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Realization of Sp(2,r) by Finite-difference Operators: the Relativistic Oscillator in an External Field

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Abstract

We solve finite-difference equations that describe a model of the relativistic linear oscillator in a homogeneous external field by Lie algebraic techniques. The corresponding wavefunctions span the discrete and continuous representations of the symplectic algebra.

1 Introduction

The quasipotential approach [1]-[3] allows the development of a relativistic quantum mechanical description [4]-[6], based on the relativistic three-dimensional configuration $\vec{r}$–space [4]. Although this approach is closely analogous to nonrelativistic quantum mechanics *à la Schrödinger*, its essential characteristic is that the relative motion wavefunction satisfies a differential-difference equation

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with step equal to the Compton wavelength of the particle, $\lambda = \hbar/mc$. For two scalar particles with equal masses m, local quasipotentials $V(\vec{r})$ lead to the equation [5]

$$[H_0(\vec{r}) + V(\vec{r})]\,\psi(\vec{r}) = E\,\psi(\vec{r})\,, \tag{1.1}$$

with the *free Hamiltonian*

$$H_0(\vec{r}) = mc^2\left[\cos h\frac{i\hbar}{mc}\partial_r + \frac{i\hbar}{mcr}\sin h\frac{i\hbar}{mc}\partial_r + \frac{\vec{L}^2}{2(mcr)^2}\exp\frac{i\hbar}{mc}\partial_r\right], \tag{1.2}$$

where $\vec{L} = -i\hbar(\vec{r}\times\vec{\nabla})$ is the orbital angular momentum operator, $\partial_r = \partial/\partial r$, and we have the finite-difference operator action $\exp(a\partial_r)\,f(r) = f(r+a)$.

In framework of the quasipotential description, the transformation between configuration and its canonically conjugate momentum space is given by the relativistic plane wave [7]

$$\xi(\vec{p},\vec{r}) = \left(\frac{p_0 - \vec{p}\cdot\vec{n}}{mc}\right)^{-1-imcr/\hbar},$$

$$\vec{r} = r\vec{n}\,, \quad \vec{n}^2 = 1, \qquad 0 < r < \infty\,, \tag{1.3}$$

rather than by the Fourier kernel $\exp(i\vec{p}\cdot\vec{r}/\hbar)$ in the nonrelativistic case. This relativistic plane wave is the generating function for the matrix elements of the principal series of unitary irreducible representations of the Lorentz group $SO(3,1)$. The momenta of positive mass particles belong to the upper sheet of the mass hyperboloid $p_0^2-\vec{p}^2 = m^2c^2$, and form a three-dimensional Lobachevsky space whose group of motion is the Lorentz group. The functions (1.3) are eigenfunctions of the Hamiltonian $H_0(\vec{r})$ in (1.2), and in the nonrelativistic limit (*i.e.*, when $r \gg \hbar/mc$ and $|\vec{p}| \ll mc$) they coincide with the Euclidean plane waves $\exp(i\vec{p}\cdot\vec{r}/\hbar)$.

Within this version of relativistic quantum mechanics, various exactly solvable phenomenological quasipotentials were considered in Refs.[4]-[6],[8]-[14]. In particular, a relativistic model of the linear oscillator was studied in detail in Refs. [8]-[10]. This model was considered for the the case of a constant external force $F(x) = -g$, *i.e.*, in the presence of a homogeneous external field $V_g(x) = gx$ in Ref.[12]. The wavefunctions were found both in the relativistic configuration x and momentum p realizations; it was shown that, in contrast with the nonrelativistic case, here one has both a discrete energy spectrum and a continuous one, depending on the value of the force $|g|$ relative to $m c\omega$, where ω is the oscillator frequency (see below). The generalized coherent states for this model were built out of the bound states for this model in [13].

