Theoretical and Mathematical Physics, Vol. 106, No. 1, 1996

DIFFERENCE EQUATION FOR ASSOCIATED POLYNOMIALS ON A LINEAR LATTICE

N. M. Atakishiyev, A. Ronveaux, and K. B. Wolf

We discuss the difference equations on a linear lattice for polynomials associated with the classical Hahn, Kravchuk, Meixner, and Charlier polynomials.

1. Introduction

The three-term recurrence relation satisfied by orthogonal polynomials can be written in matrix form using the well-known tridiagonal Jacobi matrix. For monic polynomials (with the coefficient of its highest degree term equal to one), the recurrence relation

$$p_{n+1}(x) = (x-\beta_n)\,p_n(x) - \gamma_n\,p_{n-1}(x), \qquad \gamma_n \neq 0, \qquad n \geq 1, \qquad (1.1)$$
$$p_0(x) = 1, \qquad p_1(x) = x - \beta_0,$$

corresponds to the Jacobi matrix equation

$$\mathbf{J}\,\vec{P}(x) = x\vec{P}(x), \qquad (1.2)$$

where

$$\vec{P}(x) = \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \dots \end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} \beta_0 & 1 & \cdot & \cdot & \cdot \\ \gamma_1 & \beta_1 & 1 & \cdot & \cdot \\ \cdot & \gamma_2 & \beta_2 & 1 & \cdot \\ \cdot & \cdot & \ddots & \ddots & \ddots \end{pmatrix}. \qquad (1.3)$$

This, in many ways, is equivalent to the study of the properties of orthogonal polynomials that follow from recurrence relations or from the properties of the Jacobi matrix $\mathbf{J}$. (For example, the roots of the characteristic equation for $\mathbf{J}$, cut to the first N rows and columns, coincide with the zeros of $p_N(x)$.)

Modification of the sequences $\{\beta_n, \gamma_n\}$, keeping $\gamma_n \neq 0$, generates new families of polynomials that we call $\bar{p}_n(x)$ (for a new matrix $\bar{\mathbf{J}}$), which are still orthogonal in accordance with the theorem attributed to Favard [1]. These families $\bar{p}_n(x)$, related by the modifications to $p_n(x)$, have already been investigated by many authors: for instance, the co-recursive of $p_n(x)$ by Chihara [2], Ronveaux and Marcellan [3], polynomials associated with $p_n(x)$ [4], and co-modifiers of $p_n(x)$ [5]. See also references [6–10].

Recently, a more peculiar characterization of $\bar{p}_n(x)$ has reclaimed interest: knowing the differential equation satisfied by $p_n(x)$, find a differential equation satisfied by $\bar{p}_n(x)$. In fact, it appears that in almost all situations, this is a *fourth*-order linear differential equation when $p_n(x)$ satisfies a second-order one. Polynomials $\bar{p}_n(x)$ that correspond to any finite modification of the $\{\beta_n, \gamma_n\}$ sequences can always be represented in the following way [6]:

$$\bar{p}_n(x) = A(x)\,p_n(x) + B(x)\,p^{(1)}_{n-1}(x) \qquad (1.4)$$

Visiting Scientist at IIMAS–UNAM/Cuernavaca with Cátedra Patrimonial CONACYT, México (permanent address: Institute of Physics, Azerbaijan Academy of Sciences, Baku 370143, Azerbaijan); Physique Mathématique, Facultés Universitaires Notre-Dame de la Paix, rue de Bruxelles 61, B-5000 Namur, Belgium. Republished from Teoreticheskaya i Matematicheskaya Fizika, Vol. 106, No. 1, pp. 76–83, January, 1996. Original article submitted March 1, 1995.

where $p_{n-1}^{(1)}(x)$ is the so-called ***associated polynomial*** of $p_n(x)$ (also called the *numerator* polynomial). These polynomials verify the recursion relation (1.1) with coefficients $\{\beta_{n+1}, \gamma_{n+1}\}$, and are defined by

$$p_{n-1}^{(1)}(x) = \frac{1}{\rho_0} \int_I \frac{p_n(x) - p_n(y)}{x - y} \, d\mu(y), \tag{1.5}$$

where $\rho_k = \int_I x^k \, d\mu(x)$ is the kth moment of $\rho(x)$, I is the support of the measure $d\mu(x)$, $p_0^{(1)}(x) = 1$, $p_1^{(1)}(x) = x - \beta_1$, and $A(x)$ and $B(x)$ are polynomials which are easily computed from knowledge of the modifications [6].

