# Evolution under polynomial Hamiltonians in quantum and optical phase spaces

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We analyze the difference between classical and quantum nonlinear dynamics by computing the time evolution of the Wigner functions for the simplest polynomial Hamiltonians of fourth degree in coordinate and momentum. This class of Hamiltonians contains examples which are important in wave and quantum optics. The Hamiltonians under study describe the third-order aberrations to the paraxial approximation and the nonlinear Kerr medium. Special attention is given to the quantum analog of the conservation of the volume element in classical phase space. [S1050-2947(97)05101-9]

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## I. INTRODUCTION: CLASSICAL AND QUANTUM DYNAMICS

Since the creation of quantum mechanics, efforts have been made to visualize the quantum dynamical image and to put it in better correspondence with the classical evolution. This goal can be partially achieved by using the phase-space picture in which the state of a quantum system may be represented by the quasiprobability distribution in the phase space of the corresponding classical system [1,2]. Described in this way, quantum dynamics resembles classical statistical mechanics. Clearly, this analogy is incomplete for at least two reasons: first, quasiprobability distributions may take negative values (unlike a true probability distribution), and second, the classical distribution can be localized at a point in phase space, whereas the quantum distribution must always be spread in a finite phase-space volume, in agreement with uncertainty relations. Let us consider an initial distribution which is consistent with the uncertainty relations and describes a real quantum particle. Thus we can ask: *what is the difference between classical and quantum dynamics in phase space?*

The classical dynamical law is very simple. Every element of phase space moves along the classical trajectory while preserving its volume. If at time $t=0$ the probability to find a particle in a unit volume at the point $x_0,p_0$ is $W_{\rm cl}(x_0,p_0)$, then at time $t$ the probability distribution is

$$W_{\rm cl}(x,p;t)=W_{\rm cl}(x_0(x,p,t),p_0(x,p,t)), \tag{1}$$

where $x(t)$, $p(t)$ is the classical trajectory passing through the point $x_0,p_0$ at time $t=0$. Does this picture help us to understand quantum dynamics? And if so, when can the quasiclassical approximation be efficiently used to describe a quantum system? This question is important in several branches of physics: particle quantum mechanics, quantum optics, and wave optics, because they share a common mathematical structure.

These questions are formulated naturally in the well-known language of particle quantum mechanics. They also appear in quantum optics, where a single mode of the electromagnetic field is described by the coordinate and momentum operators of the harmonic oscillator; this field oscillator interacts with an atomic system or with other field modes in the case of a nonlinear optical process. Modern quantum optics uses extensively the quasiprobability distributions [1,2], various versions of the quasiclassical approximation [3] and other quantum-mechanical tools.

Here we stress applications in wave optics. Indeed, it is well known that in the paraxial approximation, the optical Helmholtz equation reduces to the Schrödinger equation. The distance along the optical axis plays the role of the time $t$ in mechanics and, in a two-dimensional optical medium, we denote the *screen* coordinate (perpendicular to the optical axis) by $x$. The canonically conjugate momentum $p$ describes the direction of the ray at the point $x,t$. The classical limit is geometric optics. The theory of optical devices in the paraxial approximation describes the propagation of light beams as generated by Hamiltonian operators which are second-degree polynomials in $x$ and $p$; they generate linear canonical transformations of phase space. Polynomial Hamiltonians of higher degree describe the aberrations to the paraxial regime, leading to the visual deformations and unfocusing of images in optical devices [4]. In wave optics the above question can be formulated as follows: when can the optical device be well described by geometric optics, and when is it necessary to use a specifically wave optical description?

The purpose of the present paper is to compare the classical and quantum dynamics generated by the simplest nonlinear Hamiltonians. We consider Hamiltonians which are fourth-degree polynomials in the coordinate $x$ and momentum $p$. It is convenient to classify these Hamiltonians using the wave optics picture, where they describe third-order aberrations. In quantum mechanics and quantum optics, the polynomial Hamiltonians of the fourth-degree include such important examples as the anharmonic oscillator and the optical Kerr medium [5,6]. We use the Wigner quasiprobability distribution to provide a visual image of the quantum dynamics. As we shall see, it yields the closest possible common description of classical and quantum dynamics in the phase plane with the standard coordinates $x$ and $p$ [7].

In order to find parameters which can be used to determine the similarity or difference between the classical and quantum dynamics, we examine the quantum analogs of the conservation of the volume element of the classical phase plane. We do not expect the corresponding invariants to hold for arbitrary quantum processes, though we shall see that they do exist in the case of linear quantum dynamics (i.e., described by linear transformations of the Heisenberg operators). We show that the *moments* of the Wigner function [i.e., the integrals of the powers of $W(x,p)$ over the phase plane] have the desired properties [8]. The classical counterparts of these moments are invariant under any canonical transformation, as follows directly from the phase volume conservation. In quantum dynamics, these moments are preserved by linear canonical transformations but are changed by nonlinear transformations. For all the semiclassical states (described by Gaussian wave functions) these moments (in a natural normalization) are equal to unity. Their difference from unity may serve as a measure of the ''nonclassicality'' of the state. The change of these moments in the course of nonlinear quantum evolution reflects an extra growth of the quantum fluctuations over the corresponding classical level, providing information on how closely the process can be described by the quasiclassical approximation. In particular, the moments of the Wigner function can be used to detect and quantify the quantum superpositions of macroscopically distinguishable states, i.e., the so-called Schrödinger-cat states. We restrict ourselves to the case of two-dimensional phase space, where it is easy to plot and understand the graphs of the quasiprobability distributions. Only pure quantum states (i.e., those described by wave functions) will be considered here.

The paper is organized as follows. In Sec. II we recall the properties of linear canonical transformations. Sec. III contains a discussion of nonlinear canonical transformations. We review some of the previous definitions of phase volume elements for quantum states and stress the usefulness of the moments of the Wigner function. For these moments we also present an alternative formula in terms of the wave function. Secs. IV and V are the central parts of the paper; they contain the results of numerical computation of the Wigner function for single optical aberrations and the optical Kerr medium. Final comments are given in the conclusion, Sec. VI.

## II. LINEAR TRANSFORMATIONS

Time evolution in classical mechanics is a canonical transformation generated by the Hamiltonian function $h(p,x)$,

$$\dot{x}=\{x,h\}, \quad \dot{p}=\{p,h\}, \tag{2}$$

where

$$\{f,h\}=\frac{\partial f}{\partial x}\frac{\partial h}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial h}{\partial x}$$

is the Poisson bracket and the overdot indicates total time derivative. The quantum-mechanical evolution is described by the unitary transformation generated by the self-adjoint Hamilton operator $H$ through,

$$i\dot{X}=[X,H], \quad i\dot{P}=[P,H], \tag{3}$$

where $[A,B]=AB-BA$ is the commutator. We denote the operators by capitals and the classical variables by small letters. Note, that we use units where $\hbar=1$. The resulting unitary transformation can be written (both in classical and in quantum mechanics) in the exponential form $U(t)=\exp(-itH)$, where time enters as a transformation parameter. Different Hamiltonians lead to different canonical transformations.

The operators $P$ and $X$ generate rigid translations of phase space. Linear homogeneous canonical transformations [9] are generated by polynomial Hamiltonians of second degree in $P$ and $X$, i.e., linear combinations of the operators

$$P^2,(PX+XP)/2,X^2. \tag{4}$$

The harmonic-oscillator Hamiltonian $P^2/2+\omega^2X^2/2$ generates rigid rotations of phase space around the origin. The rotation by the angle $\pi/2$ is just the Fourier transformation. The generator $(PX+XP)/2$ is called the squeezing operator because it compresses phase space along one coordinate and expands it along the other; it transforms one harmonic oscillator into another with different frequency. In paraxial wave optics, $P^2/2$ generates free propagation of light rays in a homogeneous medium and $X^2/2$ corresponds to the action of a thin lens.

The Hamiltonians (4) lead to linear equations of motion that are identical in classical and quantum mechanics. In other words, the Heisenberg operator solutions to the quantum equations have the same form as the classical trajectories $p(t)$, $x(t)$. In wave optics, linear transformations describe paraxial systems; in quantum optics, they describe beam splitters, interferometers, linear amplifiers, etc.

Among the various quasiprobability distributions proposed in the literature [10,11], there is only one for which every linear quantum evolution coincides with classical evolution [given by Eq. (1)] [12]. This is the Wigner function,

$$\begin{aligned}W(x,p;t)&=2\int_{-\infty}^{+\infty}dr\Psi^*(x+r;t)e^{2ipr}\Psi(x-r;t)\\&=2\int_{-\infty}^{+\infty}dr\tilde{\Psi}^*(p+r;t)e^{-2ixr}\tilde{\Psi}(p-r;t),\end{aligned} \tag{5}$$

where $\Psi(x;t)$ and $\tilde{\Psi}(p;t)$ are solutions of the Schrödinger equation in the coordinate and momentum representations, respectively, and $\hbar=1$. Note that another normalization is often used, which differs from Eq. (5) by the factor $1/2\pi$. We include this factor into the phase volume element $dp\,dx/2\pi$, so that the marginal distributions are

$$|\Psi(x)|^2=\int W(x,p)\frac{dp}{2\pi}, \quad |\tilde{\Psi}(p)|^2=\int W(x,p)\frac{dx}{2\pi}, \tag{6}$$

and the normalization condition is $\int W dp\, dx/2\pi=1$ . In fact, the *covariance* requirement between the linear classical and quantum canonical transformations can serve to define the Wigner function [13]. For instance, the $Q$ function $Q(x,p)=|\langle\alpha|\Psi\rangle|^2$ [14], where $|\alpha\rangle$ is a coherent state with the parameter $\alpha=(x+ip)/\sqrt{2}$. This behaves as a classical distribution (1) under shifts and rotations of phase space, but has a different transformation law under the action of the squeezing operator; see, e.g., [15].