There are also algebraic approaches to the description of quantum systems by the theory of dynamical symmetries [15, 16], potential groups [17], Casimir operators with mixed spectrum [18], and the 'Euclidean connection' [19]. In these approaches, the key role is played by the dynamical algebra whose generators connect states of different representations of the symmetry subalgebra. In particular, it has been used to find the energy spectrum, the wavefunctions, and transition probabilities. Among the generators of the dynamical algebra is the Hamiltonian, the symmetry commutant algebra of the latter, and raising and lowering operators which do not commute with it. As a consequence, the eigenfunctions of the Hamiltonian form a basis of an irreducible representation of the dynamical algebra.

In Section 2, the algebraic approach is applied to solve the finite-difference equations for the relativistic linear oscillator model in a homogeneous external field. It is shown that the dynamical algebra of the oscillator model generates the $\widetilde{\mathrm{Sp}}(2,\Re)$ covering group of $Sp(2,\Re)$. The explicit form of the wavefunctions for the discrete and continuous spectra (both in the configuration and momentum realizations) are discussed in Sections 3 and 4, respectively. The Appendix contains various interesting formulas and relations for the orthogonal polynomials of Meixner–Pollaczek, Laguerre, Hermite, Meixner and Charlier.

2 The relativistic oscillator in a homogeneous external field

The wavefunction in one-dimensional configuration space x for the relativistic model of a linear oscillator in a homogeneous external field is defined as [9]

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} d\Omega_p \, \xi(p,x) \, \psi(p) \,, \tag{2.1}$$

where $d\Omega_p = mc\, dp/p_0$ is the invariant volume element in the momentum realization, $p_0^2 = m^2c^2 + p^2$, and the plane wave $\xi(p,x)$ has the form [*cf.* the kernel (1.3)]

$$\xi(p,x) = \left(\frac{p_0 - p}{mc}\right)^{-imcx/\hbar} . \tag{2.2}$$

The free Hamiltonian $H_0(x)$ and momentum P_x operators are the finite difference operators

$$H_0(x) = mc^2 \cos h \frac{i\hbar}{mc} \partial_x , \tag{2.3a}$$

$$P_x = -mc \sin h \frac{i\hbar}{mc} \partial_x , \tag{2.3b}$$

$$H_0(x)^2 - c^2 P_x^2 = m^2 c^4 , \tag{2.3c}$$

which commute with each other and have a mutual eigenfunction (2.2) with eigenvalues p_0 and p, respectively. They satisfy the following commutation relations with the relativistic coordinate x:

$$[x, P_x] = \frac{i\hbar}{mc^2} H_0(x) \,, \tag{2.4a}$$

$$[x, H_0(x)] = \frac{i\hbar}{m} P_x \,. \tag{2.4b}$$

The relativistic model of the one-dimensional harmonic oscillator studied in detail in Refs. [8, 9] has the Hamiltonian

$$\begin{aligned} \mathcal{H}^{osc}(x) &= H_0(x) + V(x) \\ &= H_0(x) + \frac{m\omega^2}{2} x \left(x + \frac{i\hbar}{mc}\right) \exp \frac{i\hbar}{mc} \partial_x . \end{aligned} \tag{2.5}$$

The "prolonged derivative" or *generalized momentum operator* [*cf.* (2.4b)] is

$$\mathcal{P}_x = \frac{im}{\hbar} [\mathcal{H}^{osc}(x), x] = P_x - \frac{1}{c} V(x) , \tag{2.6}$$

and satisfies the following commutation relations with x and $\mathcal{H}^{osc}(x)$:

$$[x, \mathcal{P}_x] = \frac{i\hbar}{mc^2} \mathcal{H}^{osc}(x) , \tag{2.7a}$$

$$[\mathcal{H}^{osc}(x), \mathcal{P}_x] = im\hbar\omega^2 \, x . \tag{2.7b}$$

Besides, for this particular potential (2.5) we have [*cf.* relation (2.3c)]

$$(\mathcal{H}^{osc}(x))^2 - c^2 \mathcal{P}_x^2 = m^2 c^4 \left(1 + \frac{\omega^2}{c^2} x^2\right) . \tag{2.8}$$