From the representation (1.4) of the perturbed polynomials $\bar{p}_n(x)$, it is possible to find the differential equation they satisfy if we know the differential equation for $p_n(x)$ [7] and the *differential relation* between $p_{n-1}^{(1)}(x)$ and $p_n(x)$. For the classical (continuous) cases, namely of Jacobi, Bessel, Laguerre, and Hermite, this differential relation is known (see below) and can be written in terms of polynomials $\sigma(x)$ of degrees less than or equal to 2, and $\tau(x)$ of degree 1 [6]. The weight function $\rho(x)$ for the measure $d\mu(x) = \rho(x)\,dx$ can be found through the Pearson differential equation

$$\frac{d(\sigma(x)\rho(x))}{dx} = \tau(x)\rho(x), \qquad \text{with} \qquad \int_I \rho(x) x^k \, dx < \infty, \quad \forall k. \tag{1.6}$$

The family $p_n(x)$ is orthogonal with respect to the weight function $\rho(x)$ and satisfies the differential equation

$$\begin{aligned} L_2[p_n(x)] &= \sigma(x)p_n''(x) + \tau(x)p_n'(x) + \lambda_n p_n(x) = 0, \\ \lambda_n &= -\frac{1}{2}n[2\tau' + (n-1)\sigma'']. \end{aligned} \tag{1.7}$$

The differential relation linking the associated polynomials to the derivative of the originating polynomial family is [8]

$$L_2^*[p_{n-1}^{(1)}(x)] = \kappa p_n'(x), \tag{1.8}$$

where L_2^* is the formal adjoint of L_2 explicitly given by [3]

$$L_2^* = \sigma(x)\frac{d^2}{dx^2} + [2\sigma'(x) - \tau(x)]\frac{d}{dx} + [\lambda_n + \sigma'' - \tau'] = L_2 + 2[\sigma'(x) - \tau(x)]\frac{d}{dx} + \sigma'' - \tau', \tag{1.9a}$$

and the constant is

$$\kappa = \sigma'' - 2\tau'. \tag{1.9b}$$

In many cases, this differential relation allows one to construct the fourth-order differential equation satisfied by the associated polynomials, which is readily obtained from Eq. (1.8) by application of the second-order differential operator that annuls $p_n'(x)$.

The aim of this article is to extend this technique to the classical (*discrete*) orthogonal polynomials of Hahn, [11][1] Kravchuk, Meixner, and Charlier, which are solutions of the difference equation [12]

$$D_2[p_n(x)] \equiv \tilde{\sigma}(x)\nabla\Delta p_n(x) + \frac{1}{2}\tilde{\tau}(x)(\Delta + \nabla)p_n(x) + \lambda_n p_n(x) = 0, \tag{1.10}$$

with the same λ_n as in Eq. (1.7). We recall the definition for the forward and backward difference operators

$$\Delta f(x) = f(x+1) - f(x), \qquad \nabla f(x) = f(x) - f(x-1), \tag{1.11a}$$

[1] We note that it was P. L. Chebyshev who introduced the Hahn polynomials with discrete orthogonality in 1875.

the corresponding "Leibnitz" rules for them

$$\Delta\big[f(x)g(x)\big] = f(x)\Delta g(x) + g(x+1)\Delta f(x), \tag{1.11b}$$

$$\nabla\big[f(x)g(x)\big] = f(x)\nabla g(x) + g(x-1)\nabla f(x), \tag{1.11c}$$

and the relation

$$\Delta\nabla = \nabla\Delta = \Delta - \nabla. \tag{1.11d}$$

The discrete weight function $\rho(x)$, under which the polynomials $p_n(x)$ are orthogonal, satisfies the Pearson-type *difference* equation [12]

$$\Delta\big[\sigma(x)\rho(x)\big] = \tau(x)\rho(x), \tag{1.12}$$

where

$$\sigma(x) = \tilde{\sigma}(x) - \frac{1}{2}\tilde{\tau}(x), \quad \tau(x) = \tilde{\tau}(x), \tag{1.13}$$

and it is assumed that all moments $\rho_k = \sum_n \rho(x_n)\, x_n^k$ are finite.

