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Holographic information in the Wigner function

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Abstract

The Wigner function of a Schrödinger-cat state f_0+f_1 is the Wigner function of f_0 plus that of f_1, plus a strongly oscillating cross-term, called the *smile function* of the cat state. We show that the marginal projection of the smile yields the transmissivity of the physical hologram of the object beam f_1 with the reference beam f_0. We review properties of linear covariance for possible applications in computer hologram interpretation or design. © 1997 Elsevier Science B.V.

1. Introduction

The Wigner function was originally applied for the understanding of quantum corrections in thermodynamic equilibrium [1]. Since then it has been used extensively in quantum mechanics because it provides a cogent interpretation of states in phase space [2]. Applications of the Wigner function to signal analysis in the space [3,4] or time domains [5] are relatively recent and new interpretations may be fruitful. In this Letter we point out that the strong oscillations of the Wigner function between two interfering signals, called the *smile function* of a Schrödinger cat state, bears interesting holographic information. We first define the terms used, then particularize one of the signals to be a plane reference beam; then we indicate that linear [6] and in particular the fractional Fourier transformations [7] of the reference beam produce covariant transformations in the hologram.

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2. Wigner function

The model of paraxial monochromatic wave optics used for one-dimensional signal analysis defines the Wigner function (for fixed wavelength $\lambda \neq 0$) as (see Ref. [8]) a sesquilinear functional of two (Lebesgue square-integrable) signals $f_0(q)$ and $f_1(q)$, $q \in \Re$ (the real line), and a function of the phase space coordinates of position and momentum, $(q,p) \in \Re^2$,

$$W(f_0,f_1|q,p) = \frac{1}{\lambda}\int_{\Re} \mathrm{d}x f_0(q-\tfrac{1}{2}x)^* \times \mathrm{e}^{-2\pi i p x/\lambda} f_1(q+\tfrac{1}{2}x). \qquad (1)$$

In the paraxial regime, the optical momentum (also called space frequency of the signal) can be identified (in a medium of unit refractive index) with the 'small' angle θ between the optical z-axis and the ray vector. Most commonly, the Wigner function is used for $f_0=f_1$ and interpreted as the phase-space quasiprobability distribution of the system in that state; when $f_0 \neq f_1$, several names are found in the literature, such as cross-correlation and Woodward ambiguity functions, that apply to (1) or its Fourier transform [8].

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The Wigner function is not a classical probability function because it can take negative values in small areas of phase space. Lohmann [4] showed that in the process of its optical production, the 'true' Wigner function is convoluted in q and in p with Gaussians $G_w(q) = (\pi w)^{-1/4}\exp(-q^2/2w)$ whose width product is minimal (i.e., satisfy the Heisenberg relation with equality), so the 'measured' Wigner function is the intensity $|(G_w * f)(q,p)|^2$ on the screen. Indeed, the Moyal identity [8] between Wigner functions and $\mathfrak{R}$-inner products $(\,,\,)$ is

$$\int_{\mathfrak{R}^2} \mathrm{d}q\,\mathrm{d}p\, W(f_0,f_0'|q,p)\, W(f_1,f_1'|q,p) = (f_0,f_1')(f_1,f_0'). \tag{2a}$$

When $f_0 = f_0'$ and $f_1 = f_1'$ this is the positive function $|(f_0,f_1)|^2$. As particular cases, when f_0 is a Gaussian G_w