From (2.6) and (2.7) it follows that the triplet x, $\mathcal{P}_x$, and $\mathcal{H}^{osc}(x)$ closes into the Lie algebra of $Sp(2,\Re)$. Indeed, the operators

$$K_0 = \frac{1}{\hbar\omega}\mathcal{H}^{osc}(x), \quad K_1 = \frac{mc}{\hbar}x, \quad K_2 = -\frac{c}{\hbar\omega}\mathcal{P}_x, \tag{2.9a}$$

obey the standard commutation relations for the $Sp(2,\Re)$ generators,

$$[K_0, K_1] = iK_2, \quad [K_2, K_0] = iK_1, \quad [K_1, K_2] = -iK_0, \tag{2.9b}$$

and provide the Casimir invariant

$$K^2 = K_0^2 - K_1^2 - K_2^2 = s(s+1)\,1, \tag{2.10a}$$

From (2.8) we see that K^2 is $(mc^2/\hbar\omega)^2\,1$ and thus $s(s+1) = \nu(\nu-1)$, with

$$\nu = \frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{mc^2}{\hbar\omega}\right)^2} \geq 1. \tag{2.10b}$$

The eigenvalue $s = -\nu$ is not necessarily an integer and corresponds to the unitary irreducible representation of the *covering group* $\widetilde{\mathrm{Sp}}(2,\Re)$ denoted $D^+(-\nu)$ [20]. The spectrum of the compact operator K_0 is bounded from below and has the values $-s+n = \nu+n$, $n = 0, 1, 2, \ldots$. Consequently, the energy levels of the model Hamiltonian (2.5) are

$$E_n = \hbar\omega(n+\nu), \qquad n = 0, 1, 2, \ldots. \tag{2.11}$$

Thus the introduction of the generalized momentum operator $\mathcal{P}_x$ leads directly to the dynamical algebra $\widetilde{\mathrm{Sp}}(2,\Re)$ for this model. We should underline that in (2.6)–(2.8) we provide companion operators to the Hamiltonian that close into an algebra, and will raise and lower the latter's eigenvalues (2.11). This is distinct from the more common (and more complicated) procedure of factorizing the Hamiltonian under consideration (*cf.* Ref. [9].)

The same line of reasoning is also valid for the relativistic model of the linear harmonic oscillator (2.5) in a homogeneous external field, governed by the difference equation

$$\mathcal{H}^g(x)\,\psi^g(x) = [\mathcal{H}^{osc}(x) + gx]\,\psi^g(x) = E\psi^g(x). \tag{2.12a}$$

Since $\mathcal{H}^{osc} = \hbar\omega K_0$ and $x = (\hbar/mc)K_1$, the equation (2.12a) can be written in terms of the $Sp(2,\Re)$ generators of the model (2.5) in the absence of external force, as

$$\left(\hbar\omega K_0 + \frac{g\hbar}{mc}K_1\right)\psi^g(x) = E\psi^g(x). \tag{2.12b}$$

From the commutation relations (2.9) for the $Sp(2,\Re)$ generators, it follows that

$$\exp^{-i\theta K_2} K_0 \exp^{i\theta K_2} = K_0\cosh\theta + K_1\sinh\theta, \tag{2.13a}$$

$$\exp^{-i\theta K_2} K_1 \exp^{i\theta K_2} = K_0\sinh\theta + K_1\cosh\theta. \tag{2.13b}$$

Therefore, by means of the unitary transformation $\psi^g(x) \mapsto \widetilde{\psi}^g(x) = \exp^{-i\theta K_2}\psi^g(x)$, the eigenvalue equation (2.12b) can be written as

$$(aK_0 + bK_1)\widetilde{\psi}^g(x) = E\widetilde{\psi}^g(x), \tag{2.14}$$

where $a = \hbar\omega\cosh\theta + (g\hbar/mc)\sinh\theta$ and $b = \hbar\omega\sinh\theta + (g\hbar/mc)\cosh\theta$. Consequently, by choosing a suitable angle θ, one of the coefficients a or b can be annulled, according to whether $|g| < mc\omega$ or $|g| \geq mc\omega$, respectively. Then, the left-hand side of (2.14) will contain only one term, either in K_0 or in K_1 up to a constant factor [15]. Let us consider these two cases separately.