In linear dynamics, the classical solution completely determines the quantum one, so one can reduce the solution of the wave equation to the solution of the corresponding classical Hamilton equations. Indeed, one may take the Wigner function describing the initial state, find its evolution from Eq. (1) and (if necessary) reconstruct the marginal distribution using Eq. (6). We shall refer to the classical probability distribution (1) evolving from the initial conditions $W(p_0,x_0;t=0)$ as the ''classical'' Wigner function. Because the classical and quantum Wigner functions evolve identically under linear dynamics, we understand that the Wigner function provides the closest common description of classical and quantum dynamics.

## III. NONLINEAR TRANSFORMATIONS

We consider now the nonlinear canonical transformations generated by the fourth-degree polynomials in $P$ and $X$. Such Hamiltonians are linear combinations of the operators

$$P^4,\{P^3X\},\{P^2X^2\},\{PX^3\},X^4, \tag{7}$$

where $\{\ldots\}$ stands for the Weyl ordering of the operators [16]. One particularly important example of polynomial Hamiltonian of fourth degree in quantum optics is $H=\frac{1}{2}(P^2+\omega^2X^2)+(\chi/4\omega^2)(P^2+\omega^2X^2)^2$, which describes the optical Kerr medium in the variables $P$ and $X$ of a single mode of the electromagnetic field.

In the nonlinear case, the classical solution does not determine the quantum dynamics, since products of $P$'s and $X$'s enter the Heisenberg equations of motion. The mean values of these products [e.g., $\langle\{PX\}(t)\rangle$] become additional variables which are absent in classical equations. Therefore, the classical and quantum Wigner functions will evolve differently.

We assume that the initial state of the system is given by a *Gaussian* wave function in the coordinate representation. Then the wave function in momentum representation, and also the Wigner function, are Gaussians. These states are also called *generalized coherent states* (GCS). Under linear evolution Gaussians remain Gaussians of possibly different parameters. Since linear evolution is the same in the classical and quantum cases, we may conclude that the GCS are quasiclassical states [17]. It is known that the only states which have an everywhere positive Wigner function are the Gaussian states [1]. Under quantum *nonlinear* evolution, the initial Gaussian loses its shape and its Wigner function must therefore take negative values in some regions of phase space. We may also expect that the quantum fluctuations *spread* the initial coherent state. In Sec. IV we consider several examples which show how these dynamical features are realized in particular nonlinear transformations.

The general picture can be summarized as follows: The initial Gaussian Wigner function is a ''hill'' in phase space. Linear evolution, both classical and quantum, moves, rotates and squeezes this hill preserving the area inside any given level curve. Classical nonlinear evolution can also *deform* the shape of the hill (with the area still kept constant). But quantum nonlinear evolution, although it moves the top of the hill in agreement with the classical picture, exhibits a new phenomenon: ''quantum oscillations'' appear at the concavities of the level curves of the hill. This is a purely quantum phenomenon and, as will be seen, is absent in the classical case. Under nonlinear evolution we may expect that the ''area'' of the hill is no longer preserved; however, it is not clear how to define this area (or phase-space volume element). As we stated in the introduction, it would be useful to formulate a quantum counterpart to the concept of classical phase volume conservation. The connected question in quantum optics can be posed as follows: *what is the most natural way to describe quantum fluctuations?*

As long as we work with linear transformations the answer is known: one may use the left-hand side of the Schrödinger-Robertson uncertainty relation (see, e.g., [18]),

$$\delta=\sigma_{xx}\sigma_{pp}-\sigma_{xp}^2\geqslant\tfrac{1}{4}, \tag{8}$$

where

$$\sigma_{xx}=(\Delta x)^2=\langle(X-\overline{X})^2\rangle,$$

$$\sigma_{pp}=(\Delta p)^2=\langle(P-\overline{P})^2\rangle,$$

$$\sigma_{xp}=\tfrac{1}{2}\langle(XP+PX)\rangle-\overline{XP}.$$

The brackets $\langle\cdots\rangle$ denote the average over a given quantum state, $\sigma_{xp}$ describes correlations between coordinate and momentum fluctuations, and $\overline{X},\overline{P}$ are the mean values of the coordinate and momentum. The equality in Eq. (8) holds for pure Gaussian states. Dodonov and Man'ko noticed that the value $\delta$ in Eq. (8) is invariant under linear canonical transformations both in classical and quantum mechanics [19]. Hence, linear dynamics do not lead to any extra growth of quantum fluctuations over the classical ones.

Under nonlinear transformations however, $\delta$ is not invariant even in classical mechanics; hence, $\delta$ cannot in general describe an element of phase-space volume because the latter must be conserved by any classical canonical transformation, linear or nonlinear. The Dodonov-Man'ko parameter $\delta$ can thus be used better to characterize the ''degree of nonlinearity'' of the system, rather than its ''degree of nonclassicality,'' as was noted in Ref. [20]. This parameter is in fact very useful to describe the short-time nonlinear behavior, even if it does not feel global effects (such as those of Schrödinger-cat states) which appear for longer times.

Let us recall some properties of Schrödinger-cat states. The quasiprobability distribution for a coherent state is a Gaussian centered at the point $(\overline{x_0},\overline{p_0})$ of the phase plane. The quantum superposition of two coherent states with mac-

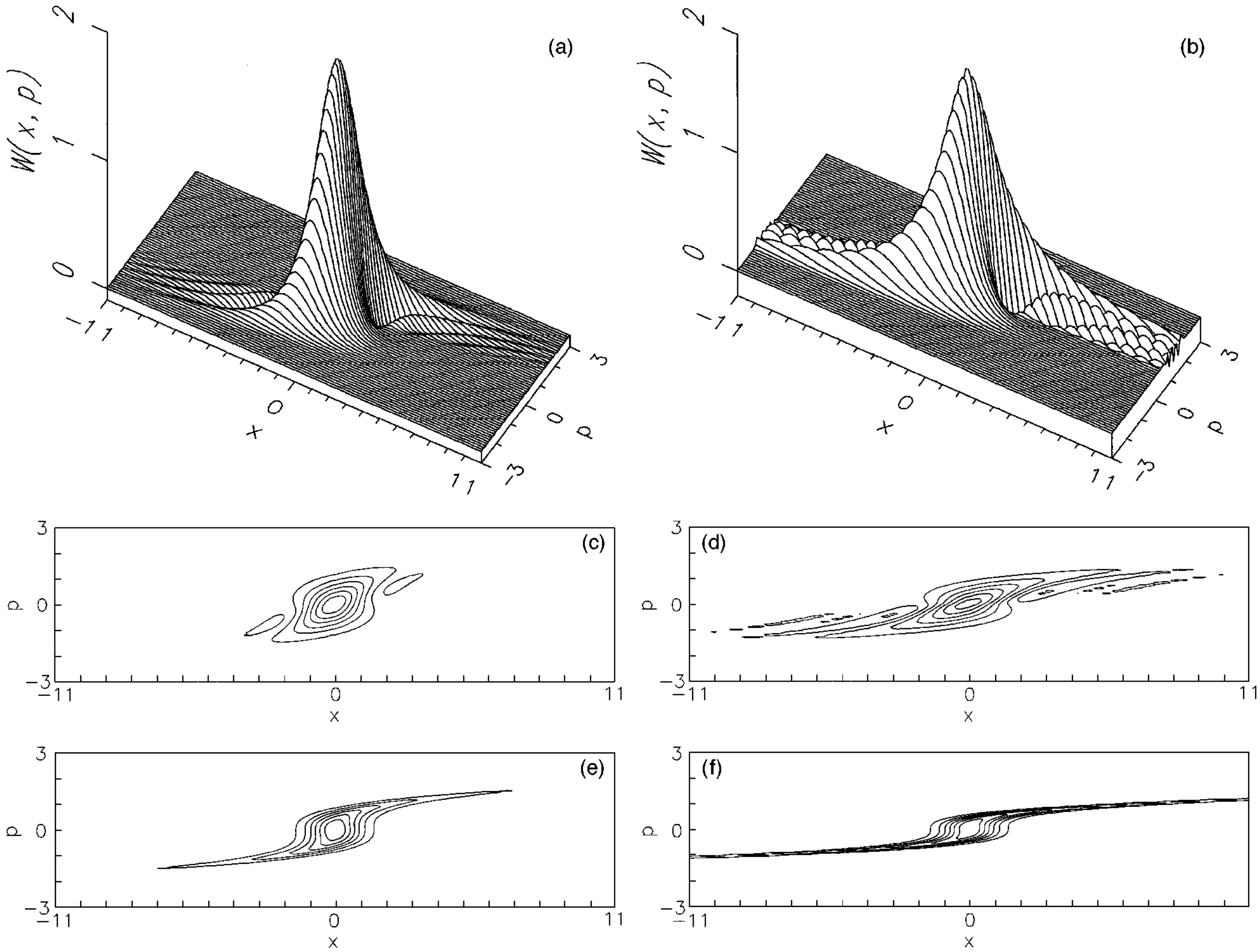


FIG. 1. Evolution of the quantum and classical Wigner functions for the Hamiltonian $P^4$ (spherical aberrations). Observe that in all our figures we use units where $\hbar=1$, and coordinates and momentum are dimensionless. (a),(b) three-dimensional plots of the quantum Wigner functions for times $t=0.5$ and $t=2.0$; (c),(d) level plots of the same quantum Wigner functions; (e),(f) level plots of the classical Wigner functions for the same time instants.

roscopically distinguishable coordinates and momenta $(\overline{x_1},\overline{p_1})$, $(\overline{x_2},\overline{p_2})$ is an example of a state [21]. Any quasiprobability distribution is different then from zero in the neighborhoods of the points $(\overline{x_1},\overline{p_1})$ and $(\overline{x_2},\overline{p_2})$; moreover, the Wigner function also shows fast oscillations at the midpoint $(\overline{x_1}+\overline{x_2})/2$, $(\overline{p_1}+\overline{p_2})/2$, which can be called the *smile* of the Schrödinger cat and reveals the coherent superposition of the states. In a statistical mixture of the same states, the oscillations are absent. The uncertainties in coordinate and momenta for the cat state have values of the order of $|\overline{x_1}-\overline{x_2}|$ and $|\overline{p_1}-\overline{p_2}|$. The parameter $\delta$ in Eq. (8) does not take into account that the particle can only occur at the neighborhood of the points $(\overline{x_1},\overline{p_1})$, $(\overline{x_2},\overline{p_2})$, and never in between.