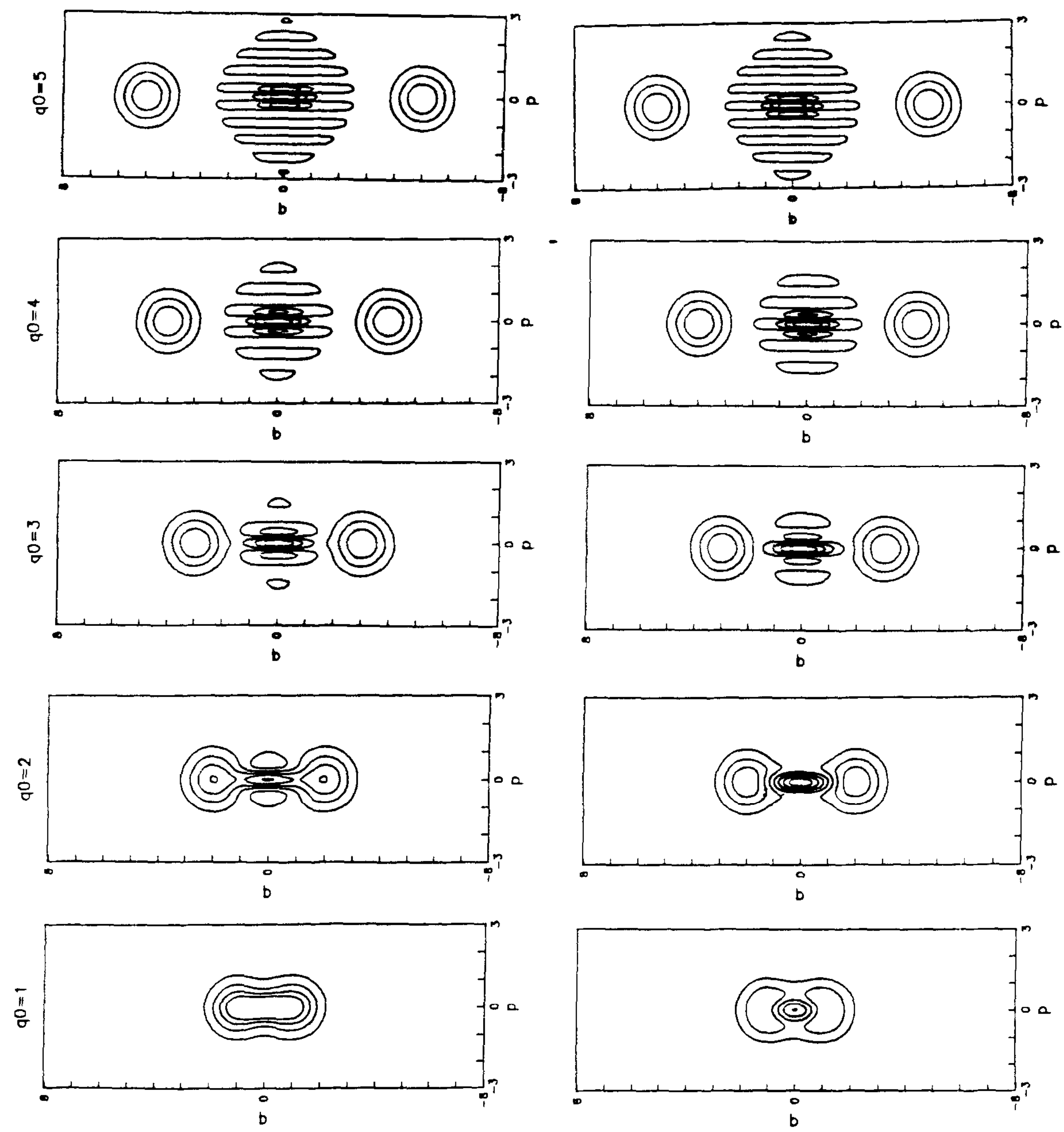

Fig. 1. Level curves of the Wigner function for a Schrödinger cat state composed of two Gaussian signals $G_w(q)$ of width $w = 1$ and peak separations of 1, 2, 3, 4 and 5 units. (a) Gaussian signals with the same sign, and (b) of opposite signs. Note that as the two Gaussians separate, the smile region remains localized around the midpoint, and increases its oscillation frequency 'across the teeth'.

and its width limits are zero or infinity, one obtains the q- or p-projections (marginal distributions) of (1),

$$\int_{\Re} \mathrm{d}p\, W(f_0,f_1|q,p) = f_0(q)^* f_1(q), \tag{2b}$$

$$\int_{\Re} \mathrm{d}q\, W(f_0,f_1|q,p) = \tilde{f}_0(p)^* \tilde{f}_1(p), \tag{2c}$$

where $\tilde{f}(p)$ is the Fourier transform

$$\tilde{f}(p) = \frac{1}{\lambda}\int_{\Re} \mathrm{d}q f(q)\, \mathrm{e}^{-2\pi i pq/\lambda},$$

$$f(q) = \int_{\Re} \mathrm{d}p \tilde{f}(p)\, \mathrm{e}^{2\pi i pq/\lambda}. \tag{2d}$$

