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Nonlinear Differential Equations as Invariants under Group Action on Coset Bundles: Burgers and Korteweg–de Vries Equation Families*

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Given a group, its coset spaces provide all homogeneous spaces for its action. A subgroup chain allows for the construction of a bundle of sections over a coset space of independent variables, where the fiber coordinates are dependent variables and all their partial derivatives up to some order, (i.e., the kth order jet). In this coset bundle, group invariants take the form of differential equations. We present two families of group–subgroup chains, one leading to various tensor Burgers-type differential equations, and the other to Korteweg–de Vries equations with an nth space derivative. Maps of the Hopf–Cole type appear in both families as transformations which intertwine the original group action to a multiplier realization of a normally extended group, yielding a new differential equation with greater symmetry. © 1986 Academic Press, Inc.

1. INTRODUCTION

The Burgers equation [1; 2 Chap. 4]

$$-cu_{qq}+u_t+uu_q=0, \tag{1.1}$$

is known to be invariant under the five-parameter set of *similarity* transformations [3, 4]

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$$t \xrightarrow{g} \bar{t} = \frac{td-b}{a-ct}, \tag{1.2a}$$

$$q \xrightarrow{g} \bar{q} = qd + (cq+x)\frac{td-b}{a-ct} + y, \tag{1.2b}$$

$$u \xrightarrow{g} \bar{u} = u(a-ct) + cq + x, \tag{1.2c}$$

which constitute a local group $\mathscr{G}_2 := \mathrm{T}_2 \wedge \mathrm{SL}(2, \mathfrak{R})$. This is the two-dimensional abelian (translation) group T_2 with parameters $\mathbf{V} := (x, y)$ in semidirect product with the group $\mathrm{SL}(2, \mathfrak{R})$ of unimodular 2×2 real matrices parametrized through $\mathbf{M} := (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$, $ad - bc = 1$; T_2 is normal in $\mathscr{G}_2$. The product law for $g_1, g_2 \in \mathscr{G}_2$, parametrized as $g := \{\mathbf{M}, \mathbf{V}\}$, is

$$\{\mathbf{M}_1, \mathbf{V}_1\}\{\mathbf{M}_2, \mathbf{V}_2\} = \{\mathbf{M}_1\mathbf{M}_2, \mathbf{V}_1\mathbf{M}_2 + \mathbf{V}_2\}. \tag{1.3}$$

The action (1.2) of $\mathscr{G}_2$ on the three-dimensional manifold $\mathfrak{R}^3$ with coordinates (u, q, t) is transitive and effective, and thus [5, 6] can be identified with the right (or left) action of this group on a coset space $\mathscr{C}_0$ by some subgroup $\mathscr{H}_0$. In [7], it was proposed to consider the subgroup $\mathscr{H}_0$ of elements

$$h = \left\{\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, (0, 0)\right\} \in \mathscr{H}_0, \tag{1.4a}$$

and to introduce coordinates on the coset space through representatives

$$c = c(u, q, t) = \left\{\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, (u, q - ut)\right\} \in \mathscr{C}_0 = \mathscr{H}_0 \backslash \mathscr{G}_2. \tag{1.4b}$$

Since any element $g \in \mathscr{G}_2$ may be decomposed as $g = hc$ uniquely, the right action of $g' \in \mathscr{G}_2$ on $\mathscr{C}_0$ may be found through performing

$$hc(u, q, t) \xrightarrow{g'} hc(u, q, t)\, g' = h'' c(\bar{u}, \bar{q}, \bar{t}). \tag{1.5}$$

This coincides with (1.2), as may be verified through elementary algebra. The Burgers equation (BE) may be seen thus group-theoretically as an invariant built out of coordinates of the coset space $\mathscr{C}_0$.

In this paper we formalize and use this construction procedure for two partial differential equation families: tensorial versions of the BE and a family of equations which includes the Korteweg–de Vries (KdV) equation

$$-cu_{qqq} + u_t + uu_q = 0. \tag{1.6}$$

These two equation families are interesting because they possess analogues of the Hopf–Cole transformation, which we interpret as an intertwining map between the obtained realization of the symmetry (similarity) group and a multiplier representation of a normal extension of this group. It may be remarkable that the intertwining we obtain for (1.6) was used by Hirota [2, Sect. 17.2; 8] in his search for multisoliton solutions of the KdV equation.

The plan of the paper is the following: In Section 2 we set up a general framework for our construction in the language of coset bundles. This is not intended to develop a general theory for the subject, since there are still few compelling steps in the construction, but only to identify them in more precise terms than those outlined for the simple BE case above.

In Section 3 we propose *ab initio* a group $\mathscr{G}_N := \mathrm{T}_N \wedge \mathrm{SL}(N, \Re)$, and a chain of subgroups of it. The right action of $\mathscr{G}_N$ on a coset space $\mathscr{C}_1 = \mathscr{H}_1 \backslash \mathscr{G}_N$ and a 2-jet over it leads to various invariants on this coset bundle, among them tensor versions of parts of the BE. Only when certain symplectic subgroups of $\mathscr{G}_N$ are considered, however, do we obtain all terms present in a tensor generalization of (1.1). The task of Section 4 is to produce a multiplier realization of the above group action. This leads to the Hopf–Cole map of tensor diffusion equations in a group-theoretic context, plus a normal (central) extension of the symmetry group.

There is a wide freedom in choosing groups, coset spaces, and coordinates. Section 5 proceeds through an informed guess to produce the KdV equation (1.6) among a family of related equations. These have an nth order q-derivative in the first summand. The symmetry group in this case has the form $\mathscr{G}^{(n)} := \mathrm{T}_2 \wedge_n (\mathrm{T}_1 \otimes \mathrm{T}_1)$ (an n-dependent semidirect product). The analogue of the Hopf–Cole transformation leads in Section 6 to a multiplier representation of a normal extension of this group. The intertwining does not lead—predictably—to a linear differential equation (except for the $n=2$ Burgers case). Instead, we obtain another nonlinear but scale-invariant equation which for $n=3$ was used by Hirota in one of his celebrated papers on the KdV equation [8]. This equation exhibits a seven-parameter symmetry group.

Some concluding remarks are offered in Section 7 on the bundle-theoretic meaning of our Hopf–Cole-type map, and on the possibility of extending this construction to other nonlinear differential equations.

2. Coset Bundles

Lie groups have a natural action on their coset spaces through right (or left) multiplication [5, Chap. 4]. If $\mathscr{G}$ is a group and $\mathscr{G} \supset \mathscr{H}_1 \supset \mathscr{H}_0$ a chain of closed subgroups, we may construct the coset spaces $\mathscr{C}_0 = \mathscr{H}_0 \backslash \mathscr{G}$, $\mathscr{C}_1 =$

$\mathscr{H}_1\backslash\mathscr{G}$, and $\mathscr{C}_d=\mathscr{H}_0\backslash\mathscr{H}_1$. As manifolds, locally, $\mathscr{C}_0=\mathscr{C}_d\times\mathscr{C}_1$. (This is true globally in our cases.) If we have local systems of coordinates for these spaces, so that

$$c_1(x)\in\mathscr{C}_1,\ c_d(u)\in\mathscr{C}_d \qquad \text{and} \qquad c_0(u,x)\in\mathscr{C}_0,\ x\in\mathfrak{R}^n,\ u\in\mathfrak{R}^m,$$

the group action of $\mathscr{G}$ on $\mathscr{C}_0$ may be found from

$$\begin{aligned} h_0 c_0(u,x) \xrightarrow{g} h_0 c_d(u)\, c_1(x)\, g \\ &= h_0 c_d(u)\, h_1(x;g)\, c_1(\bar{x}(x;g)) \\ &= h_0' c_0(\bar{u}(u,x;g), \bar{x}(x;g)). \end{aligned} \tag{2.1a}$$

Thus as in (1.2)—where x is (q,t)—the transformation nests as

$$x \xrightarrow{g} \bar{x}=\bar{x}(x;g), \qquad u\xrightarrow{g} \bar{u}=\bar{u}(u,x;g). \tag{2.1b}$$

We shall treat x as the *independent* variables—coordinates of $\mathscr{C}_1$, and u as the *dependent* variables.

A function $u(x)$ is defined by a section in the bundle $\mathscr{C}_0$ over $\mathscr{C}_1$, a surface of dimension n subject to the action of $\mathscr{G}$. Having this space of functions on $\mathscr{C}_1$ with given transformation properties, we can construct its kth-order jet bundle $\mathscr{Z}_k$ [9] and determine the action of the transformation group on it. The points $z\in\mathscr{Z}_k$ have a natural coordinate system

$$z=z(x, u(x), u_1(x),..., u_k(x)), \tag{2.2}$$

where $u_j(x)$ is the collection of all jth-order partial derivatives of the components of u with respect to the components of x.