The difficulty in describing quantum fluctuations for Schrödinger-cat states can be overcome through taking advantage of *entropy* as a measure of fluctuations [22]. Since there is no true distribution in quantum phase space, Wehrl [23] proposed to calculate the entropy using the nonnegative $Q$ function instead of the probability distribution,

$$S_Q=-\int Q\ln Q\,\frac{dp\,dx}{2\pi}. \tag{9}$$

The Wehrl entropy carries more precise information about the phase-space volume occupied by the quantum state; it is especially convenient for the description of the Schrödinger-cat states [24]. In particular, if the cat state consists of $M$ well-separated components, then $S_Q=S_0+\ln M$, where $S_0$ is the entropy of a single component. The Wehrl entropy is thus a good candidate to describe the phase-space volume occupied by the quantum state. Unfortunately, $S_Q$ is not invariant under the squeezing transformation [25]. (This follows directly from the ‘‘bad’’ behavior of the $Q$ function under the squeezing mentioned above; the Wehrl entropy overestimates quantum fluctuations in squeezed states.)

We search for a quantity that can serve to separate between classical and quantum dynamics and, from the point of view of applications, to determine if the semiclassical approximation is good or not. Recalling that linear transformations change the Wigner function covariantly in classical and quantum dynamics, we conclude that the specifically quantum features of a system are due to the nonlinear part of the dynamics, which transform an initial semiclassical state to a ‘‘highly quantum’’ one. Therefore, the parameter which distinguishes between classical and quantum dynamics also has to separate between the semiclassical and the ‘‘highly quantum’’ states [26].

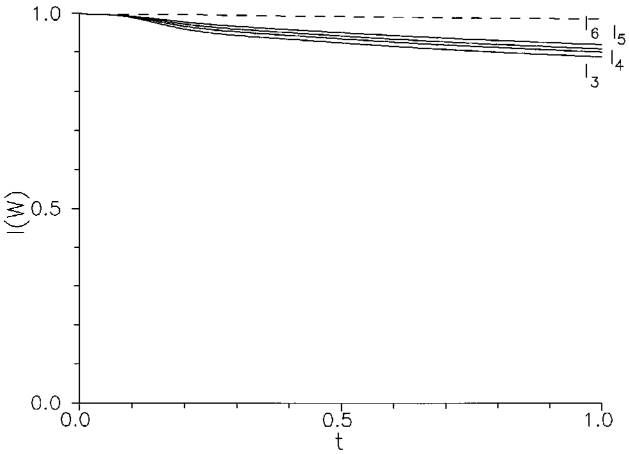


FIG. 2. Time evolution of the moments of the quantum Wigner functions for the Hamiltonian $P^4$. $I_2$ (dotted line) is shown as a quality test of our numerical computation. The decrease of the moments $I_k$ for $k\geqslant 3$ reveals the difference between classical and quantum dynamics.

We would have a measure of the classicality of the state possessing all the desirable properties if we could calculate the entropy using the Wigner function as a probability distribution. This is impossible however, since the Wigner function can take negative values (except for Gaussians). Moreover, these negative values are known to be an important manifestation of the nonclassicality of the state. (The entropy is determined as the mean value of the logarithm of the distribution, which is not well defined for negative values.) We can consider other monotonic functions beside the logarithm studying the behavior of integrals of the type

$$I=\int f(W)\frac{dp\,dx}{2\pi}, \tag{10}$$

where $f(W)$ is any monotonic function of the Wigner function [27]. It is important to note that this integral is invariant in the classical case under any canonical transformation, linear or nonlinear. To verify this, we change variables $x,p\mapsto x_0(x,p,t),p_0(x,p,t)$, where $x_0$, $p_0$ is the initial point of the classical trajectory which passes through the point $x(t)$, $p(t)$ at time $t$. Then the invariance of the integral (10) follows from the conservation of the phase-space volume under the canonical transformation [28]. In the quantum case the integrals (10) are invariant under linear transformations.

The simplest monotonic functions are the powers $W^k$. Then the integrals (10) are the moments of the Wigner function

$$I_k(t)=\frac{k}{2^{k-1}}\int W^k(p,x;t)\frac{dp\,dx}{2\pi}, \quad k=1,2,\ldots . \tag{11}$$

Corresponding quantities for true probability distributions are known as ''$\alpha$ entropies'' [29]. They obey some inequalities which reflect the uncertainty relations [30] (cf. Ref. [31]). We use here the moments of the Wigner function to characterize the spread of the Wigner function in phase space and the ''classicality'' of the corresponding quantum state.

From the normalization condition it follows that $I_1=1$. In turn, $I_2=1$ holds for any pure state. (For mixed states described by the density matrix $\rho$, the second moment gives the purity of the state, $I_2=\mathrm{Tr}(\rho^2)$ [1,32].) Therefore, only the moments $I_k, k\geqslant 3$ contain nontrivial information. It is easy to check that our normalization implies $I_k=1$ for any pure Gaussian state. Quantum linear evolution preserves the initial values of the moments. However, quantum nonlinear evolution of the initial Gaussian state may lower the values of the moments $I_k$, for $k\geqslant 3$.

The moments (11) can also be written directly in terms of the wave functions in coordinate or momentum representation (without use of the Wigner function). Indeed substituting Eq. (5) into Eq. (11) and integrating over $p$ we have,

$$I_k(t)=k\int dx\,dr_1\ldots dr_{k-1}\prod_{j=1}^{k-1}\Psi^*(x+r_j)\Psi(x-r_j)$$
$$\times\Psi^*(x-r_1-\ldots-r_{k-1})\Psi(x+r_1+\ldots+r_{k-1}).$$

This equation (and a similar one in terms of the wave function in the momentum representation) may be useful to study the analytic properties of the moments.

In Secs. IV and V we shall calculate numerically the moments $k=3,4,5,6$ for several examples of nonlinear dynamics governed by Hamiltonians of the type (7) and show that these moments indeed carry important information about the quantum state. Hence, they can be used to distinguish quasiclassical dynamics from quantum dynamics, and semiclassical states from quantum states. Moreover, they can be used to detect Schrödinger cats.

## IV. NUMERICAL RESULTS FOR MONOMIAL HAMILTONIANS (OPTICAL ABERRATIONS)

In this section we use the wave optical terminology. The Lie theory of geometrical image aberrations [4] identifies the operators (7) with the third-order aberrations in two-dimensional optical media. In geometric optics, momentum is $p=n\sin\theta$, where $n$ denotes the refractive index and $\theta$ is the angle between the ray and the optical axis. The analysis of the aberration generators as separate Hamiltonians is warranted because they represent the first nonlinear correction to some interesting physical phenomena briefly indicated below. The marginal distribution $|\Psi(x)|^2$ in Eq. (6) is the light intensity on the one-dimensional screen of coordinate $x\in\mathrm{Re}$. (The common designation of $z$ for the optical axis coordinate is replaced here by $t$, as if it were time.) We now investigate the action of aberrations on the initial *vacuum* coherent state, i.e., a Gaussian of unit width centered at the origin of phase space. The three-dimensional figures and the corresponding level plots of the Wigner function that evolves under the quantum-mechanical Hamiltonians are presented for two different time instants. The level plots of the classical Wigner functions are also shown for those times.