3. Schrödinger cat states

The so-called Schrödinger cat paradox concerns classical intuition applied to a coherent superposition of a live and a dead cat, f_0 and f_1, given by their linear combination $f_0(q)+f_1(q)$. Because of sesquilinearity, the Wigner function (1) of such a state decomposes into three real terms,

$$W(f_0+f_1,f_0+f_1|q,p) = W(f_0,f_0|q,p) + W(f_1,f_1|q,p) + S(f_0,f_1|q,p), \tag{3}$$

where

$$S(f_0,f_1|q,p) = 2\,\mathrm{Re}\, W(f_0,f_1|q,p). \tag{4}$$

In Fig. 1, the three terms are readily recognizable: the standard interpretation of (3) identifies $W(f_0,f_0|q,p)$ as the phase space distribution of one constituent state f_0, and $W(f_1,f_1|q,p)$ as that of the other, f_1. The most prominent feature of the figure, however, is the cross term (4), which oscillates strongly, and has been called the *smile function* of the Schrödinger-cat state [9].

4. Holographic reconstruction

We recall that the transmissivity of a photographic plate exposed to the object beam $f_1(q,z)$ illuminated by a distinct but coherent reference beam $f_0(q,z)$ at the standard $z=0$ plane (screen) is

$$|f_0(q)+f_1(q)|^2 = |f_0(q)|^2 + |f_1(q)|^2 + 2\,\mathrm{Re}\, f_0(q)^* f_1(q), \tag{5}$$

and produces the (one-dimensional) hologram. The first two terms in the sum are usually assumed to vary slowly across the screen and result in a constant 'background' illumination on the plate. The third term in the sum may be negative in some intervals, but its sum with the other two is of course non-negative.

When we integrate Eq. (4) over p to find its projection over q, we introduce the hologram function

$$\begin{aligned} H(f_0,f_1|q) &= \int_{\Re} \mathrm{d}p\, S(f_0,f_1|q,p) \\ &= 2\,\mathrm{Re}\int_{\Re} \mathrm{d}p\, W(f_0,f_1|q,p) \\ &= 2\,\mathrm{Re}\, f_0(q)^* f_1(q). \end{aligned} \tag{6}$$

We note that this is exactly the interference term in (5). The projection of the smile function thus contains the properly holographic information of the beam f_1 under the reference beam f_0.

5. Plane waves

Consider first the case where the two constituent wavefunctions of the Schrödinger cat state are paraxial plane waves $E_p^\lambda(q,z) = \exp(2\pi i(pq-z)/\lambda)$ in the optical space $(q,z)\in\Re^2$, of fixed wavelength $\lambda \neq 0$ and momenta (inclination angles) $p=p_0$ and p_1. At the standard screen, each wavefunction is a 'plane-wave' signal

$$\varepsilon_p^\lambda(q) = E_p^\lambda(q,z)\,|_{z=0} = \exp(2\pi i pq/\lambda), \tag{7}$$

whose wavelength *on the screen* is λ/p. From (1) we find that the Wigner function of a plane wave (7) is

$$W\left(\varepsilon_{p_0}^\lambda,\varepsilon_{p_0}^\lambda|q,p\right) = \delta(p-p_0). \tag{8}$$

Its projection on q is unity.

The Schrödinger cat state of two plane wavefunctions, $E_{p_0}^\lambda(q,z) + E_{p_1}^\lambda(q,z)$ is shown in Fig. 2a; the $z=0$ plane (line) is the standard screen and holographic plate. The corresponding Wigner function of $\varepsilon_{p_0}^\lambda(q) + \varepsilon_{p_1}^\lambda(q)$ is shown in Fig. 2b: we see the two plane-waves of the cat state marked by the δ at $p=p_0$ and $p=p_1$ as razor edges; there is also the *smile* of the cat (4):

$$S\left(\varepsilon_{p_0}^\lambda,\varepsilon_{p_1}^\lambda|q,p\right) = 2\delta\left(p-\tfrac{1}{2}[p_0+p_1]\right) \times\cos(2\pi(p_0-p_1)q/\lambda). \tag{9}$$

This is a third razor edge between the two, modulated by a trigonometric function of wavelength $\Lambda = \lambda/(p_0-p_1)$. The q-projection of the Wigner function (3) of the cat state is 2 from the two plane waves, plus $2\cos[2\pi(p_0-p_1)q/\lambda]$