The prolongation $\mathscr{T}_\mathscr{G}$ of the action of $\mathscr{G}$ to $\mathscr{Z}_k$ acts on $\mathscr{Z}_k$ transforming kth-order partial derivatives into up-to-kth-order partial derivatives. The action of $\mathscr{T}_\mathscr{G}$ is thus nested as in (2.1b) down to $\mathscr{Z}_0=\mathscr{C}_0$. We shall be interested in those (tensor-valued) functions Φ on $\mathscr{Z}_k$ which transform as

$$\Phi(z)\xrightarrow{g} \mathscr{T}_g\Phi(z)=\Phi(\bar{z})=\mu(z;g)\,\Phi(z), \tag{2.3a}$$

where μ is a *multiplier*, i.e., a tensor-valued function on $\mathscr{Z}_k\times\mathscr{G}$ which, due to the group property, satisfies

$$\mu(z; g_1 g_2)=\mu(z;g_1)\,\mu(\bar{z}(z;g_1);g_2), \qquad \mu(z;e)=\mathbf{1}. \tag{2.3b}$$

Functions Φ satisfying (2.3a)–(2.3b) will be called *invariants* on the coset bundle. The solution space of the differential equation $\Phi(z)=0$ is thus invariant in the usual sense [3] under $\mathscr{G}$, and $\mathscr{G}$ will be contained in its symmetry group. In fact, we aim to describe situations where $\mathscr{G}$ is the *maximal* symmetry group of the differential equation.

Suppose we find two invariants Φ^1 and Φ^2 which satisfy (2.3) with multipliers μ^1 and μ^2. Only when the latter are equal will also $c\Phi^1 + \Phi^2$ (c constant) be an invariant under $\mathscr{G}$. Alternatively, it may be that μ^1 and μ^2 coincide only over a subgroup $\mathscr{G}^0$ of $\mathscr{G}$, i.e., $\mu^1(z; g^0) = \mu^2(z; g^0)$ for $g^0 \in \mathscr{G}^0$ only. In that case we may replace $\mathscr{G}$ by $\mathscr{G}^0$ in all preceding statements, and consider anew the question of constructing a bundle $\mathscr{Z}_k^0$ out of the coset spaces $\mathscr{C}_1^0 = \mathscr{H}_1^0 \backslash \mathscr{G}^0$ and $\mathscr{C}_0^0 = \mathscr{H}_0^0 \backslash \mathscr{G}^0$, where $\mathscr{H}_1^0 = \mathscr{H}_1 \cap \mathscr{G}^0$ and $\mathscr{H}_0^0 = \mathscr{H}_0 \cap \mathscr{G}^0$. This reduction will apply for the tensor Burgers equations.

The above framework is sufficient to conceptualize the results of Sections 3 and 5, and those at the end of Sections 4 and 6. The connection between the two pairs, namely, the group-theoretic version of the Hopf–Cole map, will be developed heuristically in the latter Sections. We shall defer its formulation in bundle language to the concluding remarks.

3. The $\mathscr{G}_N$-, $\mathscr{G}_N^{\mathrm{Sp}}$-, and $\mathscr{G}_N^{\mathrm{Sch}}$-Tensor Burgers Equations

In the context of the preceding section we propose the following construction which will lead to tensorial extensions of the Burgers equation: let $\mathscr{G}_N := T_N \wedge \mathrm{SL}(N, \Re)$. Its elements will be parametrized as

$$g = \{\mathbf{M}, \mathbf{V}\}, \qquad \mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \qquad \det \mathbf{M} = 1, \qquad \mathbf{V} = (\mathbf{X}, \mathbf{Y}), \tag{3.1a}$$

where $\mathbf{A}$ is an $m \times m$ matrix, $\mathbf{D}$ is an $n \times n$ matrix, $\mathbf{B}$ and $\mathbf{C}$ are $m \times n$ and $n \times m$ matrices, $m + n = N$, while $\mathbf{X}$ and $\mathbf{Y}$ are m- and n-row vectors. The multiplication rule in $\mathscr{G}_N$ is the $N \times N$ generalization of (1.3), namely

$$\{\mathbf{M}_1, \mathbf{V}_1\}\{\mathbf{M}_2, \mathbf{V}_2\} = \{\mathbf{M}_1\mathbf{M}_2, \mathbf{V}_1\mathbf{M}_2 + \mathbf{V}_2\}. \tag{3.1b}$$

The group identity is $e = \{\mathbf{1}, \mathbf{0}\}$ and the inverse $\{\mathbf{M}, \mathbf{V}\}^{-1} = \{\mathbf{M}_{-1}, -\mathbf{V}\mathbf{M}^{-1}\}$.

We consider the subgroups

$$h_0 = \left\{\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, (\mathbf{0}, \mathbf{0})\right\} \in \mathscr{H}_0, \tag{3.2a}$$

$$h_1 = \left\{\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, (\mathbf{X}, \mathbf{0})\right\} \in \mathscr{H}_1, \tag{3.2b}$$

where $\det \mathbf{A} \neq 0$, and the corresponding coset spaces are parametrized through the representatives

$$c_0(\mathbf{U}, \mathbf{Q}, \mathbf{T}) := \left\{ \begin{pmatrix} \mathbf{1} & -\mathbf{T} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, (\mathbf{U}, \mathbf{Q} - \mathbf{U}\mathbf{T}) \right\} \in \mathscr{C}_0 = \mathscr{H}_0 \backslash \mathscr{G}_N, \tag{3.3a}$$

$$c_1(\mathbf{Q}, \mathbf{T}) := \left\{ \begin{pmatrix} \mathbf{1} & -\mathbf{T} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, (\mathbf{0}, \mathbf{Q}) \right\} \in \mathscr{C}_1 = \mathscr{H}_1 \backslash \mathscr{G}_N, \tag{3.3b}$$

$$c_d(\mathbf{U}) := \{\mathbf{1}, (\mathbf{U}, \mathbf{0})\} \in \mathscr{C}_d = \mathscr{H}_0 \backslash \mathscr{H}_1. \tag{3.4}$$

The group action from the right on $\mathscr{C}_0$ leads through (2.1) to the following transformation of the coordinates $\mathbf{U}$, $\mathbf{Q}$, and $\mathbf{T}$:

$$\mathbf{T} \xrightarrow{g} \bar{\mathbf{T}}(\mathbf{T}) = (\mathbf{A} - \mathbf{T}\mathbf{C})^{-1}(\mathbf{T}\mathbf{D} - \mathbf{B}), \tag{3.5a}$$

$$\mathbf{Q} \xrightarrow{g} \bar{\mathbf{Q}}(\mathbf{Q}, \mathbf{T}) = \mathbf{Q}\mathbf{D} + (\mathbf{Q}\mathbf{C} + \mathbf{X})(\mathbf{A} - \mathbf{T}\mathbf{C})^{-1}(\mathbf{T}\mathbf{D} - \mathbf{B}) + \mathbf{Y}, \tag{3.5b}$$

$$\mathbf{U} \xrightarrow{g} \bar{\mathbf{U}}(\mathbf{U}, \mathbf{Q}, \mathbf{T}) = \mathbf{U}(\mathbf{A} - \mathbf{T}\mathbf{C}) + \mathbf{Q}\mathbf{C} + \mathbf{X}. \tag{3.5c}$$

This can be seen to be the generalization of (1.2).

We denote

$$\mathbf{L} := (\mathbf{D} + \mathbf{C}\bar{\mathbf{T}})^{-1}, \qquad \mathbf{M} := \mathbf{A} - \mathbf{T}\mathbf{C}, \qquad \mathbf{S} := \mathbf{Q}\mathbf{C} + \mathbf{X}, \tag{3.6}$$

so that we can write

$$\begin{aligned} \frac{\partial Q_\mu}{\partial \bar{Q}_\nu} &= L_{\nu\mu}, & \frac{\partial T_{\alpha\mu}}{\partial \bar{Q}_\nu} &= 0, \\ \frac{\partial Q_\mu}{\partial \bar{T}_{\alpha\nu}} &= -S_\alpha L_{\nu\mu}, & \frac{\partial T_{\alpha\mu}}{\partial \bar{T}_{\beta\nu}} &= M_{\alpha\beta} L_{\nu\mu}. \end{aligned} \tag{3.7}$$

We reserve the first Greek letters $\alpha, \beta, \ldots,$ for indices in the range $\{1, 2, \ldots, m\}$, and the middle letters $\mu, \nu, \ldots,$ for those in the range $\{1, 2, \ldots, n\}$. The dependent variables $\mathbf{U} = U_\alpha$ form an m-vector under $\mathrm{SL}(N, \mathfrak{R})$, the dimensionality of the *space* variable $\mathbf{Q} = Q_\mu$ is n, and $\mathbf{T} = T_{\alpha\mu}$ is an $m \times n$ matrix. Although the interpretation of $\mathbf{T}$ as *time* is not a felicitous one, it provides one straightforward generalization of Burgers equation to tensor form. (Later we shall consider $\mathbf{T}$ to be a square symmetric matrix, and then finally allow $\mathbf{T} = t\mathbf{1}$.)