### A. Spherical aberration $H=P^4$

The first metaxial correction to paraxial free propagation is called spherical aberration. The same Hamiltonian also describes the first relativistic correction to the Schrödinger equation for a particle of nonzero mass. In Figs. 1(a)–1(f) we show the classical and quantum evolution of an initial vacuum coherent state for the time instants $t=0.5$ and $t=2.0$. The resulting states are no longer Gaussians, but are

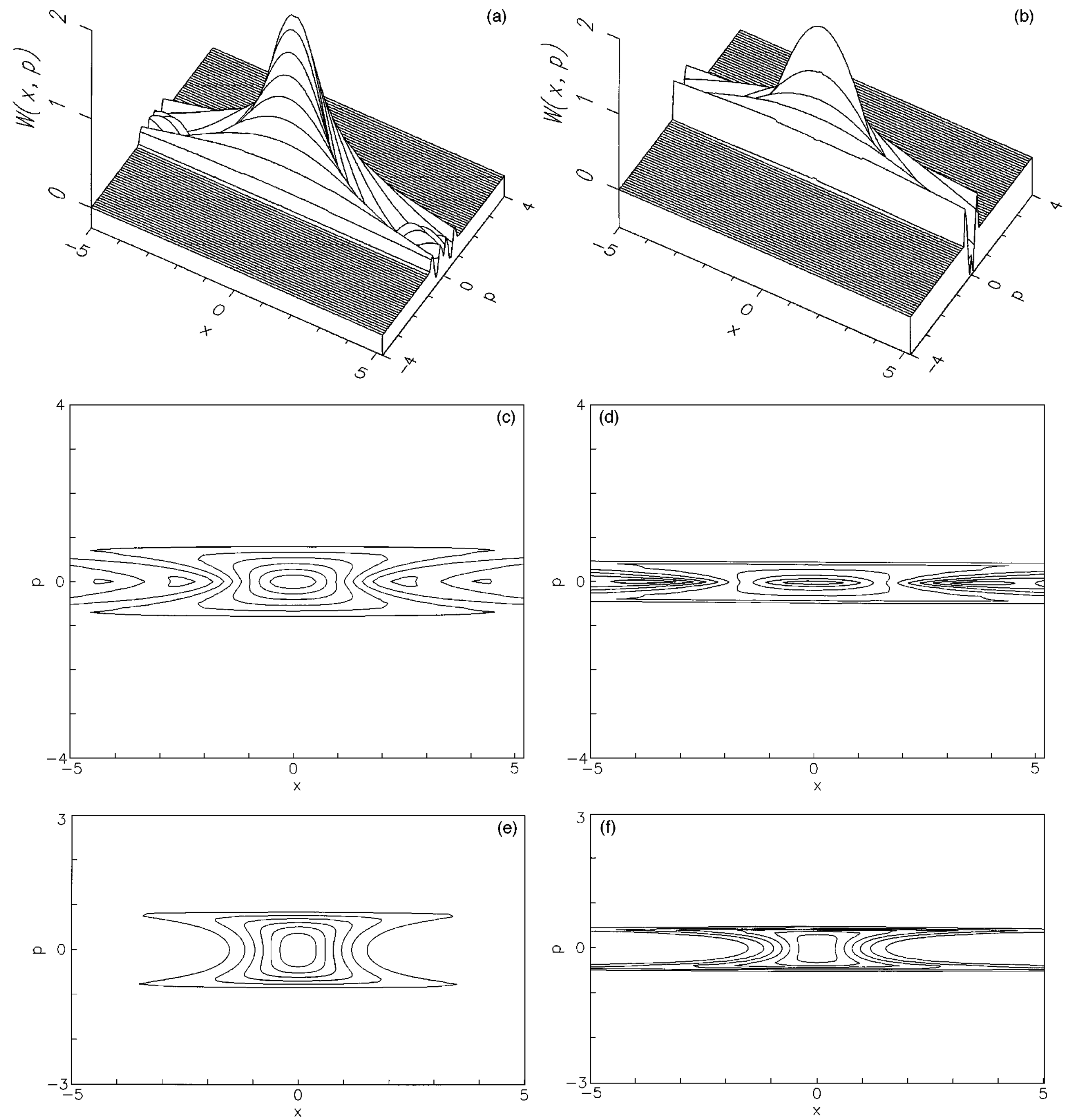


FIG. 3. The same as in Fig. 1 for the Hamiltonian $P^3X$ (coma).

represented by hills that rapidly spread in $x$. The difference between the classical and quantum cases can be seen in the additional oscillations of the quantum Wigner function, which appear in Figs. 1(a)–1(d) and are absent in Figs. 1(e) and 1(f). They are seen in the level plots as small islands forming in the concave part of the main hill; their area is considerably smaller than the area of the vacuum state. We are therefore led to call this phenomenon ''quantum oscillation.''

The behavior of the moments, shown in Fig. 2, is quite flat. There is a proportional drop in all moments beyond the second. The constancy of $I_2$ provides a reliable numerical check on the computation. The figure indicates that semiclassical states remain a good approximation to quantum states. Note that this aberration has been analytically treated in Ref. [33].

### B. Coma $H=P^3X$

The generator of this transformation is

$$H=\{P^3X\}=P^3X+i\tfrac{3}{2}P^2.$$

This Hamiltonian is also the first approximation to the *relativistic coma* phenomenon after squeezing [34]. The corre-

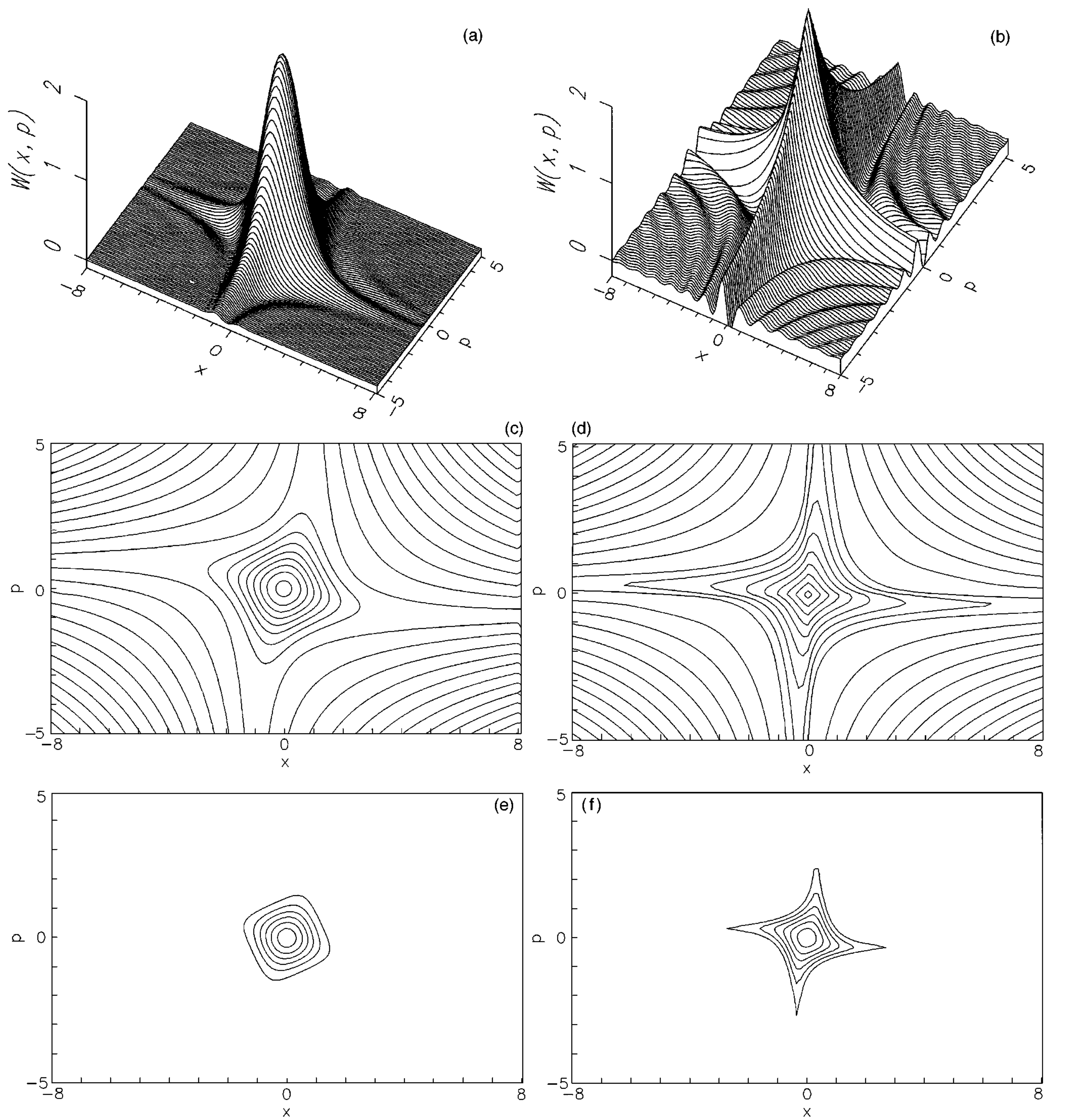


FIG. 4. Evolution of the quantum and classical Wigner functions for the Hamiltonian $P^2X^2$ (astigmatism). (a),(b) three-dimensional plots of the quantum Wigner functions for times $t=0.1$ and $t=0.5$; (c),(d) level plots of the same quantum Wigner functions; (e),(f) level plots of the classical Wigner functions for the same time instants.

sponding Schrödinger equation in momentum representation is the first-order differential equation

$$i\partial_t\tilde{\Psi}(p,t)=i(p^3\partial_p+\tfrac{3}{2}p^2)\tilde{\Psi}(p,t).$$

The exact solution to this equation reads

$$\tilde{\Psi}(p,t)=\frac{1}{(1-2p^2t)^{3/4}}\tilde{\Psi}\left(\frac{p}{\sqrt{1-2p^2t}},0\right),$$

where $\tilde{\Psi}(p,0)=\pi^{-1/4}\exp[-p^2/2]$ is the initial condition. The Wigner function has been calculated numerically from Eq. (5) to produce Figs. 3(a)–3(f). Acting in coordinate representation, i.e., on the optical screen, coma produces image caustics (which are comet shaped only in two-dimensional optical images). The signature of an image caustic in phase space is that $x_0=$constant lines cross the level plots at four points. This is seen in the wings of Figs. 3(c)–3(f). In the quantum case, ‘‘quantum oscillations’’ again occur in the

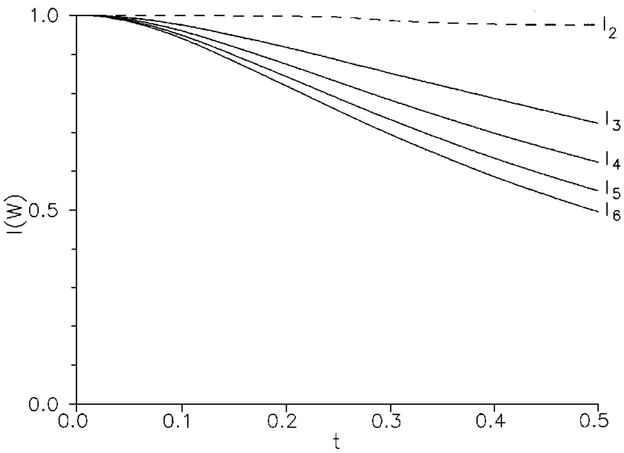


FIG. 5. The same as in Fig. 2 for the Hamiltonian $P^2X^2$ (astigmatism).

concavities of the main hill. The values of the moments drop proportionately more than in the previous aberration.