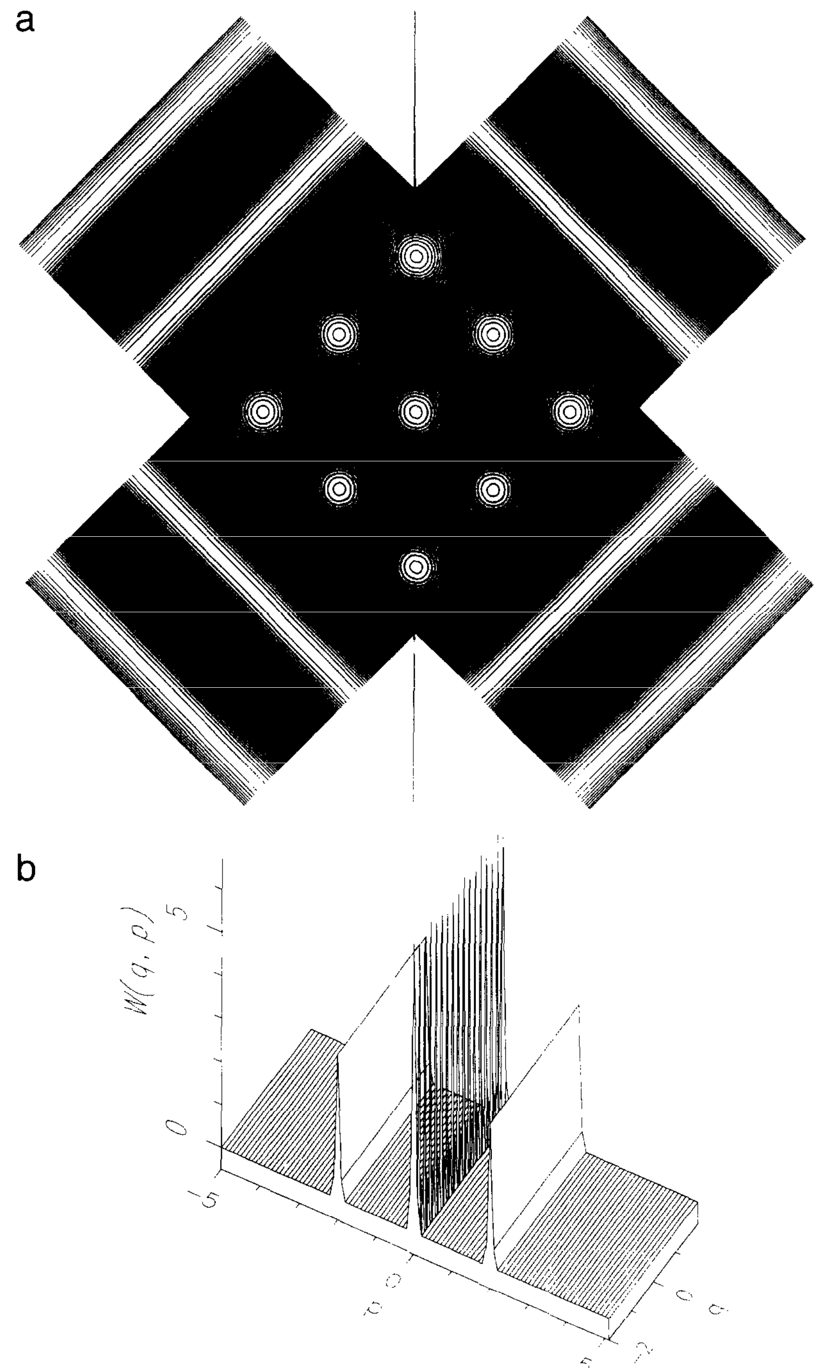

Fig. 2. (a) Two plane waves in optical space q–z and, (b) on phase space. (Actually, two narrow Gaussians $G_w(p-p_0)$ and $G_w(p+p_0)$ with $w=0.001$ and $p_0=2$.) Their smile function is the Moiré pattern and hologram at the screen.

from (9). The hologram function (6) is thus $H(\varepsilon^{\lambda}_{p_0},\varepsilon^{\lambda}_{p_1}|q) = 2[1+\cos(2\pi q/\Lambda)]$. (It ranges in [0,4]; proper normalization should bring it to the transmissivity function with range in [0,1].) Under the plane-wave reference beam $E^{\lambda}_{p_0}(q,z)$, the hologram will reproduce the object plane wave $E^{\lambda}_{p_1}(q,z)$ as a first-order beam. Seen as an operation between *signals*, this indicates that the smile function (9) defines a linear correspondence from $\varepsilon^{\lambda}_{p_0}(q)$ to $\varepsilon^{\lambda}_{p_1}(q)$.