Through (3.6)–(3.7) we can calculate the action of $\mathscr{T}_{\mathscr{G}_N}$ on the fiber coordinates of the coset bundle $\mathscr{Z}_2$, viz.,

$$\bar{U}_\alpha = U_\delta M_{\delta\alpha} + S_\alpha, \tag{3.8a}$$

$$\frac{\partial \bar{U}_\alpha}{\partial \bar{Q}_\mu} = L_{\mu\nu}\left[\frac{\partial U_\delta}{\partial Q_\nu} M_{\delta\alpha} + C_{\nu\alpha}\right], \tag{3.8b}$$

$$\frac{\partial \bar{U}_\alpha}{\partial \bar{T}_{\beta\mu}} = L_{\mu\nu}\left[\left(M_{\gamma\beta}\frac{\partial U_\delta}{\partial T_{\gamma\nu}} - S_\beta \frac{\partial U_\delta}{\partial Q_\nu}\right) M_{\delta\alpha} - C_{\nu\alpha}(U_\delta M_{\delta\beta} + S_\beta)\right], \tag{3.8c}$$

$$\frac{\partial^2 \bar{U}_\alpha}{\partial \bar{Q}_\mu \, \partial \bar{Q}_\nu} = L_{\mu\rho} L_{\nu\sigma} \frac{\partial^2 U_\delta}{\partial Q_\rho \, \partial Q_\sigma} M_{\delta\alpha}, \tag{3.8d}$$

where repeated indices are summed. We omit the expressions for $\partial^2 \bar{U}_\alpha / \partial \bar{Q}_\mu \, \partial \bar{T}_{\beta\nu}$ and $\partial^2 \bar{U}_\alpha / \partial \bar{T}_{\beta\mu} \, \partial \bar{T}_{\gamma\nu}$.

We now consider the space of functions Φ on the bundle $\mathscr{Z}_2$, and search for invariants in the sense (2.3). In using the expressions (3.8), we may construct

$$\Phi^1_{\alpha\beta\mu}(z) := \frac{\partial U_\alpha}{\partial T_{\beta\mu}} + \frac{\partial U_\alpha}{\partial Q_\mu} U_\beta \xrightarrow{g} L_{\mu\rho} M_{\delta\beta} M_{\gamma\alpha} \Phi^1_{\gamma\delta\rho}(z). \tag{3.9}$$

(This is actually a function on the bundle $\mathscr{Z}_1$.) We also have

$$\Phi^2_{\alpha\nu\mu}(z) := \frac{\partial^2 U_\alpha}{\partial Q_\nu \, \partial Q_\mu} \xrightarrow{g} L_{\mu\rho} L_{\nu\sigma} M_{\gamma\alpha} \Phi^2_{\gamma\sigma\rho}(z). \tag{3.10}$$

Additionally, one may build further invariant functions through tensor coupling of the above, or consider $\partial \Phi^1_{\alpha\beta\mu} / \partial Q_\nu$, or functions thereof. A guide for the search can be conducted for $m = 1 = n$ dimension (where $L = M$), constructing monomials which transform with the same power of M; it is straightforward to see that the invariants presented above are the only ones in $\mathscr{Z}_2$.

It may be verified that $\mathscr{G}_N$ is *not* the maximal symmetry group of either the differential equation system $\Phi^1(z) = 0$, or $\Phi^2(z) = 0$. The former—in its Schrödinger version (see below)—has been analyzed by Rosen and Ullrich [10]; the latter is linear and its symmetry group includes conformal transformations in each coordinate, linear transformations in any plane, scaling, and the addition of any fixed solution of itself. Moreover, the two invariants (3.9)–(3.10) do not define an invariant plane of functions $-c\Phi^2 + \Phi^1$, since their multipliers are different. We expect a differential equation to be economically embedded in a group to have this group as its *maximal* symmetry group. Otherwise, loss of information or redundancy may be present.

Since $\mathscr{G}_N$ does not have a single, full $\mathscr{L}_2$-invariant (in place of two different ones on a sub- and quotient-bundle), we direct our search for subgroups of $\mathscr{G}_N$ where the multipliers in (3.9) and (3.10) are equal. Due to the dimensions involved, this requires that $m = N/2 = n$, so that N must be even, and all submatrices in (3.1a) and (3.6), square. Equality of the matrix multipliers needs $\mathbf{L} = \mathbf{M}^{\mathrm{T}}$ (transpose), for all values of $\mathbf{T}$. To zeroth order in $\mathbf{T}$ this implies that the submatrices in (3.1a) must satisfy $\mathbf{DA}^{\mathrm{T}} - \mathbf{CA}^{-1}\mathbf{BA}^{\mathrm{T}} = \mathbf{1}$. To first order in $\mathbf{T}$ we must satisfy $\mathbf{CA}^{-1}\mathbf{T} = (\mathbf{CA}^{-1})^{\mathrm{T}}\,\mathbf{T}^{\mathrm{T}}$, which is possible if $\mathbf{T}$ is symmetric, reducing thus the dimension of the coset space and implying in turn (for $\mathbf{T} = \mathbf{1}$) that $\mathbf{CA}^{-1}$ is symmetric. Since $\mathbf{T}$ must remain symmetric, $\mathbf{A}^{-1}\mathbf{B}$ is symmetric, and hence also $\mathbf{D}^{-1}\mathbf{B}$ and $\mathbf{D}^{-1}\mathbf{C}$. This set of $\mathscr{G}_N$ elements constitutes the *inhomogeneous real symplectic group* $\mathscr{G}_N^{\mathrm{Sp}} := \mathbf{T}_N \wedge \mathrm{Sp}(N, \Re)$ (with even N), with elements

$$g = \left\{ \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, (\mathbf{X}, \mathbf{Y}) \right\} = \{\mathbf{M}, \mathbf{V}\}, \tag{3.11a}$$

$$\begin{gathered} \mathbf{AB}^{\mathrm{T}} = \mathbf{BA}^{\mathrm{T}}, \qquad \mathbf{AC}^{\mathrm{T}} = \mathbf{CA}^{\mathrm{T}}, \qquad \mathbf{BD}^{\mathrm{T}} = \mathbf{DB}^{\mathrm{T}}, \\ \mathbf{CD}^{\mathrm{T}} = \mathbf{DC}^{\mathrm{T}}, \qquad \mathbf{AD}^{\mathrm{T}} - \mathbf{BC}^{\mathrm{T}} = \mathbf{1}; \end{gathered} \tag{3.11b}$$

$\mathscr{G}_N^{\mathrm{Sp}}$ is the semidirect product of T_N with the $\mathrm{Sp}(N, \Re)$ group of real matrices preserving the symplectic form

$$\mathbf{MKM}^{\mathrm{T}} = \mathbf{K}, \qquad \mathbf{K} := \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ +\mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{3.11c}$$

The subgroups $\mathscr{H}_0^{\mathrm{Sp}} \subset \mathscr{H}_1^{\mathrm{Sp}} \subset \mathscr{G}_N^{\mathrm{Sp}}$ determining the new coset spaces and bundles will be the intersection of the old ones with $\mathscr{G}_N^{\mathrm{Sp}}$. This restricts (3.2)–(3.3) to $\mathbf{D} = \mathbf{A}^{\mathrm{T}-1}$ and $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$. In all ensuing equations (3.6)–(3.7) it sets $\mathbf{L} = \mathbf{M}^{\mathrm{T}}$. This restriction in the coset space dimension makes the chain rules involving $T_{\mu\nu}$ read $\partial/\partial\bar{T}_{\alpha\beta} = \frac{1}{2}(\partial T_{\mu\nu}/\partial\bar{T}_{\alpha\beta})\,\partial/\partial T_{\mu\nu}$ (summing over all values of μ and ν). It sets thus a factor of $\frac{1}{2}$ on, and symmetrizes, the right-hand sides of Eqs. (3.8) for every $\partial/\partial\bar{T}_{\alpha\beta}$ on the left-hand sides. In other words, we apply the symmetrizer to every equation involving $T_{\alpha\beta}$-derivatives, in particular (3.9).

The maximal invariant differential equation obtained in $\mathscr{L}_2^{\mathrm{Sp}}$ is thus the *symplectic (tensor) Burgers* equation system:

$$-c\frac{\partial^2 U_\alpha}{\partial Q_\beta\,\partial Q_\gamma} + \frac{\partial U_\alpha}{\partial T_{\beta\gamma}} + \frac{1}{2}\left(U_\beta \frac{\partial U_\alpha}{\partial Q_\gamma} + U_\gamma \frac{\partial U_\alpha}{\partial Q_\beta} \right) = 0. \tag{3.12}$$

Each index ranges from 1 to $N/2$.For $N = 2$ we reproduce the common BE (1.1), in $\mathscr{G}_2 := T_2 \wedge \mathrm{Sp}(2, \Re) = T_2 \wedge \mathrm{SL}(2, \Re)$; this is an accidental isomorphism for $N = 2$.

