–1 matrix element contains the inverse magnification plus the Schrödinger-quantized *coma-generating* function $p^2\,\mathbf{p}\cdot\mathbf{q}$ [18].

In reference [17] we expanded the matrix operator $\hat{B}_z^{\mathcal{H}_k}$ in series to fifth order in α, involving differential operators up to degree nine. This was applied with a symbolic computation muSIMP program to a *forward* Gaussian beam with waist at the screen, *off* the optical axis. Strictly, of course, a

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[35]This is, in fact the *average* of the two diagonal matrix elements in (14.6).

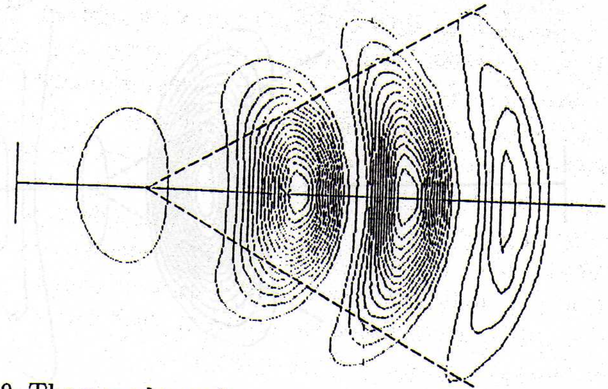


FIGURE 9. The same beam focused on a screen receding from the source at that velocity. The dashed lines indicate the geometric Seidel coma caustics at the apex. The position of the apex is shifted due to magnification and reduction for each case.

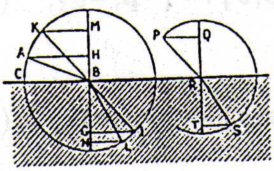
Gaussian function is not in the Helmholtz Hilbert space $\mathcal{H}_k$ because its Fourier transform has small but nonzero support outside the disk δ_n. Calculational expedience, however, makes such a beam an irresistible candidate for investigation: successive ∂_{q_j}–derivatives simply pile a q-polinomial factor in front of it. The resulting Lorentz-transformed squared function was then evaluated on a square grid, and the numerical matrix fed into a plotting algorithm that drew the spline level curves shown in Figure 8, for $\alpha = +0.3$ and Figure 9 -0.3, respectively.[36] These figures should be compared with the ones for the geometric optics phenomenon seen in the last Section. The geometric coma caustic angle (of 60°) and origin (shifted by e^α) is superposed on the figures here. With increasing truncation order of α, the single Gaussian peak unfolds into local maxima, separated by an increasing number of crescent-shaped dark fringes, whose overall features seem to stabilize by degree five in α. Comparing the figures with the pattern of diffraction in coma aberration [19], it would seem that the $\alpha = -0.3$ figure comes closer to that of pure coma than the $\alpha = +0.3$ figure. Actually, the series contains also, prominently, the magnification generator that contributes itself with circular fringes centered on the optical axis. These counteract the curved fringes of pure coma in the first figure, and reinforce those in the second.

The relativistic coma phenomenon predicted for geometric and Helmholtz optics is in a sense perplexing, since none of the two models entertains

[36]I would like to thank José Fernando Barral, of the Instituto de Astronomía, UNAM, for the indispensable help with the figures in this Section. Some ideas on applications to digital image processing were also aired and may be pursued with José Luis Morales, at the Instituto Nacional de Astrofísica, Optica y Electrónica, Tonantzintla.

a *time* variable and, *prima facie*, has nothing to do with motion. Bradley's observation of stellar aberration is fulfilled however, as far as mappings of the sphere of ray directions is concerned. Yet his experimental setup (a telescope) will *not* show the comatic phenomenon because only a *single* ray direction is involved in the formation of the stellar image, rather than a spread pencil of directions brought to focus on the moving screen. It may well be that a more appropriate environment for this effect to manifest its properties is in the field of radiating relativistic elementary particles whose disintegration products are spread and collected from large angles by means not necessarily optical. Some estimates of the relative *size* of the coma caustic have been also given in the reference.