6. Plane wave reference beams

Consider now Schrödinger cat states composed of a plane reference signal $\varepsilon^{\lambda}_{p_0}(q)$ plus a generic object beam $f(q)$ with Fourier transform $\tilde{f}(p)$ as given in (2d). The smile function of the Schrödinger cat state $\varepsilon^{\lambda}_{p_0}(q)+f(q)$ is

$$
\begin{aligned}
S\left(\varepsilon^{\lambda}_{p_0},f|q,p\right) &= 2\,\mathrm{Re}\,W\left(\varepsilon^{\lambda}_{p_0},f|q,p\right)\\
&= \frac{2}{\lambda}\,\mathrm{Re}\int_{\Re} \mathrm{d}x\,\mathrm{e}^{-2\pi i p_0(q-x/2)/\lambda}\\
&\quad \times \mathrm{e}^{-2\pi i p x/\lambda} f\left(q+\tfrac{1}{2}x\right)\\
&= 4\,\mathrm{Re}\,\mathrm{e}^{4\pi i(p-p_0)q/\lambda}\tilde{f}(2p-p_0), \qquad (11)
\end{aligned}
$$

cf. Eq. (2b). The q-projection of this smile function is the plane-wave hologram function

$$
\begin{aligned}
H\left(\varepsilon^{\lambda}_{p_0},f|q\right) &= \int_{\Re} \mathrm{d}p\,S\left(\varepsilon^{\lambda}_{p_0},f|q,p\right)\\
&= 2\,\mathrm{Re}\,\mathrm{e}^{-2\pi i p_0 q/\lambda} f(q). \qquad (12)
\end{aligned}
$$

When the signal f, although no longer a plane wave, still has a peak at momentum (space fequency) p_1 – this is indicated by f_1 as shown in Fig. 3 – then the smile function (11) will still exhibit a peak at the midpoint $\bar{p} = \frac{1}{2}(p_0+p_1)$, and can be written in the form

$$
\begin{aligned}
&S\left(\varepsilon^{\lambda}_{p_0},f_1|q,\bar{p}+\tfrac{1}{2}p'\right)\\
&\quad = 4\,\mathrm{Re}\,\mathrm{e}^{2\pi i(p'-[p_0-p_1])q/\lambda}\tilde{f}_1(p_1+p'). \qquad (13)
\end{aligned}
$$

At the peak $p=\bar{p}$ ($p'=0$), the smile function oscillates in q with the difference wavelength $\Lambda=\lambda/(p_0-p_1)$. This Moiré pattern has the width of the 'teeth' at the maximum of the smile function. The exponential factor in (11) can be set to 1 or $\pm i$ by choosing $q=0$ or $(p'/\lambda-1/\Lambda)q=\pm\frac{1}{4}$. See Fig. 4. Then the smile function provides the complex Fourier transform of the object signal along those lines:

$$
S\left(\varepsilon^{\lambda}_{p_0},f_1|0,\bar{p}+\tfrac{1}{2}p'\right) = 4\,\mathrm{Re}\,\tilde{f}_1(p_1+p'), \qquad (13a)
$$

$$
S\left(\varepsilon^{\lambda}_{p_0},f_1\left|\frac{\pm\lambda\Lambda}{4(\lambda-p'\Lambda)},\bar{p}+\tfrac{1}{2}p'\right.\right) = \pm 4\,\mathrm{Im}\,\tilde{f}_1(p_1+p'). \qquad (13b)
$$

If the smile function is known from Lohmann's Wigner function imager, Eqs. (13) yield the complex Fourier transform $\tilde{f}_1(p_1+p')$ of a Gaussian-convolved signal $f_1(q)=(G_w * f)(q)$; so the pure signal Fourier transform, $\tilde{f}(p) \sim G_{1/w}(ip)\tilde{f}_1(p)$, is the formal but simple multiplication of a growing Gaussian times the measured function $\tilde{f}_1$.