The tensor generalization (3.12) of Burgers equation may be regarded as unnattractive due to the $T_{\beta\gamma}$-derivative, which stands for time in the one-dimensional original. In order to have a scalar time variable, we may further restrict $\mathscr{G}_N^{\mathrm{Sp}}$ to a subgroup obtained by reducing $\mathrm{Sp}(N, \Re) \supset \mathrm{SL}(2, \Re) \otimes \mathrm{SO}(N/2)$. This defines $\mathscr{G}_N^{\mathrm{Sch}} := T_N \wedge [\mathrm{SL}(2, \Re) \otimes \mathrm{SO}(N/2)]$. The group elements will then have the form

$$g = \left\{ \begin{pmatrix} a\mathbf{O} & b\mathbf{O} \\ c\mathbf{O} & d\mathbf{O} \end{pmatrix}, (\mathbf{X}, \mathbf{Y}) \right\} \in \mathscr{G}_N^{\mathrm{Sch}}, \qquad \mathbf{O}\mathbf{O}^{\mathrm{T}} = \mathbf{1}, \qquad ad - bc = 1. \tag{3.13}$$

The product law is (3.1b), common to all $\mathscr{G}_N$ subgroups. The subgroups $\mathscr{H}_0^{\mathrm{Sch}} = \mathscr{H}_0 \cap \mathscr{G}_N^{\mathrm{Sch}}$ and $\mathscr{H}_1^{\mathrm{Sch}} = \mathscr{H}_1 \cap \mathscr{G}_N^{\mathrm{Sch}}$ have the form (3.2) with $\mathbf{A} = a\mathbf{O}$, $\mathbf{C} = c\mathbf{O}$, and $\mathbf{D} = a^{-1}\mathbf{O}$. They define coset spaces $\mathscr{C}_0^{\mathrm{Sch}}$ and $\mathscr{C}_1^{\mathrm{Sch}}$ with replace $\mathbf{T}$ by $t\mathbf{1}$ in (3.3). The coordinates of the first coset space are $(\mathbf{U}, \mathbf{Q}, t)$, where $\mathbf{U}$ and $\mathbf{Q}$ are still $N/2$-dimensional vectors, but t is a scalar time parameter. These replacements persist in (3.5) and (3.6), and $\mathbf{L}^{\mathrm{T}} = \mathbf{M} = (a - tc)\,\mathbf{O}$. The coset bundle $\mathscr{X}_2^{\mathrm{Sch}}$ is built upon $\mathscr{C}_1^{\mathrm{Sch}}$ as before using the transformation properties of the dependent variable $\mathbf{U}$ stemming from $\mathscr{C}_0^{\mathrm{Sch}}$. The transformation of the coordinates in $\mathscr{X}_2^{\mathrm{Sch}}$ under the action of $\mathscr{G}_N^{\mathrm{Sch}}$ may be translated from (3.8), multiplying by $\delta_{\alpha\beta}$whenever $\partial/\partial \bar{T}_{\alpha\beta}$ or $\partial/\partial T_{\alpha\beta}$ appear, summing and formally identifying $\delta_{\alpha\beta}\partial/\partial\bar{T}_{\alpha\beta}$ with $\partial/\partial\bar{t}$ and $\delta_{\alpha\beta}\partial/\partial T_{\alpha\beta}$ with $\partial/\partial t$. This transformation holds as far as the invariants themselves in (3.9), (3.10), and the differential equation (3.12). The latter we may call the *Schrödinger* (*vector*) *Burgers* equation system:

$$-c\nabla^2\mathbf{U} + \partial_t\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U} = \mathbf{0}, \tag{3.14}$$

for $N/2$-dimensional $\mathbf{U}$ and $\mathbf{Q}$. Again, for $N = 2$ this reproduces the common Burgers equation.

4. Summators, Multipliers, and the Hopf–Cole Map

In this section we present the Hopf–Cole [11, 12] map from a group-theoretic point of view for the tensor BE. The basic result is its interpretation as an intertwining between the realization (3.5) of $\mathscr{G}_N^{\mathrm{Sp}}$ and a multiplier realization of a normal (here *central*) extension of this group.

Equation (3.8a) for the transformation of the section $\mathbf{U}(x)$ (where x stands for $(\mathbf{Q}, \mathbf{T})$) has the form

$$\mathbf{U}(x) \xrightarrow{g} \bar{\mathbf{U}}(\bar{x}) = \mathbf{U}(x)\,\mathbf{M}(x; g) + \mathbf{S}(x; g). \tag{4.1}$$

This we shall call a *summator realization* of a group on the section space. Here $\mathbf{M}$ and $\mathbf{S}$ are given by (3.6). The composition property of the group leads to the well-known composition properties of the (right) multiplier factor

$$\mathbf{M}(x;\, g_1 g_2) = \mathbf{M}(x;\, g_1)\, \mathbf{M}(\bar{x}(x;\, g_1);\, g_2), \qquad \mathbf{M}(x;\, e) = \mathbf{1}. \tag{4.2a}$$

For the *summator* term, it implies

$$\mathbf{S}(x;\, g_1 g_2) = \mathbf{S}(x;\, g_1)\, \mathbf{M}(\bar{x}(x;\, g_1);\, g_2), \qquad \mathbf{S}(x,\, e) = \mathbf{0}. \tag{4.2b}$$

Any multiple of a summator term is a summator, in the same way as any power of a scalar multiplier factor is a multiplier. Due to the presence of the summator term, the space of sections $\mathbf{U}(x)$ does not here constitute a linear vector space for $\mathscr{G}_N$ action. The latter is a necessary (but not sufficient) condition for $\mathbf{U}(x)$ to be the solution of a linear differential equation satisfying (2.3).

Turning a summator into a multiplier may be achieved, *prima facie*, through taking logarithms. This is not directly possible since $\mathbf{U}$—besides being an n-vector—is multiplied under $\mathscr{G}_N$ transformations by $\mathbf{M}(x;\, g) = \mathbf{A} - \mathbf{T}\mathbf{C}$. Under the $\mathscr{G}_N^{\mathrm{Sp}}$ subgroup, where $\mathbf{L} = \mathbf{M}^{\mathrm{T}}$ for (3.6), this factor coincides with the space transformation Jacobian (3.7a), namely

$$\frac{\partial Q_\beta}{\partial \bar{Q}_\alpha} = [(\mathbf{D} + \mathbf{C}\bar{\mathbf{T}})^{-1}]_{\alpha\beta} = (\mathbf{A} - \mathbf{T}\mathbf{C})_{\beta\alpha} = [\mathbf{M}(x;\, g)]_{\beta\alpha}. \tag{4.3}$$

If we express $\mathbf{U}$ as the space gradient of a scalar function, the latter will in general transform according to a summator realization of $\mathscr{G}_N^{\mathrm{Sp}}$ with a unit multiplier factor. Summator terms $\mathbf{S}(x;\, g)$, moreover, transform under $g' \in \mathscr{G}_N^{\mathrm{Sp}}$ with the same multiplier factor $\mathbf{M}(\bar{x}(x;\, g);\, g')$ as $\mathbf{U}(x)$, but in the g-transformed independent variables $\bar{x}(x;\, g)$ [compare (4.1) and (4.2b)]. We thus propose

$$\mathrm{U}_\alpha(\mathbf{Q}, \mathbf{T}) = K \frac{\partial}{\partial Q_\alpha} \ln \phi(\mathbf{Q}, \mathbf{T}), \tag{4.4a}$$

$$\mathrm{S}_\alpha(\mathbf{Q}, \mathbf{T};\, g) = K' \frac{\partial}{\partial \bar{Q}_\alpha} \ln \sigma(\mathbf{Q}, \mathbf{T};\, g), \tag{4.4b}$$

with as yet undetermined constants K and K'. In this way, as anticipated (4.1) becomes

$$\begin{aligned}\frac{\partial}{\partial Q_\alpha} \ln \phi(\mathbf{Q}, \mathbf{T}) &\xrightarrow{g} \frac{\partial}{\partial \bar{Q}_\alpha} \ln \bar{\phi}(\bar{\mathbf{Q}}, \bar{\mathbf{T}}) \\ &= \left[\frac{\partial}{\partial \bar{Q}_\beta} \ln \phi(\mathbf{Q}, \mathbf{T})\right] \frac{\partial Q_\beta}{\partial \bar{Q}_\alpha} + \frac{K'}{K} \frac{\partial}{\partial \bar{Q}_\alpha} \ln \sigma(\mathbf{Q}, \mathbf{T};\, g) \\ &= \frac{\partial}{\partial \bar{Q}_\alpha} \ln[\sigma(\mathbf{Q}, \mathbf{T};\, g)^{K'/K}\, \phi(\mathbf{Q}, \mathbf{T})].\end{aligned} \tag{4.5}$$

