J. Phys. A: Math. Gen. 16 (1983) 2917–2922. Printed in Great Britain

Integral and Bäcklund transforms within symmetry groups of certain families of nonlinear differential equations

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Received 4 February 1983

Abstract. We consider the symmetry groups for Burgers-type and KdV-type equations, as well as their intertwining under the Hopf–Cole and Miura transformations. It is shown that Lie point symmetries of one equation in the family may be mapped into integral or Bäcklund transformations of another equation in the family.

1. Introduction

Symmetry algebras and groups provide important information about differential equations, such as those associated with the names of Hopf and Cole (Hopf 1950, Cole 1951) and Miura (1968), are important tools for solution of the equations, especially when one of them is linear. In this paper we examine the interrelation between the two in certain families of equations. We obtain integral transforms which are elements of the symmetry group of nonlinear differential equations which do not belong by themselves to the set of Lie–Bäcklund transformations (Anderson and Ibragimov 1979).

Symmetry groups for differential equations have been defined (Bluman and Cole 1974, Ovsjanikov 1978) as Lie point symmetries, namely parametric functional transformations of the dependent and independent variables, which leave a given differential equation solution space invariant. The method of construction is straightforward and leads to first-order differential operators, which close under commutation into a Lie algebra; their exponentiation yields a Lie point transformation group. Anderson and Ibragimov (1979) extend this concept non-trivially to Lie–Bäcklund transformations: infinite-order Lie tangent transformations in the infinite-order jet space. It is a fact, however, that some possible symmetry transformations are still not accounted for, namely those transformations by integral operators. Steinberg and Wolf (1979) used various tactics to obtain finite-dimensional Lie algebras of second- and higher-order differential and integral operators, and their explicit exponentiation to groups of integral transformations, for various simple linear differential equations. In this paper, the point symmetry groups of some nonlinear equations are presented. Perhaps surprisingly, equations which are related by a transformation do not have the same

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symmetry group. The main problem we address here is explanation of the differences of symmetry groups. It will be shown that the 'disappearing' symmetries become integral- or even Bäcklund-type symmetries.

2. The Burgers-type equations

The Burgers equation (1948) is the classical example of a nonlinear but linearisable (Hopf 1950, Cole 1951) equation. By Burgers-type equations we mean the family of equations of the form

$$u_y - u_{xx} + V(u, u_x) = 0. \tag{2.1}$$

Specifying V, we obtain the following well known cases:

(i) HE, the heat equation: $V = 0$;
(ii) BE, the Burgers equation: $V = uu_x$;
(iii) PBE, the potential Burgers equation: $V = \frac{1}{2}(u_x)^2$.

The transformation from the PBE to the BE is mediated by differentiation with respect to x. If $w(x, y)$ is a solution of the PBE, then

$$z(x, y) = \partial w(x, y)/\partial x \tag{2.2}$$

is a solution of the BE. The (non-unique) inverse of (2.2) transforming solutions of the BE back to solutions of the PBE is determined by the Bäcklund transformation

$$w_x = z \qquad w_y = z_x - \tfrac{1}{2}z^2 \tag{2.3a, b}$$

or equivalently, by an integral transformation

$$w(x, y) = \int^x dx'\, z(x', y) + \delta \tag{2.4}$$

where δ is an arbitrary constant.

The PBE can be further transformed to the HE through a simple functional transformation of the dependent variables. If $w(x, y)$ is a solution of the PBE, then

$$h(x, y) = \exp(-\tfrac{1}{2}w(x, y)) \tag{2.5a}$$

is a non-negative solution of the HE. Its inverse is

$$w(x, y) = -2 \ln h(x, y). \tag{2.5b}$$

Composing (2.2) and (2.5b) we obtain the celebrated Hopf–Cole transformation

$$z(x, y) = -2(\partial/\partial x) \ln h(x, y) \tag{2.6a}$$

which transforms solutions of the HE to solutions of the BE. The inverse of this is obtained through the composition of (2.4) and (2.5a), and is

$$h(x, y) = \exp\left(-\tfrac{1}{2}\int^x dx'\, z(x', y) + \delta\right). \tag{2.6b}$$

3. Symmetries of the Burgers-type equations

Applying well known methods to investigate the so-called infinitesimal symmetry operators for a given equation (Bluman and Cole 1974, Ovsjanikov 1978), we find the symmetry algebra for the BE (see e.g. Ovsjanikov 1978) which is five dimensional while the symmetry algebras for the HE (Bluman and Cole 1974) and the PBE are infinite dimensional and isomorphic.