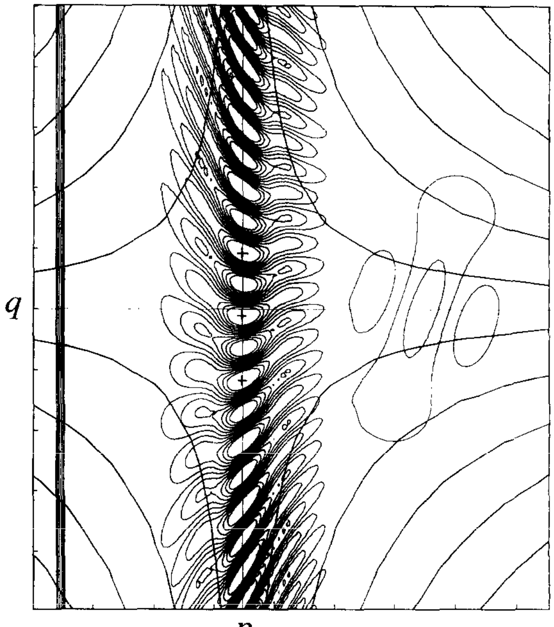

Fig. 4. The smile wavelength (width of the teeth) is Λ; the real and imaginary parts of the Fourier transform of the signal (or Gaussian-diffused measurement) are found at definite lines in phase space.

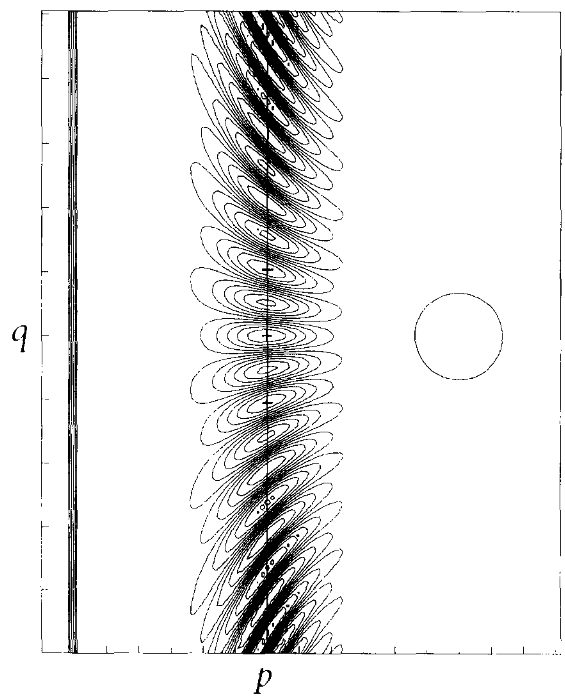

Fig. 3. Wigner function of a Schrödinger-cat state composed of a plane wave of momentum $p_0=-2$, and a signal with a peak at momentum $p_1=2$. The smile maximum is along the line $\bar{p}=\frac{1}{2}(p_0+p_1)=0$.

Whether this 'backward diffusion' process (with some cutoff) can be competitive with other Wigner function-restoring algorithms [10] or whether it will be swamped by noise cannot be answered yet.

7. Translation and linear canonical covariance

Signals $f(q)$ can be translated in phase space by means of operators in the Heisenberg-Weyl group. In what follows we consider the fixed wavelength $\lambda = 2\pi$ that we omit in the notation, and consider the operators $\exp(x\partial_q - iyq)$, $(x,y) \in \Re^2$, up to q-independent phases $e^{i\phi} = \exp[i\phi(x,y,\lambda)]$ whose precise form is unimportant here. These are the translations

$$f_T(q) = [\mathcal{T}(x,y):f](q) = e^{i\phi} e^{-iyq} f(q+x),$$
$$\tilde{f}_T(p) = e^{-i\phi} e^{ixp} \tilde{f}(p+y). \tag{14}$$

The Wigner function (1) is *covariant* under translations because it has the property

$$W(\mathcal{T}(x,y):f_0, \mathcal{T}(x,y):f_1|q,p)$$
$$= W(f_0,f_1|q+x,p+y). \tag{15}$$

For $f_0 = f_1$ this means that the Wigner function of a translated wavefunction is the translated Wigner function of the wavefunction.