Under integration with respect to $\mathbf{Q} \to^{g} \bar{\mathbf{Q}}$, this leads to the multiplier group action

$$\phi(\mathbf{Q}, \mathbf{T}) \xrightarrow{g} \bar{\phi}(\bar{\mathbf{Q}}, \bar{\mathbf{T}}) = \mu(\mathbf{Q}, \mathbf{T}; g)\, \phi(\mathbf{Q}, \mathbf{T}), \tag{4.6a}$$

where the multiplier

$$\mu(\mathbf{Q}, \mathbf{T}; g) = \nu(\mathbf{T}; g)\, \sigma(\mathbf{Q}, \mathbf{T}; g)^{K'/K} \tag{4.6b}$$

stems from the summator (4.4b) which, upon integration, yields

$$\sigma(\mathbf{Q}, \mathbf{T}; g)^{K} = \exp[(\tfrac{1}{2}\mathbf{QC} + \mathbf{X})(\mathbf{A} - \mathbf{TC})^{-1}\, \mathbf{Q}^{\mathrm{T}}], \tag{4.6c}$$

times the integration "constant" $\nu(\mathbf{T}, g)$, which may be a function of g and $\mathbf{T}$ alone. This integration function is quite crucial, as it may allow or force us to centrally extend $\mathscr{G}_N^{\mathrm{Sp}}$ by a one-parameter group, trivially as a direct product, or nontrivially as a semidirect product. We thus obtain a *scale-invariant* (although not necessarily linear) differential equation for $\phi(\mathbf{Q}, \mathbf{T})$. This equation is obtained replacing (4.4a) into the symplectic tensor BE (3.12). We express it as a gradient:

$$\frac{\partial}{\partial Q_\alpha} \phi^{-2} \left\{ \phi \left(-c \frac{\partial^2 \phi}{\partial Q_\beta\, \partial Q_\gamma} + \frac{\partial \phi}{\partial T_{\beta\gamma}} \right) + \left(c + \frac{1}{2} K \right) \frac{\partial \phi}{\partial Q_\beta} \frac{\partial \phi}{\partial Q_\gamma} \right\} = 0. \tag{4.7}$$

The transformations $\phi \mapsto \phi \tau_{\beta\gamma}(\mathbf{T})$, where τ is a tensor-valued function of $\mathbf{T}$ stemming from the arbitrariness of the integration constant in (4.4) constitute an infinite abelian normal subgroup of symmetries of the above equation system. Integrating with respect to Q_α, the integration constants will be functions $F_{\beta\gamma}$ of $\mathbf{T}$ alone. The infinite part of the symmetry group will be now lost, but through $\mathbf{F}_{\beta\gamma}(\mathbf{T}) = -\partial \ln \tau_{\beta\gamma}(\mathbf{T})/\partial T_{\beta\gamma}$, we may set the integration constants to zero. We do retain, however, the *scaling* invariance $\phi \mapsto \phi e^{z}$ for $\tau_{\beta\gamma}$ constant. The resulting (in general nonlinear) differential equation is

$$\phi \left(-c \frac{\partial^2 \phi}{\partial Q_\beta\, \partial Q_\gamma} + \frac{\partial \phi}{\partial T_{\beta\gamma}} \right) + \left(c + \frac{1}{2} K \right) \frac{\partial \phi}{\partial Q_\beta} \frac{\partial \phi}{\partial Q_\gamma} = 0. \tag{4.8}$$

Clearly, for $K = -2c$ and $\phi \neq 0$ above, we obtain the *symplectic tensor diffusion equation*

$$c \frac{\partial^2 \phi}{\partial Q_\beta\, \partial Q_\gamma} = \frac{\partial \phi}{\partial T_{\beta\gamma}}. \tag{4.9}$$

The latter is a simple, linear equation whose solutions may be found in terms of initial conditions using the techniques of canonical transforms. See [13].

If we multiply $\{\mathbf{1}, (\mathbf{X}, \mathbf{0})\}$ and $\{\mathbf{1}, (\mathbf{0}, \mathbf{Y})\}$ in both orders as applied to (4.8) or (4.9), we see that the symmetry group of these equations involves the scaling-by-e^z subgroup in semidirect product, as a central extension of $\mathscr{G}_N^{\mathrm{Sp}}$. The symmetry group of these equations is thus $\tilde{\mathscr{G}}_N^{\mathrm{Sp}} := T_1 \wedge \mathscr{G}_N^{\mathrm{Sp}} = W_N \wedge \mathrm{Sp}(N, \mathfrak{R})$, where $W_N = T_1 \wedge T_N$ is the Heisenberg–Weyl group in $N/2$ dimensions. This group $\tilde{\mathscr{G}}_N^{\mathrm{Sp}}$ may be parametrized as

$$\tilde{g} = \{\mathbf{M}, \mathbf{V}, z\} = \{\mathbf{M}, \mathbf{V}, 0\}\{\mathbf{1}, \mathbf{0}, z\} \in \tilde{\mathscr{G}}_N^{\mathrm{Sp}} = T_1 \otimes \mathscr{G}_N^{\mathrm{Sp}}, \tag{4.10a}$$

with the product law

$$\{\mathbf{M}_1, \mathbf{V}_1, z_1\}\{\mathbf{M}_2, \mathbf{V}_2, z_2\} = \{\mathbf{M}, \mathbf{V}, z\}, \tag{4.10b}$$

$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2, \qquad \mathbf{V} = \mathbf{V}_1\mathbf{M}_2 + \mathbf{V}_2, \tag{4.10c}$$

$$z = z_1 + z_2 + \tfrac{1}{2}\mathbf{V}\mathbf{K}\mathbf{V}_2^{\mathrm{T}}, \tag{4.10d}$$

where as before $\mathbf{K}$ is the symplectic metric matrix in (3.11c). The primitive equation (4.7), as we saw, has for symmetry group $\tilde{\mathscr{G}}_N^{\mathrm{Sp}}$ times a normal abelian infinite-dimensional semidirect factor; we shall not be concerned with the latter, and neither with the similar extension of the symmetry group of (4.9) due to linearity. Rather, we note that it is the generic case (4.8) which is maximally invariant under $\tilde{\mathscr{G}}_N^{\mathrm{Sp}}$, whose central extension over $\mathscr{G}_N^{\mathrm{Sp}}$ is forced upon us by the Hopf–Cole map (4.4) and subsequent integration.

Using canonical transform techniques along the lines of [7, 13], we determine the integration factor $v(\mathbf{T}; g)$, whereupon we find that under $g \in \mathrm{Sp}(N, \mathfrak{R})$, the dependent variable transforms as

$$\begin{aligned}\phi \xrightarrow{g} \bar{\phi} = \phi[\det(\mathbf{A} - \mathbf{T}\mathbf{C})]^{-c/K} \\ \times \exp[K^{-1}\{(\tfrac{1}{2}\mathbf{Q}\mathbf{C} + \mathbf{X})(\mathbf{A} - \mathbf{T}\mathbf{C})^{-1}\mathbf{Q}^{\mathrm{T}} + \tfrac{1}{2}(\mathbf{X}\bar{\mathbf{T}} + \mathbf{Y})\,\mathbf{X}^{\mathrm{T}} + z\}].\end{aligned} \tag{4.11}$$

Together with (3.5a) and (3.5b), this constitutes the (*finite* part of the) group of symmetry transformations of the differential equation (4.8).

Again we have a nonlinear group action on a manifold which is transitive and effective (*viz.* Eqs. (3.5a, 3.5b), and (4.11)), and hence again we may set up our coset bundle construction using a new subgroup chain $\mathscr{H}_0 \subset \mathscr{H}_1 \subset \tilde{\mathscr{G}}_N^{\mathrm{Sp}}$. In a pattern which we shall make explicit in the concluding section, we note that only $T_1\backslash\tilde{\mathscr{G}}_N^{\mathrm{Sp}}$ is efective on the independent variables. Hence, the coset space may be chosen through

$$\tilde{h}_1 = \left\{\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{C} & \mathbf{A}^{\mathrm{T}-1}\end{pmatrix}, (\mathbf{X}, \mathbf{0}), z\right\} \in \mathscr{H}_1, \tag{4.12a}$$

$$\tilde{c}_1(\mathbf{Q}, \mathbf{T}) := \left\{\begin{pmatrix}\mathbf{1} & -\mathbf{T}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}, (\mathbf{0}, \mathbf{Q}), 0\right\} \in \mathscr{C}_1 = \mathscr{H}_1\backslash\tilde{\mathscr{G}}_N^{\mathrm{Sp}}. \tag{4.12b}$$

The dependent variable ϕ, however, is identified with the coset representative by a new subgroup:

$$\tilde{h}_0 = \left\{ \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{A}^{\mathrm{T}-1} \end{pmatrix}, (\mathbf{X}, \mathbf{0}), c \ln \det \mathbf{A} \right\} \in \tilde{\mathscr{H}}_0, \tag{4.13a}$$

$$\tilde{c}_0(\phi, \mathbf{Q}, \mathbf{T}) := \left\{ \begin{pmatrix} \mathbf{1} & -\mathbf{T} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, (\mathbf{0}, \mathbf{Q}), K \ln \phi \right\} \in \tilde{\mathscr{C}}_0 = \tilde{\mathscr{H}}_0 \backslash \tilde{\mathscr{G}}_N^{\mathrm{Sp}}, \tag{4.13b}$$

$$\tilde{c}_d(\phi) := \left\{ \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, (\mathbf{0}, \mathbf{0}), K \ln \phi \right\} \in \tilde{\mathscr{C}}_d = \tilde{\mathscr{H}}_0 \backslash \tilde{\mathscr{H}}_1. \tag{4.13c}$$

Repeating (2.1) for (4.12)–(4.13) we obtain (3.5a), (3.5b), and (4.11). The construction of a kth order jet bundle $\tilde{\mathscr{Z}}_k$ of sections over $\tilde{\mathscr{C}}_1$ proceeds as before. Upon finding the prolongation of the group on the fiber coordinates (analogues of (3.8)) we verify that (4.8) is an invariant in $\tilde{\mathscr{Z}}_2$ and (4.7) an invariant in $\tilde{\mathscr{Z}}_3$.