6.15 Reflection, refraction, and concluding remarks



As, for example, if there passes a ray in Air from A *to* B *that finds at* B *the surface of Glass* CBR, *it detours to* I *in this Glass; and another from* K *to* B *detours to* L; *and another from* P *to* R *that detours to* S; *there must be the same proportion between the lines* KM *and* LN, *or* PQ *and* ST, *than between* AH *and* IG, *but not the same between the angles* KBM *and* LBN, *or* PRQ *and* SRT, *than between* ABH *and* IBG.

René Descartes, *Discourse on the Method*
Second Discourse: *On Dioptrics*

Throughout this paper we have considered the radius of the Descartes sphere of ray directions to be a fixed number n; in passing to Helmholtz optics we defined the wavenumber in the medium to be $k = nk_0$, with k_0 the wavenumber for $n = 1$. The need for this unit is not in Euclidean optics of a single homogeneous medium, but to allow for the phenomenon of refraction, where at least *two* homogeneous media are involved. This we do here starting with reflections, and ending with some concluding remarks on what has been done to establish a theory of global optics with wavization, and what remains to be done.

In geometric optics, we recall from Section 5, a single light ray is a coset in the Euclidean group $\{\mathbf{p}, \sigma; \mathbf{q}\}$, a point in the space $\wp = \mathcal{H}^{\text{geom}}\backslash\mathcal{E}_3$. In Helmholtz optics, Sections 9 and 10, a single plane wave is the irreducible component $k = nk_0$ of a coset in $\mathcal{H}^{\text{wf}}\backslash\mathcal{E}_3$, characterized by $\vec{p}$ or $\{\mathbf{p}, \sigma\}$ [see Eqs. (10.7)], and realized by the two-function $\mathbf{W}_{\mathbf{p},\sigma}(\mathbf{q})$ (amplitude and normal derivative) at the $z = 0$ screen $\mathbf{q} \in \Re^2$ given in (10.8). We may define the following two physical operations on the two-disk projected Descartes sphere of ray directions:

$$\text{Reflection} \quad \nearrow \mapsto \nwarrow, \qquad \mathbf{R} : \{\mathbf{p}, \sigma\} \mapsto \{\mathbf{p}, -\sigma\}, \tag{15.1a}$$

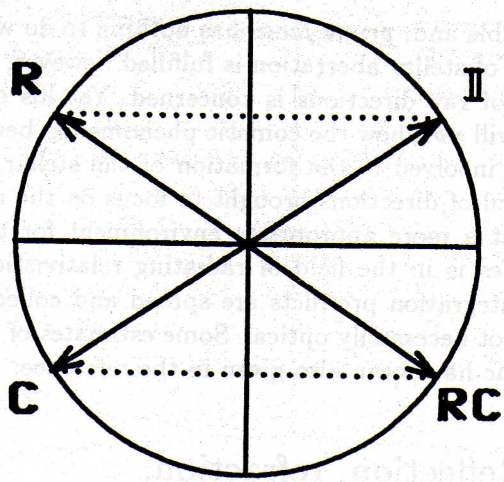


FIGURE 10. Four rays on the Descartes sphere: the original ray **I**, the reflected ray **R**, the conjugate ray **C**, and the reflected-conjugate ray **RC**.

$$\text{Conjugation} \quad ↗ \mapsto ↙, \qquad \mathsf{C} : \{\mathbf{p}, \sigma\} \mapsto \{-\mathbf{p}, -\sigma\}, \qquad (15.1b)$$

and their product

$$↗ \mapsto ↘, \qquad \mathsf{RC} : \{\mathbf{p}, \sigma\} \mapsto \{-\mathbf{p}, \sigma\}. \qquad (15.1c)$$