The solutions of the classical equations of motion are

$$x_0 = x(1-2p^2t)^{3/2}, \quad p_0 = \frac{p}{\sqrt{1-2p^2t}},$$

where the trajectory $x=x(t)$, $p=p(t)$ begins at the point $x_0, p_0$. We notice that the whole initial momentum range $-\infty < p_0 < \infty$ is mapped into the interval $|p(t)| < 1/\sqrt{2t}$; no points map beyond this interval. At the quantum level, the Wigner function at time $t$ is zero outside the strip $|p| < 1/\sqrt{2t}$ and the normalization condition involves the integration only over this strip. This squeezing in the momentum variable corresponds to the forward compression of ray directions under relativistic boost of the screen in geometric optics [34].

### C. Astigmatism $H = P^2X^2$

Astigmatism can be characterized classically as a hyperbolic *torsion* of phase space stemming from a radius-dependent differential hyperbolic rotation. (For two-dimensional images there is also the *curvature of field* aberration; in our one-dimensional case it coalesces with astigmatism.)

The Weyl-ordered Hamiltonian in the coordinate representation has the form

$$H = \{P^2X^2\} = -x^2\partial_x^2 - 2x\partial_x - \tfrac{1}{2}.$$

(Other quantization schemes will differ only in the additive constant.) The Green function for this Hamiltonian can be found exactly, both in coordinate or momentum representation. However, it is more convenient to solve numerically the differential equation for the wave function and then to find the Wigner function by integration.

In Figs. 4(a)–4(f) we see a cross-symmetric hill developing out of the initial vacuum coherent state for times $t=0.1$ and $t=0.5$. The quantum case again shows ''quantum oscillations'' that are much stronger now. In Figs. 4(c) and 4(d) we show, among others, the zero-level curves which, due to the shape of the ''quantum oscillations,'' appear as if they were hyperbolas. The behavior of the moments is shown in Fig. 5; they decrease much faster than in the other aberrations. (Note also that these figures are computed for shorter times than those of the other aberrations.)

### D. Distortion $H = PX^3$

The Hamiltonian in the coordinate representation is

$$H = \{PX^3\} = X^3P - i\tfrac{3}{2}X^2 = -i(x^3\partial_x + \tfrac{3}{2}x^2).$$

The differential equation for distortion in the coordinate representation has the same form as that for coma in the momentum representation, with a change in the sign of time $t \to -t$. Distortion and coma are Fourier conjugate of each other [36] and, thus, the evolution under distortion corresponds to backward comatic dynamics.

The classical and quantum Wigner functions are shown in Figs. 6(a)–6(f). As we saw above, coma compresses phase space along the momentum axis. Correspondingly, distortion will expand phase space along the coordinate axis, as can be seen from the classical trajectories,

$$x = \frac{x_0}{\sqrt{1-2x_0^2t}}, \quad p = p(1-2x_0^2t)^{3/2}.$$

These trajectories reach infinity in finite time: at time $t$, the points which initially have coordinates $|x_0| < 1/\sqrt{2t}$ will still be in the finite plane, while the points $x_0 = \pm 1/\sqrt{2t}$ map to infinity. The points $|x_0| > 1/\sqrt{2t}$ will disappear from the classical phase space and so do not contribute to the quantum solution. As a result, the normalization of the wave function is not preserved. This unpleasant property of the distortion Hamiltonian has been pointed out by Klauder [35]. Correspondingly, the moments $I_1$ and $I_2$ are not constant in this case, as we see in Fig. 7.

### E. Pocus $H = X^4$

This aberration has received its playful name [36] because of its $p$-unfocusing effect. It is the Fourier transform of spherical aberration: it spreads rays in momentum and leaves the position coordinate invariant (so it does not affect the geometric image quality and is not included in the traditional Seidel classification [37]), but multiplies the wave function by a phase $e^{itx^4}$.

The evolution of the Wigner function can be found from spherical aberration by the Fourier rotation of the phase plane plus time inversion. It is shown in Figs. 8(a)–8(f) for the time instants $t=0.5$ and $t=2$. The moments $I_k$ are invariant under this transformation and are the same as in Fig. 2. The effect of pocus on classical phase space and on the quantum Wigner function is on par with all other nonlinear transformations.

## V. OPTICAL KERR MEDIUM

A successful model of active optical media in which self-interaction of the field takes place is the Kerr medium [5,6,38–40]. Its Hamiltonian is a harmonic oscillator describing a single quantized mode of the electromagnetic field

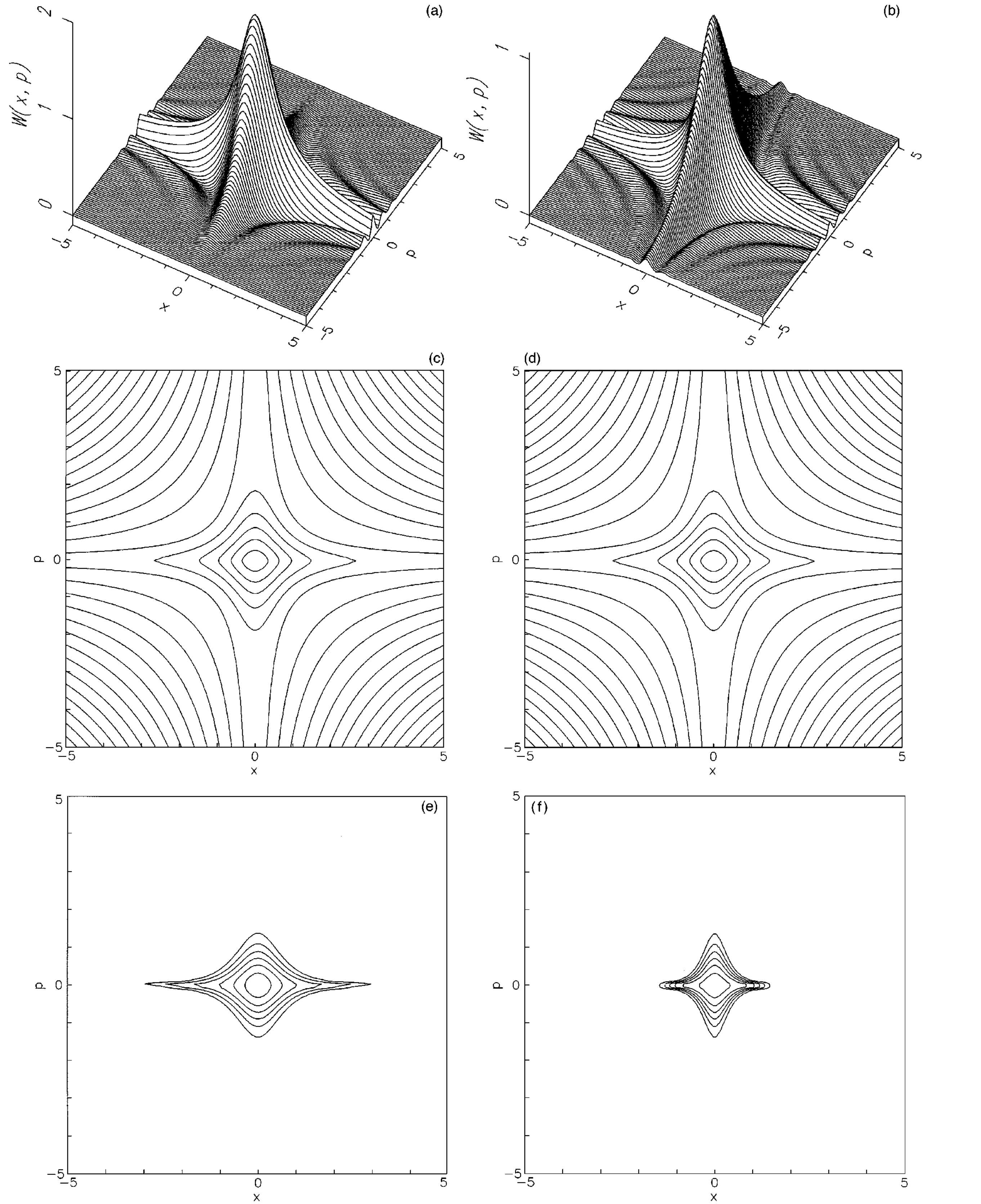


FIG. 6. The same as in Fig. 1 for the Hamiltonian $PX^3$ (distortion).