Further, Moshinsky and García-Calderón [11] showed that the Wigner function (1) is also covariant under linear canonical transformations. These are integral transforms [6] in (2:1) correspondence with 2×2 matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$\det M = 1$, of the form

$$f_M(q) = \left[\mathcal{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right](q) = \int_{\Re} dq' \, C_M(q,q') \, f(q'), \tag{16a}$$

with integral kernel (for $b \neq 0$ and up to a q-independent phase $e^{i\phi}$)

$$C_M(q,q') = \frac{e^{i\phi}}{\sqrt{2\pi b}} \exp\left[i\left(\frac{a}{2b}q'^2 - \frac{1}{b}q'q + \frac{d}{2b}q^2\right)\right]. \tag{16b}$$

Paraxial imaging systems are represented by $b = 0$ matrices whose canonical transform collapses to

$$\left[\mathcal{C}\begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix} f\right](q) = e^{i\phi} a^{-1/2} e^{icq^2/2a} f(q/a). \tag{16c}$$

In quantum optics this is called a *squeezing* transformation, and is performed by relativistic means; in phase-space optics it is realized by a magnifying (reducing) lens system.

For linear canonical transformations, covariance holds as

$$W\left(\mathcal{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f_0, \mathcal{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f_1 | q,p\right)$$
$$= W(f_0,f_1|aq+bp,cq+dp). \tag{17}$$

For $f_0 = f_1$ this only says that the Wigner function of a wavefunction transformed by first-order optics is the Wigner function of the original wavefunction, only subject to the linear transformation of the underlying phase space. (This correspondence does not hold beyond linear transformations; for nonlinear ones see Ref. [12].) In particular, the fractional Fourier transform is

$$\mathcal{C}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix};$$

it brings about a rotation of phase space by θ around the origin; for $\theta = \frac{1}{2}\pi$ it corresponds, but for normalization, to (10).

8. Covariance of the smile function and hologram

When the signal is a Schrödinger-cat state composed of a reference and object beams, covariance relates the separate transformations of the two. The hologram function will follow only for point transformations $q'(q)$. From Eqs. (3), (4), (15) and (17), we find the relations

$$H(\mathcal{T}(x,y):f_0,f_1|q) = H(f_0,\mathcal{T}(-x,-y):f_1|q+x), \tag{18a}$$

$$H\left(\mathcal{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix}:f_0,f_1|q\right)$$
$$= \int_{\Re} dp \, S\left(f_0,\mathcal{C}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}:f_1|aq+bp,cq+dp\right), \tag{18b}$$

$$H\left(\mathcal{C}\begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix}:\varepsilon_{p_0},f_1|q\right)$$
$$= H\left(\varepsilon_{p_0},\mathcal{C}\begin{pmatrix} 1/a & 0 \\ -c & a \end{pmatrix}:f_1|aq\right). \tag{18c}$$

Eq. (18a) states that the hologram of a translated object with a similarly translated reference beam is simply the translation of the original the hologram, and that a hologram illuminated with a rotated reference beam will produce a similarly rotated object beam. Eq. (18b) tells us that the linear transformation of phase space makes the smile function of $\mathcal{C}(M^{-1})f_1$ with the reference beam f_0 coincident with the smile function of f_1 under the beam

$\mathscr{C}(M)f_0$, obtained by transforming f_0 with a linear system. When the reference beam is a plane wave

$$\varepsilon_{p_0}(q) = e^{-i\pi/4}\sqrt{2\pi}\,C\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(p_0,q)$$

of wavelength $\lambda = 2\pi$, and since canonical transforms compose as their matrices multiply [6], its

$$\mathscr{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

transform is

$$\left[\mathscr{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix}:\varepsilon_{p_0}\right](q)$$
$$= e^{-i\pi/4}\sqrt{2\pi}\,C\begin{pmatrix} b & -a \\ d & -c \end{pmatrix}(p_0,q)$$
$$= \frac{1}{\sqrt{a}}\exp i\left(-\frac{b}{2a}p_0^2 + \frac{1}{a}p_0 q - \frac{c}{2a}q^2\right). \qquad (19)$$