To construct the difference equation satisfied by the perturbed family $\bar{p}_n(x)$ in (1.4), or equivalently, to study the properties of the Jacobi matrix $\bar{\mathbf{J}}$, we have to search for a difference operator—let us call it D_2^*—that plays the same role as L_2^* in Eq. (1.8). To do this, we shall use, *mutatis mutandis*, the techniques previously applied to the classical (continuous) case [8], with the same notations.

2. Difference relation for associated classical discrete polynomials

It is well known that functions of the *second kind*

$$q_n(x) = \frac{1}{\rho_0}\sum_{j=0}^{N-1} \frac{p_n(y_j)}{y_j - x}\rho(y_j), \tag{2.1}$$

with a zero moment ρ_0 of the weight defined in (1.12), are nonpolynomial solutions of the recurrence relation (1.1) [7]. The summations run over all points of the discrete orthogonality measure y_j $(j = 0, 1, \ldots, N-1)$. On the other hand, the function $Q_n(x) = q_n(x)/\rho(x)$ satisfies the same difference equation (1.10) for the polynomial $p_n(x)$ [13]. In this way, the link between $Q_n(x)$ and $r_n(x) \equiv p_{n-1}^{(1)}(x)$ can be exploited, which follows from the relations (1.5) and (2.1), i.e.,

$$r_n(x) = \rho(x)\big[Q_n(x) - Q_0(x)p_n(x)\big]. \tag{2.2}$$

We note that with the aid of (1.11d) and (1.13), it is more convenient to rewrite Eq. (1.10) for $p_n(x)$ and $Q_n(x)$ as

$$\sigma_+(x)Q_n(x+1) + \sigma_-(x)Q_n(x-1) + \big[\lambda_n - \sigma_+(x) - \sigma_-(x)\big]Q_n(x) = 0, \tag{2.3}$$

where

$$\sigma_+(x) = \tilde{\sigma}(x) + \frac{1}{2}\tilde{\tau}(x) = \sigma(x) + \tau(x), \tag{2.4}$$
$$\sigma_-(x) = \tilde{\sigma}(x) - \frac{1}{2}\tilde{\tau}(x) = \sigma(x).$$

To find a difference analogue of Eq. (1.8) for $r_n(x)$, let us first compute $r_n(x \pm 1)$:

$$r_n(x\pm 1) = \rho(x\pm 1)\big[Q_n(x\pm 1) - Q_0(x\pm 1)\,p_n(x\pm 1)\big]. \tag{2.5}$$

From the difference equation for the weight function (1.12), we deduce that

$$\rho(x \pm 1) = \frac{\sigma_{\pm}(x)}{\sigma_{\mp}(x \pm 1)}\rho(x), \tag{2.6}$$

and therefore, $r_n(x \pm 1)$ can now be written as

$$r_n(x \pm 1) = \frac{\sigma_{\pm}(x)}{\sigma_{\mp}(x \pm 1)}\rho(x)\big[Q_n(x \pm 1) - Q_0(x \pm 1)p_n(x \pm 1)\big]. \tag{2.7}$$

The equations satisfied by $Q_n(x)$ and $Q_0(x)$ suggest that the two relations

$$R_1 \equiv \sigma_-(x+1)r_n(x+1) = \sigma_+(x)\rho(x)\big[Q_n(x+1) - Q_0(x+1)p_n(x+1)\big], \tag{2.8}$$

$$R_2 \equiv \sigma_+(x-1)r_n(x-1) = \sigma_-(x)\rho(x)\big[Q_n(x-1) - Q_0(x-1)p_n(x-1)\big], \tag{2.9}$$