of frequency $\omega$, plus a self-interaction term with a coupling constant $\chi$, [6]. It has the form

$$H=\frac{1}{2}(P^2+\omega^2X^2)+\frac{1}{\omega^2}\chi(P^2+\omega^2X^2)^2, \tag{12}$$

in units where $\hbar=1$. In quantum electrodynamics, the Hamiltonian is usually written in terms of the photon number operator $\hat{n}=a^\dagger a$ as $H=\omega\hat{n}+\chi\hat{n}^2$ [here $\omega$ is shifted related to Eq. (12)]. It is clear that the harmonic-oscillator Hamiltonian and the total Kerr Hamiltonian (12) have the same

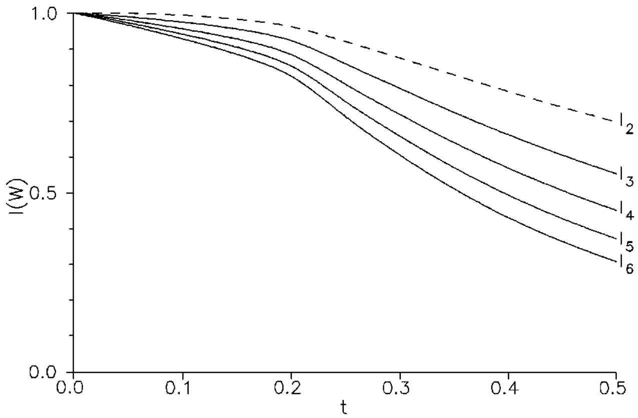


FIG. 7. The same as in Fig. 2 for the Hamiltonian $PX^3$ (distortion).

eigenvectors. The photon number is conserved but there is a nontrivial evolution of the field phase.

The time evolution of the Wigner function under the Kerr Hamiltonian is shown in Figs. 9(a)–9(f) and the corresponding evolution of the moments is shown in Figs. 10. In these figures we choose $\chi = 1$. The first term in the Hamiltonian (12) leads to the ''fast'' rotation of the graphs with angular frequency $\omega$; we work in the interaction picture, which subtracts this rotation.

The Wigner function of the initial Gaussian state is shown in Fig. 9(a). It is centered at the point $\overline{x} = \sqrt{2\overline{n}} \simeq 5.7$, indicated by the radial distance to the origin and $\overline{p} = 0$ (it is thus *not* the vacuum state) corresponding the Glauber coherent state of photon number $\overline{n} = 16$. For small time $t = 0.02$, the Gaussian is first stretched and rotated in the phase plane as shown in Fig. 9(b); all the moments, shown in Figs. 10, are still close to unity and so the state is still nearly semiclassical. It is squeezed in a definite direction in phase plane, however. This squeezing can be seen clearly in Fig. 9(b). (Note that in the graphs of the $Q$ function it would be more difficult to visually notice squeezing since the hills would be ''fatter.'') We can use the propagation in a linear medium by the bare harmonic-oscillator Hamiltonian to achieve the best squeezing in the field coordinate or momentum [39].

As time advances, Fig. 9(c) shows that the hill is stretched along a circle (*not* along a straight line); the angular range of the hill spreads and we see a crescent. The deformation of the top of the hill is still semiclassical. However, the shape of the hill is already sufficiently bent for the ''quantum oscillations'' to appear. As long as the moments $I_k$ are still $\sim 1$ in Figs. 10, these ''quantum oscillations'' are weak and their contribution to the phase-space volume is small. The area of the hill increases slowly while the angular spread grows faster, so we may expect a radial (*amplitude*) squeezing. It actually occurs slightly away from the radial direction, but

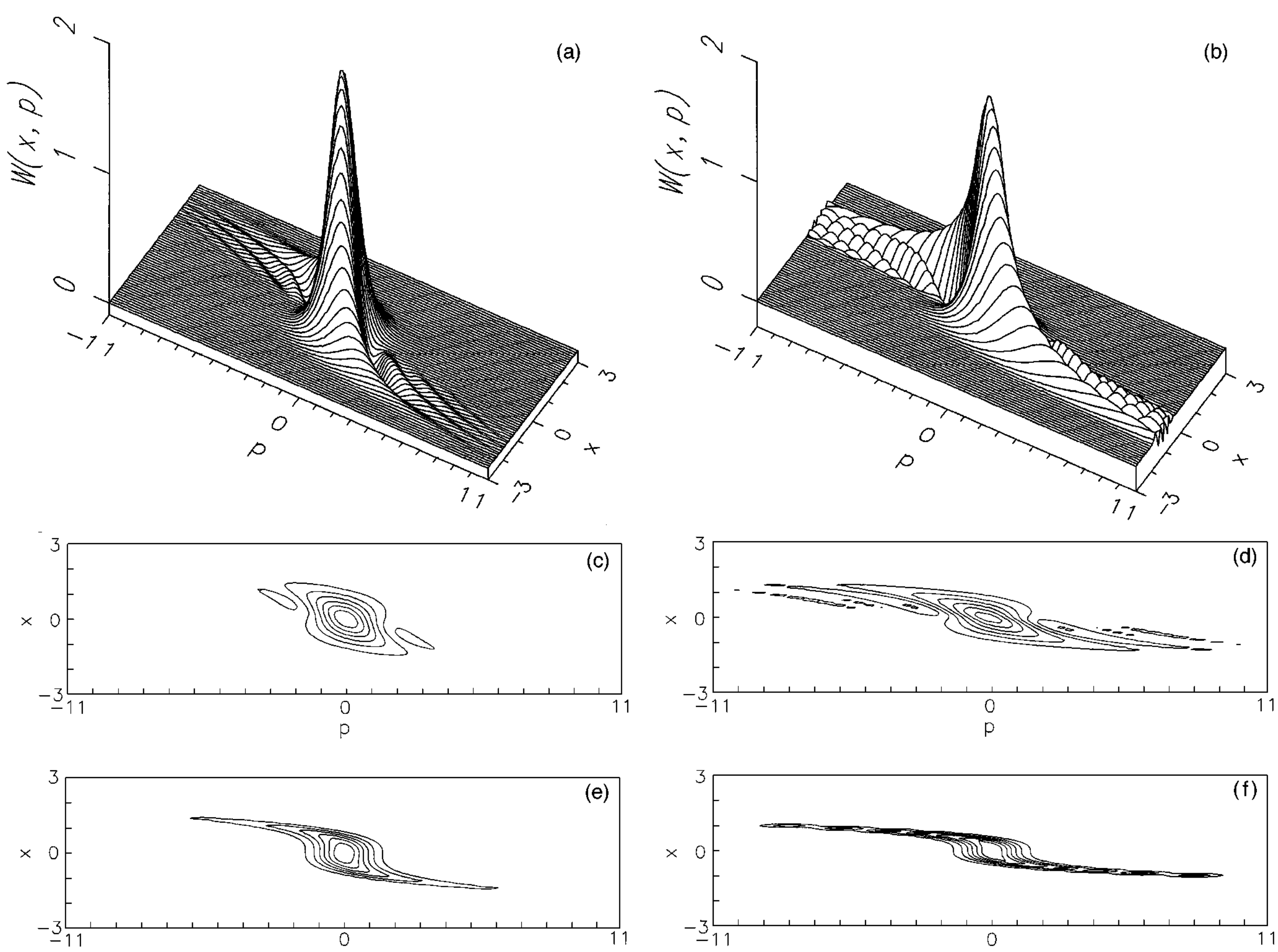


FIG. 8. The same as in Fig. 1 for the Hamiltonian $X^4$ (pocus).