The Hopf–Cole map (4.4a) is thus a transformation from the bundle $\mathscr{Z}_2$ of the last section, onto *equivalence classes* of $\tilde{\mathscr{Z}}_3$: orbits in $\tilde{\mathscr{Z}}_3$ under the T_1 extension in $\tilde{\mathscr{G}}_N^{\mathrm{Sp}} = T_1 \wedge \mathscr{G}_N^{\mathrm{Sp}}$, such that the base spaces $\tilde{\mathscr{C}}_1$ and $\mathscr{C}_1$ are transformed in the same way. The map is defined essentially through the fiber coordinate relation (4.4a) prolonged in the natural way to higher derivatives. Although there seems to be no compelling restriction on the position of $\mathscr{H}_0$ in $\mathscr{G}_N^{\mathrm{Sp}}$, $\tilde{\mathscr{H}}_0$ must be such that $\tilde{\mathscr{G}}_N^{\mathrm{Sp}}$ act effectively on $\tilde{\mathscr{C}}_0$, so that the T_1 extension over $\mathscr{G}_N^{\mathrm{Sp}}$ have a nontrivial orbit.

We may repeat the above construction for the $\mathscr{G}_N^{\mathrm{Sch}}$ subgroup of $\mathscr{G}_N^{\mathrm{Sp}}$ given in (3.13), leading to the same Hopf–Cole transformation (4.4), which intertwines the summator realization of this group to a multiplier one of $\tilde{\mathscr{G}}_N^{\mathrm{Sch}} = T_1 \wedge \mathscr{G}_N^{\mathrm{Sch}}$. The Schrödinger vector Burgers equation (3.14) is thus transformed to the $N/2$-dimensional vector equation

$$\phi(-c\nabla^2\phi + \partial_t\phi) + (c + \tfrac{1}{2}K)\, \nabla\phi \cdot \nabla\phi = 0 \tag{4.14}$$

for scalar t. The conclusions on the symmetry group and a parallel coset bundle construction obviate any further remark on this case.

5. Coset Bundles for a Family of KdV-type Equations

In this section we consider the family of differential equations

$$-cu_{(n)q} + u_t + uu_q = 0, \tag{5.1}$$

where $u_{(n)q}$ denotes $\partial^n u/\partial q^n$ and c is a constant. This family includes the Burgers and KdV equations for $n = 2$ and $n = 3$, respectively. The similarity group for the latter is well known and suggests that we examine the four-parameter solvable groups $\mathcal{G}^{(n)}$ whose elements we denote

$$g=\{\mathbf{M},\mathbf{V}\}\in\mathcal{G}^{(n)},\qquad \mathbf{M}=\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix},\qquad \mathbf{V}=(x,y), \tag{5.2a}$$

with the product law

$$\{\mathbf{M}_1,\mathbf{V}_1\}\{\mathbf{M}_2,\mathbf{V}_2\}=\{\mathbf{M}_1\mathbf{M}_2,\, a_2^{1-2/n}\mathbf{V}_1\mathbf{M}_2+\mathbf{V}_2\}. \tag{5.2b}$$

In principle n could be any real number, but we shall use only natural n's. The structure of the group is

$$\mathcal{G}^{(n)}:=T_2^{(x,y)}\wedge_n(T_1^{(b)}\otimes T_1^{(a)})$$

with one n-dependent semidirect product, and where $T_1^{(s)}$ indicates that the subgroup parameter is s.

We now propose the subgroups $\mathcal{G}^{(n)}\supset\mathcal{H}_1^{(n)}\supset\mathcal{H}_0^{(n)}$ and coset space representatives

$$h_1=\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix},(x,0)\right\}\in\mathcal{H}_1^{(n)}, \tag{5.3a}$$

$$c_1(q,t):=\left\{\begin{pmatrix} 1 & -t \\ 0 & 1\end{pmatrix},(0,q)\right\}\in\mathcal{C}_1^{(n)}=\mathcal{H}_1^{(n)}\backslash\mathcal{G}^{(n)}, \tag{5.3b}$$

$$h_0=\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix},(0,0)\right\}\in\mathcal{H}_0^{(n)}, \tag{5.4a}$$

$$c_0(u,q,t):=\left\{\begin{pmatrix} 1 & -t \\ 0 & 1\end{pmatrix},(u,q-ut)\right\}\in\mathcal{C}_0^{(n)}=\mathcal{H}_0^{(n)}\backslash\mathcal{G}^{(n)}, \tag{5.4b}$$

$$c_d(u):=\{\mathbf{1},(u,0)\}\in\mathcal{C}_d=\mathcal{H}_0^{(n)}\backslash\mathcal{H}_1^{(n)}. \tag{5.4c}$$

(Compare with (3.2–3.3) and note that here also $c_0(u,q,t)=c_d(u)\,c_1(q,t)$.) The group action from the right on the $\mathcal{C}_0^{(n)}$ coset coordinates follows as in (2.1) leading to

$$t\xrightarrow{g}\bar{t}(t)=ta^{-2}-ba^{-1}, \tag{5.5a}$$

$$q\xrightarrow{g}\bar{q}(q,t)=qa^{-2/n}+txa^{-2}-xba^{-1}+y, \tag{5.5b}$$

$$u\xrightarrow{g}\bar{u}(u)=ua^{2-2/n}+x. \tag{5.5c}$$

(For $n=2$ compare with the BE case (3.5) with $N=2$, $\mathbf{C}=\mathbf{0}$.)

We now consider sections $u(q, t)$ in the bundle $\mathscr{C}_0^{(n)}$ over $\mathscr{C}_1^{(n)}$ and thereby build the nth jet bundle $\mathscr{Z}_n$ over $\mathscr{C}_1^{(n)}$ with fiber coordinates $\{u(q, t), u_1(q, t),..., u_k(q, t)\}$. The prolongation $\mathscr{T}_{\mathscr{G}^{(n)}}$ of $\mathscr{G}^{(n)}$ on $\mathscr{Z}_n$ can be obtained from (5.5c) using

$$\partial_{\bar{q}} = a^{2/n}\partial_q, \qquad \partial_{\bar{t}} = a^2\partial_t - xa^2\partial_q. \tag{5.6}$$

Invariant functions on $\mathscr{Z}_n$ are

$$\Phi^1(z) := u_t + uu_q, \qquad \Phi^2(z) := u_{(n)q}. \tag{5.7}$$

They have the same multiplier $\mu(z; g) = a^{4-2/n}$ (independent of z) and hence any linear combination $-c\Phi^2 + \Phi^1$ provides a differential equation (5.1) having $\mathscr{G}^{(n)}$ in its symmetry group. For $n = 2$ this is the Burgers equation whose full symmetry group is larger: $\mathscr{G}_2 \supset \mathscr{G}^{(2)}$. For $n = 3$ it is the KdV equation whose full symmetry group is precisely $\mathscr{G}^{(3)}$. This is true also for higher n's. The family of equations (5.1) is thus displayed as coset bundle invariants for the above group chains.

6. Symmetry Group Extension and Hopf–Cole Maps for KdV-type Equations

The summator-to-multiplier map for the KdV family of equations (5.1) can be obtained as in Section 4. The differences with the previous case are: (i) in (5.4) the multiplier $a^{2-2/n}$ and summator x do not depend on the coordinates (q, t), and (ii) the multiplier is not equal to the space transformation Jacobian, but to its $(n-1)$th power. The latter implies that the analogue of the Hopf–Cole map for this equation family is

$$u(q, t) = K\frac{\partial^{n-1}}{\partial q^{n-1}} \ln \phi. \tag{6.1}$$

Under this map, the group action (5.5c) becomes a multiplier realization. For $n = 2$ it is the ordinary Hopf–Cole map. For $n = 3$ (and $K = -12c$) it is the map used by Hirota [8] in his search for multisoliton solutions of the KdV equation. The general-n case leads to nonlinear and rather unwieldy equations for ϕ. We may, however, display the KdV ($n = 3$) case explicitly:

$$\begin{aligned}\partial_q \phi^{-2}\{&\phi(-c\phi_{qqq} + \phi_t)_q - \phi_q(-c\phi_{qqq} + \phi_t) - 3c(\phi_{qq}^2 - \phi_q\phi_{qqq}) \\ &+ [6c + \tfrac{1}{2}K][\phi_{qq}^2 + \phi^{-2}\phi_q^2(\phi_q^2 - 2\phi\phi_{qq})]\} = 0.\end{aligned} \tag{6.2}$$

Hirota's choice $K=-12c$ and q-integration with a null constant simplifies the expression to [2, Sect. 17.2; 8]

$$\phi(-c\phi_{qqq}+\phi_t)_q-\phi_q(-c\phi_{qqq}+\phi_t)-3c(\phi_{qq}^2-\phi_q\phi_{qqq})=0 \tag{6.3}$$

(usually written for $c=-1$). In spite of its complicated appearance, (6.2)—and (6.3) as well—have symmetry groups larger than $\mathscr{G}^{(3)}$. This is due to the two (in general $n-1$) integrations in the map (6.1). A similar situation was analyzed between (4.7) and (4.8), and lead to the extension of the symmetry group of the new over the original equation.