The symmetry group obtained by exponentiation of the symmetry algebra for the BE is

$$x \overset{g}{\mapsto} \bar{x} = xd + (cx + \zeta)(yd - b)/(a - cy) + \gamma \tag{3.1a}$$

$$y \overset{g}{\mapsto} \bar{y} = (yd - b)/(a - cy) \tag{3.1b}$$

$$z \overset{g}{\mapsto} \bar{z} = z(a - cy) + (cx + \zeta) \tag{3.1c}$$

where a, b, c, d, γ, $\zeta \in R$ and $ad - bc = 1$. It can be shown that this is a nonlinear representation of the goup ISL(2, R) that is a semidirect product of the simple real group SL(2, R) and the two-dimensional Abelian group.

A finite-dimensional part of the PBE symmetry group can be obtained from (3.1) using the Bäcklund transformation (2.3). The independent variables x, y transform equally as for the BE while for the dependent one we get

$$w \overset{g}{\mapsto} \bar{w} = w + \tfrac{1}{2}(a - cy)^{-1}[cx^2 + 2x\zeta + \zeta^2(yd - b)] - \ln(a - cy) + \delta \tag{3.2}$$

where a, b, ... are the same as in the previous case and δ is a real number. The transformations (3.1a), (3.1b) and (3.2) form a nonlinear representation of WSL(2, R): $= W_2 \otimes$ SL(2, R) (see e.g. Wolf 1979, p 420) where W_2 is the three-dimensional Heisenberg–Weyl group.

The transformation (3.2) can be mapped by (2.5) to the transformation of the dependent variable of the HE

$$h \overset{g}{\mapsto} \bar{h} = (a - cy)^{1/2} \exp\left(\frac{cx^2 + 2x\zeta + \zeta^2(yd - b)}{-4(a - cy)} - \tfrac{1}{2}\delta\right) h \tag{3.3}$$

which together with (3.1a) and (3.1b) form a finite-dimensional part of the HE symmetry group.

The infinite-dimensional part of the symmetry groups of the PBE and HE do not appear in this way, the reason being that the subgroup of point transformations belonging to the infinite-dimensional parts corresponds to integral transformations of the dependent variable in Burgers' equation.

Let us explain this feature in detail. The infinite-dimensional subgroups originate in the linear superpositions

$$h(x, y) \overset{g(H)}{\mapsto} \bar{h}(x, y) = h(x, y) + H(x, y) \tag{3.4}$$

where H, which is assumed to be a solution of the HE, plays the role of group parameter. The transformation of w derived from the transformation of h is

$$w(x, y) \overset{g(H)}{\mapsto} \bar{w}(x, y) = -2 \ln[\exp(-\tfrac{1}{2}w(x, y)) + H(x, y)], \tag{3.5}$$

The integral symmetry transformation of the corresponding BE solution, found from (3.5), (2.2) and (2.4), is

$$z(x,y) \overset{g(H,\delta)}{\mapsto} \bar{z}(x,y) = (\partial/\partial x)\bar{w}(x,y)$$
$$= \frac{\partial}{\partial x}\left\{-2\ln\left[\exp\left(-\tfrac{1}{2}\int^{x} dx'\, z(x',y)+\delta\right)+H(x,y)\right]\right\} \tag{3.6}$$

where δ is a real number and H is a non-negative solution of the HE. The transformations (3.6) form a group parametrised by δ and H. The generators of the symmetry transformations (3.6) are not first-order differential operators and therefore cannot be discovered by the usual methods for studying symmetries. The infinitesimal Lie point symmetries of the BE give no hint in this sense that the BE is linearisable, but for the PBE this is obvious. It means that even though the linearisation procedure is effected by (2.5), the important part of the Hopf–Cole transformation from the point of view of symmetry groups is the inverse (2.4) of the derivative transformation (2.2). It is this which transforms the BE into the PBE which has an infinite-dimensional symmetry group, the same as the linear heat equation.

The transformation (3.6) can be expressed as the integral superposition of solutions (cf Taflin 1981) of the BE: if z and Z are solutions of the BE then

$$\bar{z}(x,y) = -2\,\partial_x\left\{\ln\left[\exp\left(-\tfrac{1}{2}\int^{x} dx'\, z(x',y)+\delta\right)+\exp\left(-\tfrac{1}{2}\int^{x} dx'\, Z(x',y)\right)\right]\right\} \tag{3.7}$$

is also a solution of the BE.