We have called the two former operations *physical* because the first corresponds to ordinary reflection by a mirror in the $z = 0$ plane, and the second to reflection in a phase-conjugation mirror, that reverses the directions of rays and wavefronts. They both belong to the component of the *orthogonal* group O(3) disconnected from the identity **I**, and may be realized by 3×3 matrices diag$(1, 1, -1)$ and diag$(-1, -1, -1)$, respectively, of determinant -1. Their product **RC** = **CR** is a proper $\mathcal{R}_3$ rotation by π around the z–axis; also, $\mathbf{R}^2 = \mathbf{C}^2 = \mathbf{I}$. See Figure 10. The small arrow diagrams in (15.1) express our intuition in geometric optics regarding reflections and inversions at the screen. In Helmholtz optics reflection **R** reverses the sign of the second, normal derivative component in (10.8)[37] while conjugation **C** is *complex* conjugation. Note that these operations act exclusively on ray direction, *i.e.*, on points of the Descartes sphere; they do *not* affect **q**, neither in the geometric nor Helmholtz cases.

In geometric optics the operator **I** + **C** produces a world of *nondirected* "⤢" rays[38] while in Helmholtz optics it leads to purely *real* wavefunctions.

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[37]The operator of reflection on the Helmholtz Hilbert space $\mathcal{H}_k$ is realized by the 2×2 matrix diag$(1, -1)$.

[38]We may see them as having Wigner-type distributions that have the same value for $\vec{p}$ and $-\vec{p}$ at every **q**. Alternatively, this property could be introduced

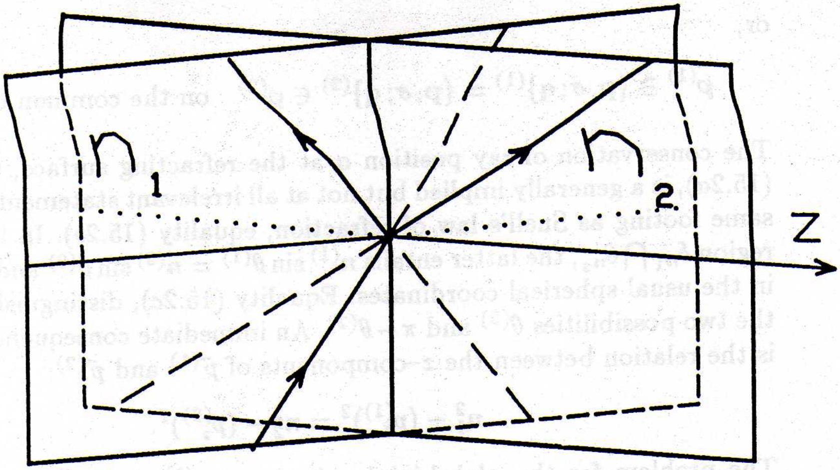

FIGURE 11. The joining of two homogeneous media with two different refractive indices at the reference (1–dim) screen.

The operator **I**+**R** correspondingly creates a "⋇" world where every optical being is accompanied by its reflection by the $z = 0$ screen-turned-mirror; Helmholtz two-functions in particular have zero normal derivative there [see p_z appear in the second component in (10.8); with **I**−**R** the first component is made to vanish]. We should not conclude that there is "nothing behind the mirror"—whatever that means; only that the distribution of rays or wavefunctions are forced to satisfy certain *boundary conditions* at the $z = 0$ plane. Indeed, through linear combinations of **I**, **C**, **R**, and **CR** acting on distributions on $\wp$ or wavefunctions in $\mathcal{H}_k$, we may describe four-ray $\Re^3$ situations "⋇" where the $z = 0$ plane is a special submanifold where boundary conditions can be imposed.

After these reflections, consider *two* homogeneous media, characterized by *different* refractive indices n_1 and n_2, joined at their $z = 0$ plane. We use the word *joined* rather than *separated*, because we have the paradigm of two homogeneous *interpenetrating* media, and postulate that quantities, ray distributions or wavefunctions, match at the reference plane. See the sketch of the two-dimensional analogue in Figure 11.