The analysis of the integration/extension process may be carried out as in Section 4. Here we prefer to follow this process on the Lie-algebraic level since it seems to yield a better insight into the structure of the extended group in the generic case. We thus introduce the generators of the Lie-algebra of $\mathscr{G}^{(n)}$ according to the parameters (5.2a). Using an obvious notation, these are

$$J_a=-2t\partial_t-(2/n)\,q\partial_q+(2-2/n)\,u\partial_u,\qquad J_b=-\partial_t,\qquad J_y=\partial_q, \tag{6.4a}$$

$$J_x=t\partial_q+\partial_u. \tag{6.4b}$$

The abelian normal subgroup $T_2^{(x,y)}$ corresponds to the subalgebra generated by J_x and J_y, which commute. The particular choice of null integration constants for (6.1) maps ∂_u to $[K(n-1)!]^{-1}\,q^{n-1}\phi\partial_\phi$, and $u\partial_u$ to nought. Hence, (6.4) are mapped to the set of operators

$$\tilde{J}_a=-2t\partial_t-(2/n)\,q\partial_q+v\phi\partial_\phi,\qquad \tilde{J}_b=-\partial_t,\qquad \tilde{J}_y=\partial_q, \tag{6.5a}$$

$$\tilde{J}_x=t\partial_q+[K(n-1)!]^{-1}\,q^{n-1}\phi\partial_\phi, \tag{6.5b}$$

respectively, where v is an arbitrary constant. The commutators of (6.5) follow those of (6.4), except for $[\tilde{J}_x,\tilde{J}_y]$ which is no longer zero, but $[K(n-2)!]^{-1}\,q^{n-2}\phi\partial_\phi$. If $n=2$ (the BE case), the latter generator is central and the group extension thereby complete. For $n\geqslant 3$, this is not the case: through repeated commutation with J_x, J_y, and $\tilde{J}_b$ we obtain $\frac{1}{2}n(n-1)$ extra operators

$$\tilde{J}_{kl}:=(Kl!)^{-1}\,t^kq^l\phi\partial_\phi,\qquad 0\leqslant k+l\leqslant n-2, \tag{6.5c}$$

which together with (6.5a) and (6.5b) constitute the Lie algebra of the extended symmetry group $\tilde{\mathscr{G}}^{(n)}\supset\mathscr{G}^{(n)}$ of the differential equation (6.2) satisfied by ϕ. The subset (6.5c) is a normal abelian subalgebra, and $\tilde{J}_{00}$ is the center for $\tilde{\mathscr{G}}^{(n)}$. The one-parameter central extension of Section 4 is thus specific to the $n=2$ BE case; the $n=3$ KdV case extends $\mathscr{G}^{(3)}$ to $\tilde{\mathscr{G}}^{(3)}$ by three parameters. Nonzero choices of the integration constants (functions $F(t)$) used in (6.4)–(6.5) would lead to different extended algebras involving operators $F(t)\, q^{n-k}\phi\partial_\phi$, $k=2,\ldots,n$ and their repeated commutators with (6.5a)–(6.5b). The present choice (6.5c) is minimal and keeps the dimension of the extension finite.

The action of the various one-parameter subgroups generated by each of the operators in (6.5) on the $\mathscr{C}_0$ bundle coordinates (ϕ, q, t) is as follows. For t and q it is the same as (5.5a)–(5.5b), the extension $\exp(\sum_{k,l}\zeta^{kl}\tilde{J}_{kl})$ being the stability group of every point; hence only $\mathscr{G}^{(n)}$ acts effectively on this submanifold. For the new bundle coordinate ϕ, $\exp(b\tilde{J}_b + y\tilde{J}_y)$ is the stability subgroup while the rest of the extended group elements act as

$$\left\{\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha}\end{pmatrix}, \mathbf{0}, [\mathbf{0}]\right\}\phi := \exp(\alpha\tilde{J}_a)\,\phi = e^{\alpha\nu}\phi, \tag{6.6a}$$

$$\{\mathbf{1}, (x, y), [\mathbf{0}]\}\,\phi := \exp(x\tilde{J}_x + y\tilde{J}_y)\,\phi = \phi\exp\left(\frac{x}{K}\sum_{m=1}^{n}\frac{q^{n-m}(xt+y)^{m-1}}{(n-m)!\,m!}\right). \tag{6.6b}$$

The latter expression yields the consistent integration constants of (6.1) under this two-parameter group. Finally,

$$\{\mathbf{1}, \mathbf{0}, [\{\zeta^{kl}\}]\}\,\phi := \exp\left(\sum_{k,l}\zeta^{kl}\tilde{J}_{kl}\right)\phi = \phi\exp\left(K^{-1}\sum_{k,l}\zeta^{kl}t^k q^l/l!\right). \tag{6.6c}$$

These equations thus provide a multiplier realization of $\tilde{\mathscr{G}}^{(n)}$ on the new $\mathscr{C}_0$ bundle coordinates. The prolongation $\mathscr{T}_{\mathscr{G}^{(n)}}$ on the bundle of sections $\mathscr{Z}_n$ is straightforward through the use of (5.6).

The fact that $\tilde{\mathscr{G}}^{(n)}$—or any other infinite normal extension of $\mathscr{G}^{(n)}$ obtained through various integration-constant functions of t alone as described above—is the full symmetry group of (6.2) is obvious by construction. For the q-integrated form of the same equation, in particular (6.3), this statement is true only for $\tilde{\mathscr{G}}^{(n)}$: any term $F(t)\,\phi^2$ replacing zero on the right-hand side, would not be invariant under the full group, but could be produced out of exponentiating the generators in other extensions.

As was done in the last section, the $\tilde{\mathscr{G}}^{(n)}$ group multiplication law extending (5.2) may be obtained explicitly. For this one may use the symbols in (6.6) plus various Baker–Campbell–Hausdorff relations to move normal subgroups through the semidirect products. For $n=2$ (the BE case) we obtain a subgroup of (4.13) for $N=2$. For the $n=3$ KdV case, the expression is still sufficiently compact to merit its explicit display. Let

$$\tilde{g} = \{\mathbf{M}, \mathbf{V}, [\zeta]\} = \{\mathbf{M}, \mathbf{V}, [0]\}\{\mathbf{1}, \mathbf{0}, [\zeta]\} \in \tilde{\mathscr{G}}^{(3)}, \tag{6.7a}$$

with $\mathbf{M}$ and $\mathbf{V}$ as in (5.1a), and $[\zeta] := [\zeta^{01}, \zeta^{10}, \zeta^{00}]$. Then, the product law appears as

$$\{\mathbf{M}_1, \mathbf{V}_1, [\alpha_1, \beta_1, \gamma_1]\}\{\mathbf{M}_2, \mathbf{V}_2, [\alpha_2, \beta_2, \gamma_2]\} = \{\mathbf{M}, \mathbf{V}, [\alpha, \beta, \gamma]\}, \tag{6.7b}$$

$$\mathbf{M} = \mathbf{M}_1 \mathbf{M}_2, \qquad \mathbf{V} = a_2^{1/3} \mathbf{V}_1 \mathbf{M}_2 + \mathbf{V}_2, \tag{6.7c}$$

$$\alpha = a_2^{-2/3} \alpha_1 + \alpha_2 + \tfrac{1}{2} \mathbf{V} \mathbf{K} \mathbf{V}_2^{\mathrm{T}}, \tag{6.7d}$$

$$\beta = a_2^{-2} \beta_1 + \beta_2 - a_2^{-2/3} x_2 \alpha_1 + \tfrac{1}{6}(x + x_2)\, \mathbf{V} \mathbf{K} \mathbf{V}_2^{\mathrm{T}}, \tag{6.7e}$$

$$\gamma = \gamma_1 + \gamma_2 - a_2^{-2/3} y_2 \alpha_1 + a_2^{-3} b_2 \beta_1 + \tfrac{1}{6}(y + y_2)\, \mathbf{V} \mathbf{K} \mathbf{V}_2^{\mathrm{T}}, \tag{6.7f}$$

where $\mathbf{K}$ is the 2×2 symplectic metric matrix in (3.11c).