be summed to yield

$$\begin{aligned} R_1 + R_2 &\equiv \big[\sigma_-(x+1)r_n(x+1) + \sigma_+(x-1)r_n(x-1)\big] = \\ &= -\big[\lambda_n - \sigma_+(x) - \sigma_-(x)\big]Q_n(x)\rho(x) - \\ &- \Big\{\sigma_+(x)Q_0(x+1)p_n(x+1) - \big[\sigma_+(x)\Delta Q_0(x) - \sigma_-(x)Q_0(x)\big]p_n(x_1)\Big\}\rho(x) \end{aligned} \tag{2.10}$$

or

$$R_1 + R_2 \equiv -\big[\lambda_n - \sigma_+(x) - \sigma_-(x)\big]r_n(x) - \sigma_+(x)\rho(x)\Delta Q_0(x)\big[p_n(x+1) - p_n(x-1)\big]. \tag{2.11}$$

The difference relation we search for now reads

$$\begin{aligned} &\big[\sigma_-(x+1)r_n(x+1) + \sigma_+(x-1)r_n(x_1) + \{\lambda_n - \sigma_+(x) - \sigma_-(x)\}r_n(x)\big] = \\ &\quad = -\rho(x)\sigma(x)\nabla Q_0(x)\big[p_n(x+1) - p_n(x_1)\big], \end{aligned} \tag{2.12a}$$

where we have used the equality

$$\sigma_+(x)\Delta Q_0(x) = \sigma(x)\nabla Q_0(x), \tag{2.12b}$$

which follows from (2.3) for $n = 0$, i.e., when $\lambda_0 = 0$. We may write (2.12a) in operator form as

$$\begin{aligned} D_2^*\big[r_n(x)\big] &= \big[\sigma_-(x+1)\Delta - \sigma_+(x_1)\nabla + \lambda + \Delta\sigma_-(x) - \nabla\sigma_+(x)\big]r_n(x) = \\ &= -\rho(x)\sigma(x)\nabla Q_0(x)\big[(\Delta + \nabla)p_n(x)\big]. \end{aligned} \tag{2.13}$$

In analogy with the continuous case, the difference operator D_2^* can be identified as the formal adjoint of D_2 in (1.10), namely

$$\begin{aligned} D_2^* f(x) &\equiv \Delta\nabla\big[\tilde{\sigma}(x)f(x)\big] - \frac{1}{2}(\Delta + \nabla)\big[\tilde{\tau}(x)f(x)\big] + \lambda_n f(x) = \\ &= \left\{D_2 + \big[\tilde{\sigma}'(x) - \tilde{\tau}(x)\big](\Delta + \nabla) + (\tilde{\sigma}'' - \tilde{\tau}')\left[\frac{1}{2}(\Delta - \nabla) + 1\right]\right\} f(x), \end{aligned} \tag{2.14}$$

where we have used the involution property $\Delta^* = -\nabla$. In the limit when the lattice step $h \to 0$, Eq. (2.14) coincides with the relation between L_2 and L_2^* in (1.9a).

Now it remains only to simplify the last expression in (2.13) by employing the fact that the factor $\rho(x)\sigma(x)\nabla Q_0(x)$ is the constant κ'. This becomes evident upon writing the difference equation (1.10) in the self-adjoint form [12]:

$$\Delta\big[\sigma(x)\rho(x)\nabla Q_n(x)\big] + \lambda_n\rho(x)Q_n(x) = 0, \qquad \text{with} \qquad \lambda_0 = 0. \tag{2.15}$$

The value of the constant κ' can be found from

$$Q_0(x)\rho(x) = \frac{1}{\rho_0}\sum_{j=0}^{N-1}\frac{\rho(x_j)}{x_j - x}, \tag{2.16}$$

which follows from the definition of $Q_0(x)$ and relation (2.1). Computing $\nabla Q_0(x)$ from Eq. (2.16), we obtain

$$\rho_0\sigma(x)\rho(x)\nabla Q_0(x) = \sigma(x)\sum_{j=0}^{N-1}\frac{\rho(x_j)}{x_j - x} - [\sigma(x_1) + \tau(x-1)]\sum_{j=0}^{N-1}\frac{\rho(x_j)}{x_j + 1 - x}. \tag{2.17}$$