Its Wigner function is a Dirac razor-edge δ on the plane $(q,p) = (aq + bp_0, cq + dp_0)$, $q \in \Re$. In particular, the fractional Fourier transform rotates the phase space about the origin carrying with it the Wigner function plot of the states and reminding us of the Radon transform of a density function that reconstitutes the object from its shadows; in this case of the smile function, though. Eq. (18c) covers the limit case of squeezing transformations $q' = aq$ for $a > 0$ and typically $a = -1$ for imaging systems by one lens. Because in (7) and (19) the plane-wave exponent is really $2\pi ipq/\lambda$, (18c) also describes the change of wavelength from 2π to $\lambda = 2\pi a$. Whereas in quantum mechanics the scale between position and momentum coordinates is fixed by nature to the Planck constant $\hbar = h/2\pi$, in optics the reduced wavelength $\lambda/2\pi$ has no such restriction.

9. Concluding remarks

We bring attention to the fact that the strong oscillation region in the Wigner function of Schrödinger cat states bears information on the hologram between their constitutents. When the reference beam is a plane wave, its Wigner function is a razor edge δ in the phase plane and some particularly simple relations ensue between the object beam and its smile function. Covariance allows us to translate and tilt this razor edge, thereby obtaining holograms for linear canonical (paraxial optical) transforms of the object beam. These are relations which may be useful both for the interpretation or for the design of holograms.

We leave open the choice of regarding the signal or the hologram function as known by computation or by experiment, and the other to be found. Having a standard reference signal, the smile function of a Schrödinger-cat state presents holographic information of the object state. This is particularly transparent in the paraxial wave optical model; it may clarify similar strategies for close-to-Wigner functions obtained in the time domain [5] and the quantum optics of squeezed light [12].

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References

[1] E. Wigner, Phys. Rev. 40 (1932) 749; M. Hillery, R.F. O'Connell, M.O. Scully, E.P. Wigner, Phys. Rep. 106 (1984) 121.
[2] H.-W. Lee, Phys. Rep. 259 (1995) 147.
[3] M.J. Bastiaans, J. Opt. Soc. Am. 69 (1979) 1710.
[4] A. Lohmann, Optics Comm. 42 (1980) 32; H.O. Bartelt, K.-H. Brenner, H. Lohmann, Optics Comm. 32 (1980) 32; H. Bartelt, K.-H. Brenner, Israel J. Techn. 18 (1980) 260; K.-H. Brenner, H. Lohmann, Optics Comm. 42 (1982) 310.
[5] L. Cohen, Proc. IEEE 77 (1989) 941; K.W. DeLong, R. Trebino, D.N. Fittinghoff, C.L. Ladera, SPIE 2377 (1995) 44; J. Paye, A. Migus, J. Opt. Soc. Am. B (1996) .
[6] K.B. Wolf, Integral Transforms in Science and Engineering (Plenum Press, New York, 1979) Part IV.
[7] D. Mendlovic, H.M. Ozaktas, J. Opt. Soc. Am. A 10 (1993) 1875; A.W. Lohmann, J. Opt. Soc. Am. A 10 (1993) 2181; H.M. Ozaktas, D. Mendlovic, J. Opt. Soc. Am. A 10 (1993) 2521; D. Mendlovic, H.M. Ozaktas, A.W. Lohmann, Appl. Optics 33 (1994) 6188; D. Mendlovic, Y. Bitran, R.G. Dorsch, A.W. Lohmann, J. Opt. Soc. Am. A 12 (1995) 1665.
[8] W. Schempp, in: Lie Methods in Optics, Eds. J. Sánchez-Mondragón, K.B. Wolf, Lecture Notes in Physics, Vol. 250 (Springer, Berlin, 1986) pp. 1.
[9] K.B. Wolf, Optics Comm. 132 (1996) 343.
[10] C. Gonzalo, J. Bescos, L.R. Berriel-Valdós, P. Artal, Appl. Optics 26 (1990) 751.
[11] G. García-Calderón, M. Moshinsky, J. Phys. A 13 (1980) L185.
[12] N.M. Atakishiyev, S.M. Chumakov, A.L. Rivera, K.B. Wolf, Phys. Lett. A 215 (1996) 128; A.L. Rivera, N.M. Atakishiyev, S.M. Chumakov, K.B. Wolf, Evolution under polynomial Hamiltonians in quantum and optical phase spaces, Phys. Rev. A (1997), accepted.