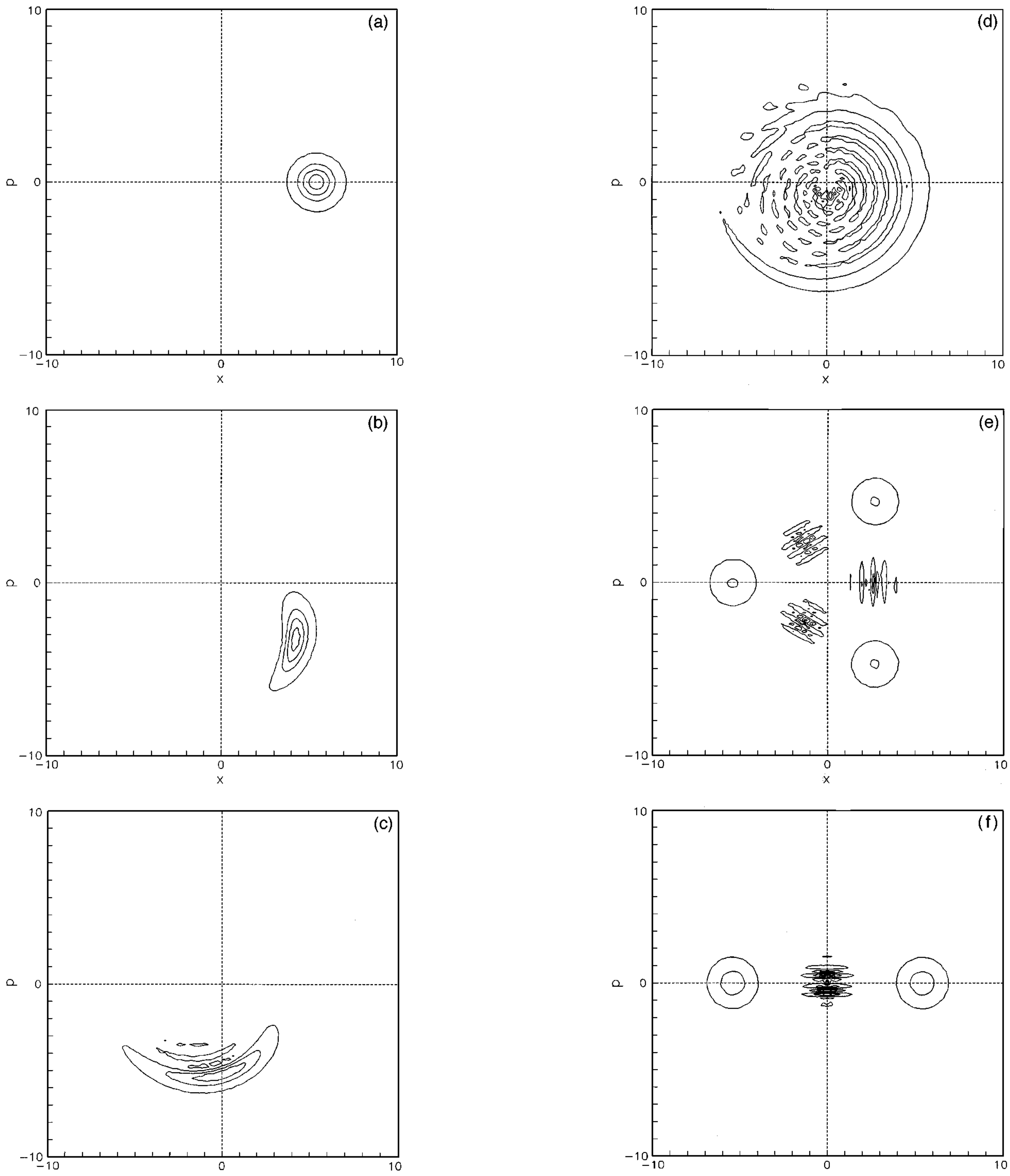


FIG. 9. Evolution of the quantum Wigner function for the Hamiltonian $\chi(P^2/\omega+X^2\omega)^2$ (Kerr media). The initial state is a coherent one described by a Poisson distribution with $\overline{n}=16$. (a) $t=0$; (b) $t=0.02$; (c) $t=0.05$; (d) $t=0.2$; (e) $t=\pi/3$; and (f) $t=\pi/2$; we see Schrödinger cats for times $t=\pi/3$ and $t=\pi/2$.

can be transformed into amplitude squeezing if we shift the origin of phase plane, so as to put it at the center of curvature of the crescent. Physically, this can be realized by placing the nonlinear Kerr medium inside one arm of a Mach-Zehnder interferometer, as was proposed by Kitagawa and Yamamoto [6]. In this way strong squeezing in the photon-number fluctuations can be achieved. The Kerr amplitude squeezing can be further enhanced by electing as initial state an already squeezed state [40]. (Note that the ''quantum oscillations'' are invisible in the graphs of the $Q$ function used in Ref. [6].)

As time evolves further [see Fig. 9(d)] the angular spread reaches $2\pi$ and the ''quantum oscillations'' become comparable to what remains of the original crescent (the classical hill) and occupy the whole interior. It becomes clear that these ''quantum oscillations'' are due to self-interference in

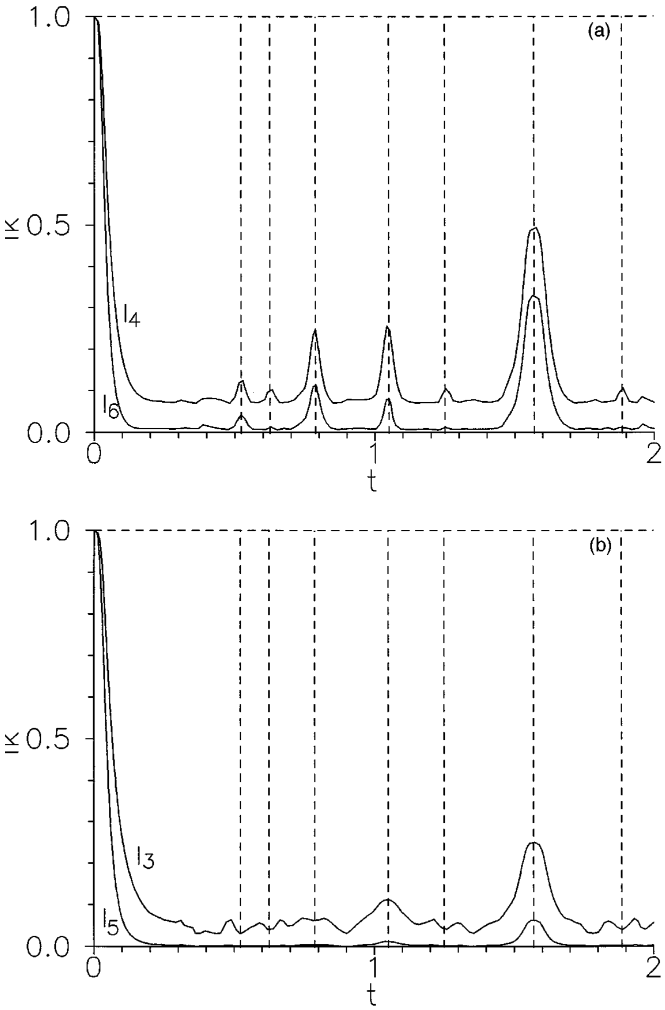


FIG. 10. Time evolution of the moments of the quantum Wigner functions for the Kerr Hamiltonian. (a) Even moments $I_2$ (dotted line), $I_4$, and $I_6$. (b) Odd moments $I_1$ (dotted line), $I_3$, and $I_5$. Dashed vertical lines correspond to time instants $\pi/6$, $\pi/5$, $\pi/4$, $\pi/3$, $2\pi/5$, $\pi/2$, and $3\pi/5$, when Schrödinger cats appear.

phase space: different parts of the hill create interference fringes when meeting each other.

At some definite time instants, the self-interference leads to *standing waves* along the circle. These waves are formed in the Kerr medium at times $\chi t = L\pi/M$, where $L,M$ are mutually prime integers, $L < M < \sqrt{\bar{n}}$. These are the Schrödinger cats [21]; see Figs. 9(e) and 9(f). The cat state in the Kerr medium at time $\chi t = L\pi/M$ has $M$ very well pronounced components. This is a consequence of the integer spectrum of the Kerr interaction Hamiltonian $\hat{n}^2$. The self-interference phenomenon appears also in the Jaynes-Cummings [41] and Dicke models [42]. It has been shown that the field in both models, for special initial conditions, can be described by the effective Hamiltonian $H_{\rm Dicke} \sim \sqrt{\hat{n}+1/2}$ [42], i.e., the square root of the harmonic-oscillator Hamiltonian. The Dicke Hamiltonian thus generates evolution which is in a sense similar to the Kerr one (cf., [43]); however, the effective Hamiltonian does not have an integer spectrum and Schrödinger cats are not so well pronounced. We emphasize that the sharp interference fringes in the smiles between the cat components of Figs. 9(e) and 9(f) would be absent if the state were a statistical mixture of the same components. The $Q$ function does not show any structure between the states and, hence, will not distinguish between coherent superposition and statistical mixture of components.

Most of the information contained in the Wigner function plots can be restored from the graphs of the moments $I_3$ to $I_6$ shown in Fig. 10. The time instants at which we can expect amplitude squeezing are those where the initial peak still conserves its identity and the moments are still close to unity. When the Wigner function shows complicated ''quantum oscillations,'' moments are kept in their lowest, steady values. The times when Schrödinger cats appear correspond to the well-pronounced peaks of the moments. One can estimate the maximum values of the moments at these peaks for well-separated cat components.

When a cat state consists of two separate components centered at points $x_1, p_1$ and $x_2, p_2$, so that the wave function in the coordinate representation has the form

$$\Psi(x) = \alpha\Psi_1(x) + \beta\Psi_2(x),$$

where $\alpha$ and $\beta$ are the amplitudes of the components and $\overline{\alpha\beta} = |\alpha\beta| e^{-i\phi}$, then the Wigner function has the form

$$W = |\alpha|^2 W^{(1)} + |\beta|^2 W^{(2)} + |\alpha\beta| W^{(12)}, \tag{13}$$

where $W^{(1)}$ and $W^{(2)}$ are the Wigner functions of the separate components and $W^{(12)}$ is the contribution of the ''smile'' region. Let us suppose for simplicity that the cat components are Glauber coherent states and that $p_1 = p_2 = 0$. Then we have

$$W^{(12)} = 4\exp[-(x-x_c)^2 - p^2]\cos[p(x_1 - x_2) + \phi],$$

with $x_c = \frac{1}{2}(x_1 + x_2)$. The Wigner function (13) is exponentially small everywhere except for the neighborhoods of the points $(x_1,0)$, $(x_2,0)$, and the midpoint $x_c$. When these neighborhoods do not significantly overlap, the integrals $I_k$ will consist of the contributions for these three points, and we have

$$I_k = |\alpha|^{2k} I_k^{(1)} + |\beta|^{2k} I_k^{(2)} + |\alpha\beta|^k I_k^{(12)},$$

where $I_k^{(1)}$ and $I_k^{(2)}$ are the moments corresponding to the first and the second components and

$$I_k^{(12)} \approx C_{k/2}^k + O(\exp[-(x_1 - x_2)^2/4]), \quad k \text{ even}$$

$$I_k^{(12)} \approx O(\exp[-(x_1 - x_2)^2/16]), \quad k \text{ odd.}$$

The binomial coefficient $C_{k/2}^k = k!/[(k/2)!]^2$ can be approximated by $C_{k/2}^k \sim 2^k (2/\pi k)^{1/2}$ for large $k$. Neglecting the exponentially small terms and taking into account that for a single coherent state $I_k$ is unity, we have

$$I_k \approx |\alpha|^{2k} + |\beta|^{2k} + |\alpha\beta|^2 C_{k/2}^k, \quad k \text{ even,}$$

$$I_k \approx |\alpha|^{2k} + |\beta|^{2k}, \quad k \text{ odd.} \tag{14}$$

If the cat state has $M$ well-separated components and all the $M(M-1)/2$ ''smile'' regions are also well separated from each other and from the components, then the sums in the above equations have $M$ terms corresponding to the components and the even-$k$ moments will have $M(M-1)/2$ additional terms. In Fig. 9, only two- and three-component cat states can be considered to be well separated. Correspondingly, Eqs. (14) give the correct numerical values of the moments $I_k$ at the peaks for times $\pi/2$ and $\pi/3$; see Fig. 10.