In geometric optics, let us indicate the coordinates of single rays in the two $\wp$'s by superindices $^{(1)}$ and $^{(2)}$. The *conservation* statements will be

$$\mathbf{q}^{(1)} = \mathbf{q}^{(2)} \in \Re^2, \qquad \delta_{n_1} \ni \mathbf{p}^{(1)} = \mathbf{p}^{(2)} \in \delta_{n_2}, \qquad \sigma^{(1)} = \sigma^{(2)}. \quad (15.2a, b, c)$$

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as part of the $\mathcal{H}^{\text{geom}}$ symmetry group.

or,

$$\wp^{(1)} \ni \{\mathbf{p}, \sigma; \mathbf{q}\}^{(1)} = \{\mathbf{p}, \sigma; \mathbf{q}\}^{(2)} \in \wp^{(2)} \quad \text{on the common domain.} \tag{15.2d}$$

The conservation of ray position $\mathbf{q}$ at the refracting surface, the equality (15.2*a*), is a generally implied but not at all irrelevant statement; it is on the same footing as Snell's law of refraction, equality (15.2*b*). In the common region $\delta_{n_1} \bigcap \delta_{n_2}$, the latter entails $n^{(1)} \sin\theta^{(1)} = n^{(2)} \sin\theta^{(2)}$ and $\phi^{(1)} = \phi^{(2)}$ in the usual spherical coordinates. Equality (15.2*c*), distinguishes between the two possibilities $\theta^{(2)}$ and $\pi - \theta^{(2)}$. An immediate consequence of (15.2*b*) is the relation between the z–components of $\vec{p}^{(1)}$ and $\vec{p}^{(2)}$:

$$n_1^2 - (p_z^{(1)})^2 = n_2^2 - (p_z^{(2)})^2. \tag{15.3}$$

The problem for the *global* joining between $\wp^{(1)}$ and $\wp^{(2)}$ is that in the region between the union and the intersection of the δ_n's, one of the p_z's must be imaginary.

A similar set of conservation statements may be made for Helmholtz two-functions in the form

$$\mathbf{F}^{(1)}(\mathbf{q}) = \mathbf{F}^{(2)}(\mathbf{q}), \text{ for all } \mathbf{q} \in \Re^2. \tag{15.4}$$

If we work with the plane waves in (10.8), in linear combination with their reflections through R $C_j^{\nearrow}\mathbf{W}_{\mathbf{p}^{(j)},\sigma^{(j)}}(\mathbf{q}) + C_j^{\nwarrow}\mathbf{W}_{\mathbf{p}^{(j)},-\sigma^{(j)}}(\mathbf{q})$, we find also Snell's law in the form (15.2*b*) and with the consequence (15.3). There are also two relations for the four linear combination coefficients, $C_1^{\nearrow} + C_1^{\nwarrow} = C_2^{\nearrow} + C_2^{\nwarrow}$ and $p_z^{(1)}(C_1^{\nearrow} + C_1^{\nwarrow}) = p_z^{(2)}(C_2^{\nearrow} + C_2^{\nwarrow})$. The assumption that a transmitted wave is not accompanied by its transmitted reflection provides the ratio of the coefficients of transmission and reflection. As is also well known, this exercise provides the interpretation of imaginary p_z's as *evanescent* waves, exponentially decreasing beyond the boundary where total internal reflection occurs [19]. Such solutions are outside the Hilbert space $\mathcal{H}_k$. Four-wave situations are of interest in the case of dynamic holograms, but this would take us beyond the intended scope of this monograph.

In Figures 12, 13, and 14 we present the three subcases of global refraction for $n_1 < n_2$, between air to the left and glass to the right, say. For $\sigma_1 = +1 = \sigma_2$ rays, we have the traditional rendering in Fig. 12 of refraction into denser media, in company with some reflection back into air that is indicated by dotted lines. In Fig. 13 we picture a $\sigma_1 = -1 = \sigma_2$ ray, for which the same configuration applies with $z = 0$ serving as interface for *right-to-left* glass-to-air refraction, entailing some reflection back into glass. Beyond $\sin\theta^{(2)} = n_1/n_2$, the conservation laws (15.2) *et seq.* imply total internal reflection, as for the ray in Fig. 14, with the presence of an evanescent wave of imaginary p_z lying on one of the branches of the equilateral hyperboloid extending beyond the inner sphere. The three processes occur in global refraction. To the right, corresponding Feynman diagrams