4. KdV-type equations and their symmetries

Another type of equations whose symmetries we study for non-point transformations are the Korteweg–de Vries (KdV)

$$z_y + z_{xxx} + 6zz_x = 0 \tag{4.1}$$

and modified KdV (MKdV) equations

$$v_y + v_{xxx} - 6v^2 v_x = 0. \tag{4.2}$$

The solutions of the KdV and MKdV equations are related by the Miura transformation (Miura 1968)

$$z = \pm v_x - v^2. \tag{4.3}$$

The inverse of this, discovered by Lamb (1974), is the Bäcklund transformation

$$v_x = \pm(z+v^2) \qquad v_y = \mp z_{xx} - 2[vz_x \pm z(z+v^2)]. \tag{4.4}$$

The symmetry group of the KdV equation is four-dimensional and its action on the independent and dependent variables is

$$x \overset{g}{\mapsto} \bar{x} = \alpha x + 6\delta\alpha^3 y + \beta \tag{4.5a}$$

$$y \overset{g}{\mapsto} \bar{y} = \alpha^3 y + \gamma \qquad z \overset{g}{\mapsto} \bar{z} = \alpha^{-2} z + \delta \tag{4.5b, c}$$

where $\alpha, \beta, \gamma, \delta \in R$, $\alpha > 0$ are group parameters. The action of the three-dimensional symmetry group of the MKdV is

$$x \overset{g}{\mapsto} \bar{x} = \alpha x + \beta \qquad y \overset{g}{\mapsto} \bar{y} = \alpha^3 y + \gamma \qquad v \overset{g}{\mapsto} \bar{v} = \alpha^{-1} v. \tag{4.6a, b, c}$$

Now we investigate the correspondence between the actions (4.5) and (4.6) given by Miura–Bäcklund transformation (4.3) and (4.4). It is easy to show that the MKdV symmetry group is transformed into the subgroup of the KdV equation defined by $\delta = 0$. This is evident for the action on the independent variables, while for the dependent one, it follows from (4.6) and (4.3) that

$$z = \pm v_x - v^2 \overset{g}{\mapsto} \bar{z} = \pm \bar{v}_{\bar{x}} - \bar{v}^2 = \alpha^{-2}(\pm v_x - v^2) = \alpha^{-2} z. \tag{4.7}$$

In order to find which transformations of the MKdV solution correspond to the one-parameter group we use the Miura transformation (4.3) and its inverse (4.4), both of which hold in the transformed as well as in the untransformed variables. This implies

$$\bar{v}_{\bar{x}} = \bar{v}_x = \pm(\bar{z} + \bar{v}^2) = \pm(z + \delta + \bar{v}^2) = v_x \pm (\bar{v}^2 - v^2 + \delta) \tag{4.8a}$$

$$\begin{aligned} \bar{v}_{\bar{y}} &= \mp \bar{z}_{\bar{x}\bar{x}} - 2(\bar{v}\bar{z}_{\bar{x}} + \bar{z}\bar{v}_{\bar{x}}) \\ &= v_y - 2(\bar{v} - v)(\pm v_{xx} - 2v_x v) \pm 2(\bar{v}^2 - v^2 + \delta)(v^2 \mp v_x - \delta) - 2\delta v_x. \end{aligned} \tag{4.8b}$$

These equations together with

$$\bar{x} = x + 6\delta y \qquad \bar{y} = y \tag{4.8c}$$

define a generalised Bäcklund symmetry of the MKdV equation which has appeared as a consequence of the Lie point symmetry of the KdV equation. It is interesting that the group parameter δ has become the parameter of the Bäcklund transformation.

5. Concluding remarks

This paper has in common with Steinberg and Wolf (1979) the quest to provide examples of 'higher' symmetries, meaning groups of transformations in the solution space, of certain differential equations (here nonlinear ones) which include the usual Lie point symmetries as a proper subgroup. The new symmetries are integral or Bäcklund transformations.

Since linear differential equations always possess an infinite-dimensional normal symmetry subgroup due to the linear superposition of solutions, this infinite-dimensional part should appear in the symmetry group of nonlinear but linearisable equations. This proves to be true for the Burgers (and can be shown also for the Liouville) equation, but since the corresponding infinite-dimensional part contains only integral transformations, these are not easy to detect.

The case of the KdV-type equation shows that Bäcklund transformations may be obtained as a consequence of point symmetry groups.

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