To evaluate the constant, we may let $x \to \infty$ on both sides of this equation. We write first the polynomials $\sigma(x)$ and $\tau(x)$ explicitly as

$$\sigma(x) = \frac{1}{2}\sigma'' x^2 + \sigma'(0)x + \sigma(0), \qquad \tau(x) = \tau' x + \tau(0). \tag{2.18}$$

The asymptotic development of $f(x)$ and $f(x-1)$ with $f(x) = \sum_{j=0}^{N-1}\rho(x_j)/(x_j - x)$ gives, up to terms proportional to $1/x^2$,

$$f(x) \approx -\frac{\rho_0}{x} - \frac{\rho_1}{x^2}, \qquad f(x-1) \approx -\frac{\rho_0}{x} - \frac{\rho_0 + \rho_1}{2}, \tag{2.19}$$

where ρ_1 is the first moment. From this it follows that, for large x, the right-hand side of (2.17) behaves as a constant, $\left(\tau' - \frac{1}{2}\sigma''\right)\rho_0$, and therefore,

$$\lim_{x\to\infty}\rho(x)\sigma(x)\nabla Q_0(x) = \tau' - \frac{1}{2}\sigma'' = \kappa' = -\frac{1}{2}\kappa. \tag{2.20}$$

Substituting (2.20) into Eq. (2.13) leads to the sought-for difference relation

$$D_2^*\left[p_{n-1}^{(1)}(x)\right] = \frac{1}{2}\kappa(\Delta + \nabla)p_n(x) \tag{2.21}$$

between $r_n(x) = p_{n-1}^{(1)}(x)$ and $p_n(x)$. In the limit when the lattice step $h \to 0$, (2.21) coincides with the differential relation (1.8) because in this limit, $D_2^* \to L_2^*$ and $\Delta + \nabla \to 2d/dx$.

The discrete Chebyshev orthogonal polynomials $t_n(x)$ are derived for $\tilde{\sigma}(x) = x(N-x) + \frac{1}{2}N$ and $\tilde{\tau}(x) = N - 2x$. They have a uniform weight $\rho(x_j) = 1$ and are Hahn polynomials $Q_n(x;\alpha,\beta,N)$ with $\alpha = \beta = 0$ (see p. 29–30 in [14]). As the number N of discrete points tends to infinity, $t_n(x)$ coincide with the Legendre polynomials

$$\lim_{N\to\infty} t_n\left(\frac{1}{2}(1-x)N\right) = P_n(x). \tag{2.22}$$

It is interesting to note that $D_2^* = D_2$, for the discrete Chebyshev polynomials, because $\tilde{\sigma}'(x) = \tilde{\tau}(x)$. The last relation also holds for the classical Legendre polynomials, for which $L_2^* = L_2$.

3. Fourth order equation satisfied by the associated polynomials

In this section, we address the problem of finding the difference equation satisfied by the discrete associated polynomials $p_n^{(1)}(x)$, defined from the difference relation (2.21), where the originating polynomials $p_n(x)$ satisfy the difference equation (1.10). We rewrite (2.21) for convenience as

$$P_n(x) \equiv 2\kappa^{-1}D_2^*\left[p_{n-1}^{(1)}(x)\right] = (\Delta + \nabla)p_n(x). \tag{3.1}$$

This difference equation for $P_n(x)$ should not contain the originating polynomials $p_n(x)$; it turns out to be of fourth order. For this reason, we recall that the hypergeometric-type difference Eq. (1.10) or, equivalently, equation (2.3) satisfied by the originating polynomials, is of the form

$$\left[\sigma_+(x)\Delta - \sigma_-(x)\nabla + \lambda_n\right]p_n(x) = 0, \tag{3.2}$$

with λ_n and $\sigma_\pm(x)$ defined in (1.7) and (2.4), respectively.