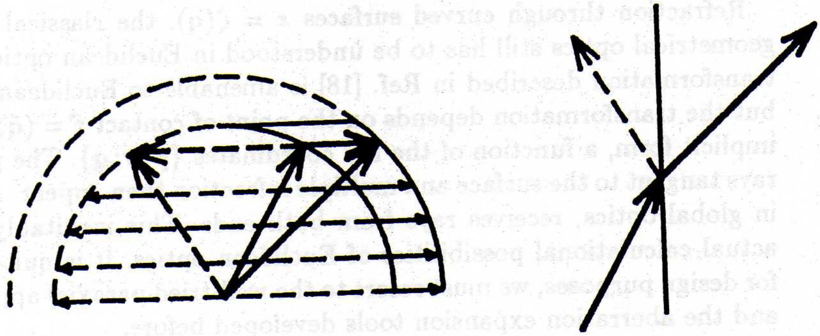

FIGURE 12. The diagram of Descartes for refraction from one lighter medium into one denser.

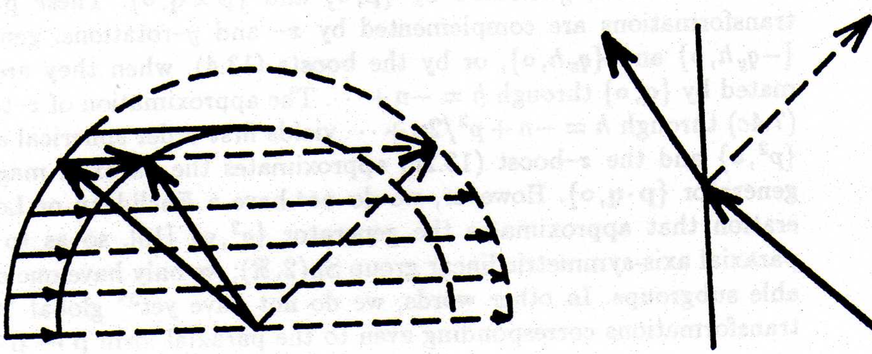

FIGURE 13. The two media with rays issuing from the right, out of the denser medium.

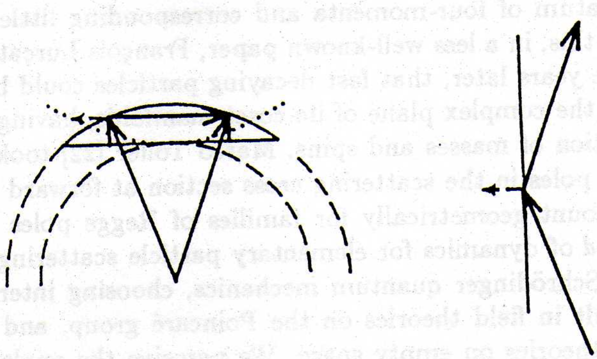

FIGURE 14. Internal reflection for rays comming from the denser medium, and the evanescent ray.

are drawn, borrowing the implications of a field theory on the coset spaces of the Euclidean group.

Refraction through curved surfaces $z = \zeta(\mathbf{q})$, the classical problem of geometrical optics still has to be understood in Euclidean optics. The *root* transformation described in Ref. [18] is amenable to Euclidean treatment, but the transformation depends on the point of contact $\vec{r} = (\bar{\mathbf{q}}, \zeta(\bar{\mathbf{q}}))$ in an implicit form, a function of the ray coordinates $\{\mathbf{p}, \sigma; \mathbf{q}\}$. The problems of rays tangent to the surface and multiple refraction then appear. A telescope, in global optics, receives rays from both ends. This inevitably limits the actual calculational possibilities of Euclidean optics. It is quite clear that for design purposes, we must resort to the well-tried paraxial approximation and the aberration expansion tools developed before.