## VI. CONCLUSIONS

The difference between classical and quantum dynamics is connected with the phenomenon of self-interference in phase space. For quasiperiodic motion the latter leads to the Schrödinger-cat states. Such states can be produced when the quantized electromagnetic field propagates inside the optical Kerr medium [5,21]. It is a ''global phenomenon'' since the quantum state spreads over all the phase volume allowed by conservation laws, and occurs usually at times longer than the period of fast oscillation of the system.

We have shown here that quantum nonlinear dynamics also differs from its classical counterpart for shorter times, i.e., when the state is still well localized in phase space. The nonclassicality is manifest in the ''quantum oscillations.'' The higher moments of the Wigner function can be used as numerical parameters to measure this difference.

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[1] M. Hillery, R.F. O'Connel, M.O. Scully, and E.P. Wigner, Phys. Rep. **106**, 121 (1984).
[2] V. Tatarski, Usp. Fiz. Nauk **139**, 587 (1983) [Sov. Phys. Usp. **26**, 535 (1983)]; Hai-Woong Lee, Phys. Rep. **259**, 147 (1995).
[3] See, for example, S.M. Chumakov, A.B. Klimov, and J.J. Sánchez-Mondragón, Phys. Rev. A **49**, 4972 (1994).
[4] A.J. Dragt, E. Forest, and K.B. Wolf, in *Lie Methods in Optics*, edited by J.J. Sánchez-Mondragón and K.B. Wolf, Lecture Notes in Physics Vol. 250 (Springer-Verlag, Heidelberg, 1986), p. 104.
[5] G.S. Milburn, Phys. Rev. A **33**, 674 (1986).
[6] M. Kitagawa and Y. Yamamoto, Phys. Rev. A **34**, 3974 (1986).
[7] We use here the standard definition of the Wigner function which is connected with the Heisenberg-Weyl dynamical group. If the photon number and the phase are chosen as the coordinates in phase space, it naturally leads to a new definition of the Wigner function; see J. P. Bizarro, Phys. Rev. A **49**, 3255 (1994). J.A. Vaccaro, Opt. Commun. **113**, 421 (1995).
[8] N.M. Atakishiyev, S.M. Chumakov, K.B. Wolf, and A.L. Rivera, in *Proceedings of the IV Wigner Symposium*, edited by N.M. Atakishiyev, T.H. Seligman, and K.B. Wolf (World Scientific Publishing, Singapore, 1996), p. 166.
[9] M. Moshinsky and C. Quesne, J. Math. Phys. **12**, 1772 (1971).
[10] R. Graham, F. Haake, H. Haken, and W. Weidlich, Z. Phys. **213**, 21 (1968).
[11] K.E. Cahill and R.J. Glauber, Phys. Rev. **177**, 1857 (1969); **177**, 1882 (1969).
[12] G. García-Calderón and M. Moshinsky, J. Phys. A **13**, L185 (1980).
[13] Hui Li, Phys. Lett. A **188**, 107 (1994).
[14] K. Husimi, Proc. Phys. Math. Soc. Jpn. **22**, 264 (1940).
[15] M. Hillery, M. Freyberger, and W. Schleich, Phys. Rev. A **51**, 1792 (1995).
[16] The Weyl ordering is naturally associated with the Wigner function. Any self-adjoint ordering scheme gives the same result for fourth-degree monomials, except for $p^2q^2$, which only exhibits different additive constants (of units $\hbar^2$). It turns out, therefore, that the Wigner function, being sesquilinear in the wave functions, will be insensitive to the ordering scheme. For higher-degree monomials, we recall that $\int(dx\,dp/2\pi)p^n x^m = \overline{\{P^n X^m\}}$.
[17] Note that the set of GCS includes the squeezed states. Squeezing is usually considered a nonclassical attribute of a state. However, we are interested in the dynamical behavior, i.e., in the transformation properties of the states. All GCS can be transformed among themselves by linear transformations which are essentially semiclassical. That is why we consider GCS as semiclassical states.
[18] V.V. Dodonov, E.V. Kurmyshev, and V.I. Man'ko, Phys. Lett. A **79**, 150 (1980).
[19] V.V. Dodonov and V.I. Man'ko, in *Group Theoretical Methods in Physics*, (Harwood Academic, Chur, Switzerland, 1985), Vol. 1, p. 591; V.V. Dodonov and O.V. Man'ko, in *Group Theoretical Methods in Physics*, Proceedings of the Third Seminar, edited by M.A. Markov (Nauka, Moscow, 1986), Vol. 2, p. 432.
[20] A.J. Dragt, F. Neri, and G. Rangarajan, Phys. Rev. A **45**, 2572 (1992).
[21] Z. Bialynicka-Birula, Phys. Rev. **173**, 1207 (1968); B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 1055 (1986).
[22] V.V. Dodonov and V.I. Man'ko, in *Invariants and Evolution of Nonstationary Quantum Systems*, Vol. 183 of the Proceedings of Levedev Physics Institute, edited by M.A. Markov (Nova Science, Commack, NY, 1989).
[23] A. Wehrl, Rev. Mod. Phys. **50**, 221 (1978).
[24] I. Jex and A. Orlowski, J. Mod. Opt. **41**, 2301 (1994); A. Orlowski, H. Paul, and G. Kastelewicz, Phys. Rev. A **52**, 1621 (1995).
[25] C.H. Keitel and K. Wódkiewicz, in *Proceedings of the Second International Workshop on Squeezed States and Uncertainty Relations, Moscow, May 22-29, 1992*, edited by D. Ham, Y.S. Kim, and V.I. Man'ko (NASA Conference Publication, 1993), p. 259.
[26] A measure of ''nonclassicality'' of a state (different from the one proposed here) was introduced in the following articles: C.-T. Lee, Phys. Rev. A **52**, 3374 (1995); N. Lütkenhaus and

S.M. Barnett, *ibid.* **51**, 3340 (1995).

[27] The definition of entropy uses the logarithm function to have an additive quantity for systems consisting of noninteracting subsystems. However, we work with a single degree of freedom which cannot be further separated into ''subsystems.'' Thus the additive property of the logarithm seems to be dispensable for our goals.

[28] Note that we cannot introduce under the sign of integral (10) any additional dependence on $p$ and $x$, different from the one contained in the Wigner function, because it will destroy the invariance under classical canonical transformations.

[29] A. Wehrl, Rep. Math. Phys. **10**, 159 (1976).

[30] H. Maassen and J.B.M. Uffink, Phys. Rev. Lett. **60**, 1103 (1988).

[31] B. Daeubler, Ch. Miller, H. Risken, and L. Schoendorff, Phys. Scr. **T48**, 119 (1993); I. Bialynicki-Birula, M. Freyberger, and W. Schleich, Phys. Scr. **T48**, 113 (1993).

[32] V. Bužec, C.H. Keitel, and P.L. Knight, Phys. Rev. A **51**, 2575 (1995).

[33] V.I. Man'ko and K.B. Wolf, in *Lie Methods in Optics* (Ref. [4]) , p. 207; R. Baltin and V.I. Man'ko, in *ICONO 95: Atomic and Quantum Optics: High-Precision Measurements*, Proceedings of SPIE 2799, edited by S.N. Bagaev, A.S. Chirkin (1996), p. 262.

[34] K.B. Wolf, J. Opt. Soc. Am. A **10**, 1925 (1993).

[35] J.R. Klauder, in *Lie Methods in Optics* (Ref. [4]), p. 183.

[36] K.B. Wolf, J. Opt. Soc. Am. A **5**, 1226 (1988).

[37] H. Buchdahl, *Optical Aberration Coefficients* (Dover, New York, 1968).

[38] G.G. Milburn and C.A. Holmes, Phys. Rev. Lett. **56**, 2237 (1986); E.A. Akhundova and M.A. Mukhtarov, J. Phys. A **28**, 5287 (1995).

[39] R. Tanas, in *Coherence and Quantum Optics V*, edited by L. Mandel and E. Wolf (Plenum, New York, 1984), p. 645.

[40] K. Sundar, Phys. Rev. Lett. **75**, 2116 (1995).

[41] I.Sh. Averbukh, Phys. Rev. A **46**, R2205 (1992).

[42] S.M. Chumakov, A.B. Klimov, and J.J. Sanchez-Mondragon, Opt. Commun. **118**, 529 (1995).

[43] S.M. Chumakov, A.B. Klimov, and C. Saavedra, Phys. Rev. A **52**, 3153 (1995).