Let us now multiply equation (3.1) by $\sigma_+(x)$ and use (3.2), i.e.,

$$\sigma_+(x)P_n(x) = \sigma_+(x)(\Delta+\nabla)p_n(x) = \left[\sigma_-(x)\nabla-\lambda_n\right]p_n(x)+\sigma_+(x)\nabla p_n(x) = \left[2\tilde{\sigma}(x)\nabla-\lambda_n\right]p_n(x), \tag{3.3}$$

where we have used $\sigma_+(x) + \sigma_-(x) = 2\tilde{\sigma}(x)$. Next, we apply Δ to both sides of (3.3), obtaining

$$\begin{aligned}\Delta\sigma_+(x)P_n(x) &= 2\tilde{\sigma}(x+1)(\Delta-\nabla)p_n(x) + 2\left[\Delta\tilde{\sigma}(x)\right]\nabla p_n(x) - \lambda_n\Delta p_n(x) = \\ &= \left[2\tilde{\sigma}(x-1) - \lambda_n\right]\Delta p_n(x) - 2\tilde{\sigma}(x)\nabla p_n(x),\end{aligned} \tag{3.4}$$

where we have used relations in (1.11b) and (1.11d). Analogously, we multiply Eq. (3.1) by $\sigma_-(x)$ and apply ∇ to obtain

$$\begin{aligned}\nabla\sigma_-(x)P_n(x) &= 2\tilde{\sigma}(x-1)(\Delta-\nabla)p_n(x) + 2\left[\nabla\tilde{\sigma}(x)\right]\Delta p_n(x) + \lambda_n\nabla p_n(x) = \\ &= 2\tilde{\sigma}(x)\Delta p_n(x) - \left[2\tilde{\sigma}(x-1) - \lambda_n\right]\nabla p_n(x).\end{aligned} \tag{3.5}$$

We now have three equations relating $P_n(x)$ and $p_n(x)$: Eqs. (3.1), (3.4), and (3.5). This allows us to eliminate $\Delta p_n(x)$ and $\nabla p_n(x)$. We multiply (3.4) by $2\tilde{\sigma}(x)$ and subtract (3.5) multiplied by $\left[2\tilde{\sigma}(x+1)-\lambda_n\right]$,

$$\begin{aligned}&2\tilde{\sigma}(x)\Delta\sigma_+(x)P_n(x) - \left[2\tilde{\sigma}(x+1) - \lambda_n\right]\nabla\sigma_-(x)P_n(x) = \\ &= \left\{-4\tilde{\sigma}(x)^2 + \left[2\tilde{\sigma}(x+1) - \lambda_n\right]\left[2\tilde{\sigma}(x-1) - \lambda_n\right]\right\}\nabla p_n(x).\end{aligned} \tag{3.6}$$

Similarly, we multiply (3.5) by $2\tilde{\sigma}(x)$ and add (3.4) multiplied by $\left[2\tilde{\sigma}(x-1) - \lambda_n\right]$ to obtain

$$\begin{aligned}&2\tilde{\sigma}(x)\nabla\sigma_-(x)P_n(x) - \left[2\tilde{\sigma}(x-1) - \lambda_n\right]\Delta\sigma_+(x)P_n(x) = \\ &= \left\{4\tilde{\sigma}(x)^2 - \left[2\tilde{\sigma}(x+1) - \lambda_n\right]\left[2\tilde{\sigma}(x-1) - \lambda_n\right]\right\}\Delta p_n(x).\end{aligned} \tag{3.7}$$

We subtract (3.7) from (3.6), noting that the resulting right-hand side will contain the factor of $(\Delta+\nabla)p_n(x)$, namely

$$\begin{aligned}\Sigma(x) &\equiv 4\tilde{\sigma}(x)^2 - \left[2\tilde{\sigma}(x+1) - \lambda_n\right]\left[2\tilde{\sigma}(x-1) - \lambda_n\right] = \\ &= \left(2\tilde{\sigma}'(x)\right)^2 + 4(\lambda_n - \sigma'')\tilde{\sigma}(x) - (\lambda_n - \sigma'')^2,\end{aligned} \tag{3.8}$$

which, due to (3.1), is $P_n(x)$.