The last part of Section 4 identified the dependent variable satisfying the tensor diffusion equation with a coset space coordinate represented by the center of the symmetry group $\mathscr{G}_N^{\mathrm{Sp}}$ (cf. Eq. (4.15c)). The same construction can be made with the solutions of Eq. (6.2) [and Hirota's equation (6.3) for $K = -12c$]. The subgroup chain $\tilde{\mathscr{G}}^{(3)} \supset \tilde{\mathscr{H}}_1^{(3)} \supset \tilde{\mathscr{H}}_0^{(3)}$ and the coset space parameters are the analogues of (5.3)–(5.4) for $\tilde{\mathscr{G}}^{(3)}$, namely,

$$\tilde{h}_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, (x, 0), [\alpha, \beta, \gamma] \right\} \in \tilde{\mathscr{H}}_1^{(3)}, \tag{6.8a}$$

$$\tilde{c}_1(q, t) := \left\{ \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, (0, q), [0, 0, 0] \right\} \in \tilde{\mathscr{C}}_1^{(3)} = \tilde{\mathscr{H}}_1^{(3)} \backslash \tilde{\mathscr{G}}^{(3)}, \tag{6.8b}$$

$$\tilde{h}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, (x, 0), [\alpha, \beta, -12vc \ln a] \right\} \in \tilde{\mathscr{H}}_0^{(3)}, \tag{6.9a}$$

$$\tilde{c}_d(\phi) := \{\mathbf{1}, \mathbf{0}, [0, 0, K \ln \phi]\} \in \tilde{\mathscr{C}}_d = \tilde{\mathscr{H}}_0^{(3)} \backslash \tilde{\mathscr{H}}_1^{(3)}, \tag{6.9b}$$

where the coset representatives of $\tilde{\mathscr{C}}_0^{(3)} = \tilde{\mathscr{H}}_0^{(3)} \backslash \tilde{\mathscr{G}}^{(3)}$ may be clearly built as $\tilde{c}_0(\phi, q, t) = \tilde{c}_d(\phi)\, \tilde{c}_1(q, t)$. The parameter v is the ϕ-scaling multiplier parameter for $\tilde{J}_a$ in (6.6a). The right action of the group $\tilde{\mathscr{G}}^{(3)}$ (Eqs. (6.7)) on the above coset space coordinates has thus the differential equations (6.2) and (6.3) for its coset bundle invariant. The latter is Hirota's equation when $K = -12c$.

7. Concluding Remarks

Whenever our starting point is a differential equation which possesses a symmetry group $\mathscr{G}$ larger than that of pure independent variable transformations, a subgroup chain may be found so that the differential equation becomes a coset bundle invariant, and the structure presented in Section 2 applies. This is due basically to the fact that the dependent and independent variables constitute a homogeneous space for the symmetry group, and all homogeneous spaces are coset spaces [5, 6]. Some ingenuity seems to be necessary, however, to find concrete coordinates in specific realizations. In Sections 3–6 we have given four families of examples of this (Eqs. (3.12)–(3.14), (4.8)–(4.9), (5.1), and (6.2)–(6.3)).

In the language of coset bundles, the generalized Hopf–Cole map appears to conform to the following pattern: (i) We produce an *extension* of the symmetry group $\mathscr{G}$ to a larger group $\tilde{\mathscr{G}} = \mathscr{E} \wedge \mathscr{G}$, $\mathscr{E}$ being normal in $\tilde{\mathscr{G}}$; in our examples $\mathscr{E}$ is, furthermore, abelian and possesses a one-dimensional center for $\tilde{\mathscr{G}}$. We assume a subgroup chain $\mathscr{H}_0 \subset \mathscr{H}_1 \subset \mathscr{G}$ has been found for the equation as described in Section 2. (ii) We then build $\tilde{\mathscr{H}}_1 = \mathscr{E} \wedge \mathscr{H}_1$, so that $\tilde{\mathscr{C}}_1 = \tilde{\mathscr{H}}_1 \backslash \tilde{\mathscr{G}}$—the space parametrized by the independent variables x—is a homogeneous space for $\tilde{\mathscr{G}}$ on which only the $\mathscr{G}$ subgroup of $\tilde{\mathscr{G}}$ acts effectively, and so that its action be identical to the original action of $\mathscr{G}$ on $\mathscr{C}_1 = \mathscr{H}_1 \backslash \mathscr{G}$. There does not appear to be any reason in principle for this to occur, but it is the case in the two families of examples treated here. Lastly (iii) a subgroup $\tilde{\mathscr{H}}_0 \subset \tilde{\mathscr{H}}_1$ must be found so that $\tilde{\mathscr{G}}$ can act effectively on $\tilde{\mathscr{C}}_0 = \tilde{\mathscr{H}}_0 \backslash \tilde{\mathscr{G}}$—the space parametrized by the dependent variables ϕ. If $\tilde{\mathscr{G}}$ has a center, this must be involved in the coset representative so that the action of $\tilde{\mathscr{G}}$ be effective; here it has been arranged so that it produces the subgroup of scalings.

These three steps refer to the coset spaces for the two groups. We assume we have constructed a coset bundle of sections $\mathscr{Z}_k$ for the first equation as in Section 2, and we do likewise for the second subgroup chain described above, to obtain a second bundle of sections $\tilde{\mathscr{Z}}_{k'}$ subject to the prolonged action of $\tilde{\mathscr{G}}$. We then search for invariants $\tilde{\Phi}(\tilde{z})$ on the $\tilde{\mathscr{Z}}_{k'}$ bundle $z\{x, \phi(x), \phi_1(x),..., \phi_{k'}(x)\}$. We may expect $k' \geqslant k$.

The differential equation $\tilde{\Phi}(\tilde{z}) = 0$ will have for symmetry group $\tilde{\mathscr{G}} \supset \mathscr{G}$, and a one-to-one mapping M should exist between the old bundle coordinates z and the orbits of $\tilde{z}$ under the extension subgroup $\mathscr{E} \subset \tilde{\mathscr{G}}$. If the independent variables x in $\mathscr{C}_1$ and $\tilde{\mathscr{C}}_1$ are the same, this mapping M will only involve nontrivially the fiber coordinates $\mathscr{C}_1 \backslash \mathscr{Z}_k$ and $\tilde{\mathscr{C}}_1 \backslash \tilde{\mathscr{Z}}_{k'}$. This framework would seem to include the possibility of very general Bäcklund-type transformations [14].

For the examples at hand, two simplifying features occur. First, the mapping M is of the form $u(x) = M(\tilde{z})$ prolonged to the rest of $\mathscr{Z}_k$ by differentiation, and so the extension subgroup $\mathscr{E}$ is recognizable as the kernel of M. Second, the invariant $\tilde{\Phi}$ may be written as a gradient: $\tilde{\Phi}(\tilde{\mathscr{Z}}_{k'}) = \Phi(M(\tilde{\mathscr{Z}}_{k'})) = \partial_x \Psi(\tilde{\mathscr{Z}}_{k'-1})$, and the differential equation $\Psi(\tilde{\mathscr{Z}}_{k'-1}) = 0$ still exhibits a symmetry group $\tilde{\mathscr{G}}$ with a finite-dimensional extension over $\mathscr{G}$.

Relating the solutions of a nonlinear differential equation to the solutions of a second such equation with a larger symmetry group seems to be conducive to certain valuable results. The Burgers-to-diffusion transformation through the Hopf–Cole map, KdV-to-primitive KdV, and modified KdV-to-KdV through the Miura map [15] (the latter not included in this paper: see [14] exemplify one-parameter extensions. The Hirota map for the KdV equation exemplifies a three-parameter extension. Beyond this, one should note the fruitful use of group representation and coupling theory on homogeneous spaces to expect that group invariants on coset bundles may provide additional insight into nonlinear differential equations.

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References

1. J. M. Burgers, A mathematical model illustrating the theory of turbulence, *Adv. in Appl. Mech.* **1** (1948), 171–199.
2. G. B. Whittham, "Linear and Nonlinear Waves," Wiley, New York, 1974.
3. G. W. Bluman and J. D. Cole, Similarity methods for differential equations, *Appl. Math. Sci.* **13** (1974).
4. G. W. Bluman, private communication.
5. A. O. Barut and R. Rączka, "Theory of Group Representations and Applications," Chap. 4, Theorem 1, Polish Sci., Warsaw, 1977.
6. S. Helgason, "Differential Geometry and Symmetric Spaces," Chap. II, Theorem 3.2, Academic Press, New York, 1962.
7. K. B. Wolf, Nonlinear group action and differential equations, presented at the "Topical Amer. Math. Soc. Meeting," Albuquerque, N.M., November 1976.
8. R. Hirota, Exact solutions of the Korteweg–de Vries equation for multiple collisions of solitons, *Phys. Rev. Lett.* **27** (1971), 1192–1194.
9. R. Hermann, "The Geometry of Non-Linear Differential Equations, Bäcklund Transformations, and Solitons, Part A," Interdisciplinary Math., Vol. XII, Math.-Sci., Brookline, Mass., 1976.
10. G. Rosen and G. W. Ullrich, Invariance group of the equation $\partial \mathbf{u}/\partial t = -\mathbf{u} \cdot \nabla \mathbf{u}$, *SIAM J. Appl. Math.* **24** (1973), 286–288.
11. E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.* **3** (1950), 201–230; J. D. Cole, On a quasilinear parabolic equation occurring in thermodynamics, *Quart. Appl. Math.* **9** (1951), 225–236.
13. S. Steinberg and K. B. Wolf, Groups of integral transforms generated by Lie algebras of second- and higher-order differential operators, *Nuovo Cimento A* **53** (1979), 149–177.
14. L. Hlavatý, K. B. Wolf, and S. Steinberg, Integral and Bäcklund transforms within symmetry groups of certain families of nonlinear differential equations, *J. Phys. A: Math. Gen.* **16** (1983), 2917–2922.
15. R. Miura, The Korteweg–de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation, *J. Math. Phys.* **9** (1968), 1202–1204.

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