We must thus point out the following avenue of the Euclidean theory to incorporate lenses, that can be best explained in geometric optical terms. Euclidean operations that are well defined both in Euclidean and in Heisenberg-Weyl paraxial optics [see (7.4)–(7.5)] are x– and y–translations, and z–rotations, generated by $\{\mathbf{p}, \circ\}$ and $\{\mathbf{p} \times \mathbf{q}, \circ\}$. These paraxial $\mathcal{E}_2$ transformations are complemented by x– and y–rotations, generated by $\{-q_y h, \circ\}$ and $\{q_x h, \circ\}$, or by the boosts (13.4), when they are approximated by $\{\mathbf{q}, \circ\}$ through $h = -n + \cdots$. The approximation of z–translation (7.4*c*) through $h = -n + p^2/2n + \cdots$ yields first-order spherical aberration $\{p^2, \circ\}$ and the z–boost (13.2*b*) approximates the paraxial magnification generator $\{\mathbf{p} \cdot \mathbf{q}, \circ\}$. However, we do *not* have a Euclidean or Lorentz operation that approximates the generator $\{q^2, \circ\}$ [18], so as to have the paraxial axis-symmetric linear group $\mathrm{Sp}(2, \Re)$; we only have one of its solvable subgroups. In other words, we do not have yet[39] global "thin-lens" transformations corresponding even to the paraxial form $\mathbf{p} \mapsto \mathbf{p} + \alpha \mathbf{q}$.

Slightly over fifty years ago, Eugene P. Wigner published his classical paper [20] classifying the irreducible representations of the Poincaré group; possible free particles in Nature were identified as their bases, for each stratum of four-momenta and corresponding little group spins. Building on this, in a less well-known paper, François Lurçat [21] proposed, twenty-five years later, that fast decaying particles could be represented by poles on the complex plane of its coset manifolds, having a Breit-Wigner distribution of masses and spins. Marco Toller [22] took this picture to search for poles in the scattering cross section at forward momentum transfer to account geometrically for families of Regge poles that should provide a *kind* of dynamics for elementary particle scattering. Now, unlike the case in Schrödinger quantum mechanics, choosing interactions is notably difficult in field theories on the Poincaré group, and they remain basically as theories on empty space. We perceive the analogy with the Euclidean

[39] See the chapter by V.I. Man'ko and K.B. Wolf in this Volume, that addresses this problem and gives a solution.(Note added in proof).

group, in itself much simpler than the Poincaré group, in that homogeneous space rays and waves, polarized or scalar, are well described. But otherwise *well-known elementary interactions such as refraction by a plane, already* need their complex extension. One may hope that processes with a hidden symmetry, such as scattering by a refracting sphere, by a Maxwell fish-eye medium, say, or bound systems such as parabolic- or elliptic-profile fibers could be analyzed in terms analogous to hidden symmetry in quantum mechanics, keeping ray direction on its now complex Descartes sphere.

In Euclidean optics we *do* have transformations that globalize some of the second-order aberrations in the x and y–boosts, and for Seidel spherical aberration $\{(p^2)^2, \circ\}$ (in z–translations) and circular coma $\{p^2\mathbf{p}\cdot\mathbf{q}, \circ\}$ (in z–boosts). They have been constructed both in geometric and in wave optics, and generate an infinite dimensional solvable subgroup of all passive optical transformations. With this limitation and extent we answer the two questions posed in the first Section. Other issues that have been raised in the intervening material should be left for further development.

6.16 Acknowledgements

I would like to thank the participants of the Lie Methods in Optics II workshop for the generous hearing and comments on material from this contribution, when it was in embrionary form. In particular, I am indebted to Dr. R. Simon, who stayed on for a week at IIMAS–Cuernavaca, and provided the key remarks and silences for the structure to take form. Dr. Simon's visit to Mexico was made possible by the support from the **Third World Academy of Sciences** (Trieste) Fellowship program, and from the **Fondo de Fomento Educativo BCH** (México DF), as *Distinguished Visitor* of CIFMA AC. As I worked through the subject, I benefited from frequent consultations with Dr. François Leyvraz, Instituto de Física/Cuernavaca, that I wish to acknowledge.

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