Hence, the polynomial $P_n(x)$ defined in (3.1) satisfies the second-order difference equation

$$\begin{aligned}\tilde{D}_2\left[P_n(x)\right] = &\left[2\tilde{\sigma}(x) + 2\tilde{\sigma}(x-1) - \lambda_n\right]\Delta\sigma_+(x)P_n(x) + \\ &+ \left[2\tilde{\sigma}(x) + 2\tilde{\sigma}(x+1) - \lambda_n\right]\nabla\sigma_-(x)P_n(x) + \Sigma(x)P_n(x) = 0.\end{aligned} \tag{3.9}$$

From here and (3.1), the associated polynomials $r_n(x) = p_{n-1}^{(1)}(x)$ therefore satisfy the factorized fourth-order difference equation

$$\tilde{D}_2\left[D_2^*(r_n(x))\right] = 0. \tag{3.10}$$

In the limit when the step h of the linear lattice under consideration tends to zero, (3.10) coincides with the fourth-order differential equation for the associated polynomials,

$$\left[L_2 + \sigma'(x)\frac{d}{dx} + \tau'\right] L_2^* p_{n-1}^{(1)}(x) = 0, \tag{3.11}$$

which was discussed in detail in [3].

4. Conclusions

The difference relation (2.21) obtained for the *first* associated orthogonal polynomials on a *linear lattice* can probably be extended to *nonuniform lattice* cases. These are of fundamental importance in constructing the representations of quantum groups. For the first associated polynomials, the basic property that the function $Q_n(x) = q_n(x)/\rho(x)$ (see (2.1)) satisfies the same difference equation as $p_n(x)$ is still true for nonlinear lattices. For the higher associated polynomials, a different approach should be considered, as in the continuous cases [4,10]. Work is in progress.

Acknowledgments. One of the authors (A.R.) would like to acknowledge the kind hospitality and support of IIMAS–UNAM for the short stay in Cuernavaca during which this article was written. We are also grateful to Rafael González Campos, Escuela de Ciencias Físico-Matemáticas of the Universidad Michoacana de San Nicolás de Hidalgo, for the invitation to a stimulating workshop in Morelia. This research was carried out under the support of DGAPA–UNAM Project IN104293.

REFERENCES

1. T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York (1978).
2. T. S. Chihara, *Proc. Amer. Math. Soc.*, **8**, 899–905 (1957).
3. A. Ronveaux and F. Marcellán, *J. Comp. Appl. Math.*, **25**, 105–109 (1989).
4. S. Belmehdi and A. Ronveaux, *Rend. Math., Series VII*, **11**, 313–326 (1991).
5. A. A. Ronveaux, S. Belmehdi, J. Dini, and P. Maroni, *J. Comp. Appl. Math.*, **29**, 225–231 (1990).
6. A. Zarzo, A. Ronveaux, and E. Godoy, *J. Comp. Appl. Math.*, **49**, 349–359 (1993).
7. A. Ronveaux, "Fourth-order differential equations related to classical orthogonal polynomials," in: *Proceedings of the VIGO Symposium on Orthogonal Polynomials and Their Applications* (A. Cachafeiro and E. Godoi, eds.), Universidad de Santiago de Compostela, Spain (1988), pp. 159–169.
8. A. Ronveaux, *J. Phys. A*, **21**, L749–L753 (1988).
9. H. A. Slimm, *J. Math. Anal. Appl.*, **136**, 1–19 (1988).
10. A. Ronveaux, "Fourth-order differential equations and orthogonal polynomials of the Laguerre-Hahn class," in: *Orthogonal Polynomials and Their Applications* (C. Brezinski *et al.*, eds.) (IMACS Annals on Computing and Applied Mathematics, Nos. 1–4; XIV), Vol. 9, J. C. Baltzer AG, Scient. Publ. Co., Basel (1991), pp. 379–385.
11. P. L. Chebyshev, *Complete Collected Works* [in Russian], Vol. 3, Moscow, USSR Academy of Sciences (1948).
12. A. F. Nikiforov and V. B. Uvarov, "Classical orthogonal polynomials of a discrete variable on nonuniform lattices," Preprint No. 17, Keldysh Institute of Applied Mathematics, Moscow (1983); A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer, Berlin (1991).
13. N. M. Atakishiyev and S. K. Suslov, *Rev. Mex. Fís.*, **34**, 152–167 (1988).
14. R. Koekoek and R. F. Swarttouw, "The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue," Report No. 94–05, Delft University of Technology, Delft (1994).