3 The states of the discrete spectrum

In the case $|g| < mc\omega$ we can choose $\tanh\theta_g = -g/mc\omega$, leading to $b = 0$ and $a = (\hbar/mc)\times\sqrt{(mc\omega)^2 - g^2} = \hbar\omega\sin\phi_g$, where the angle ϕ_g is defined through $g = mc\omega\cos\phi_g$, $0 < \phi_g < \pi$. Equation (2.14) is then reduced to the eigenvalue equation for the compact generator,

$$\hbar\omega\sin\phi_g\, K_0\widetilde{\psi}^g(x) = E\widetilde{\psi}^g(x), \tag{3.1}$$

having discrete spectrum $n+\nu$. Therefore, the energy spectrum for the bound states of the model (2.12) is [*cf.* Eq. (2.11)]

$$E_n = \hbar\omega(n+\nu)\sin\phi_g, \qquad n = 0, 1, 2 \ldots. \tag{3.2}$$

where ν is related to the Casimir operator eigenvalue $\nu(\nu-1)$ in (2.10). Since (3.1) coincides with the equation for the relativistic linear oscillator in the absence of external force (2.5), the unitary transformation in the case when $|g| < mc\omega$

$$\psi^g(x) = S_g\widetilde{\psi}^g(x), \qquad S_g = \exp(i\theta K_2) \tag{3.3}$$

relates bound states for arbitrary g with bound states for $g = 0$ [13], *i.e.*,

$$\psi_n^g(x) = S_g\widetilde{\psi}_n^g(x) = S_g\psi_n^0(x). \tag{3.4}$$

We note that in the nonrelativistic limit, the operator S_g coincides with the unitary shift operator $\exp(x_0\, d/dx)$, $x_0 = g/m\omega^2$, and the relation (3.4) reflects the fact that in quantum mechanics the presence of the external field $V_g(x) = gx$ does not change the form of the harmonic oscillator potential, but only shifts its equilibrium position, namely $\psi_n^g(x) = \exp(x_0\, d/dx)\psi_n^0(x) = \psi_n^0(x+x_0)$ [21].

To find an explicit form [12] for the wavefunctions ψ_n^g one can start with the momentum realization, in which the Hamiltonian (2.12*a*) is a second-order differential operator, and to define them in the x-realization afterwards by using the transformation (2.1). It is also possible to look directly for solutions of the difference equation (2.12) via expansion in generalized powers by using the method of undetermined coefficients (see, *e.g.*, [22]). In the next Section, however, it will be shown that this particular difference equation is simply solved with the aid of Gauss' recursion relation for the hypergeometric functions $F(\alpha,\beta;\gamma;z)$ [23], *i.e.*,

$$\alpha(1-z)\,F(\alpha+1,\beta;\gamma;z) + (\alpha-\gamma)\,F(\alpha-1,\beta;\gamma;z)$$
$$+[\alpha(2-z)+\gamma-\beta z]\,F(\alpha,\beta;\gamma;z) = 0. \tag{3.5}$$

The reason for leaving this derivation to the next Section is that the continuous case turns out to be more *general*, *i.e.*, it will contain also the discrete case.

The orthonormalized eigenfunctions of (2.12*a*) in the momentum realization are given [12] in terms of the generalized Laguerre polynomials [23] by

$$\psi_n^g(p) = c_n^g\eta^\nu\exp(i\eta e^{i\phi_g})\,L_n^{2\nu-1}(2\eta\sin\phi_g)\,,$$
$$\eta = \frac{c}{\hbar\omega}(p_0+p), \qquad c_n^g = \sqrt{\frac{n!}{mc\Gamma(2\nu+n)}}\; i^n(2\sin\phi_g)^\nu. \tag{3.6}$$

The orthonormalized wavefunctions in the x-realization (3.4), related with (3.6) by the transformation (2.1), are expressed through the Meixner-Pollaczek polynomials $P_n^\nu(x;\varphi)$ and their weight function [24] as

$$\psi_n^g(x) = \tilde{c}_n^g P_n^\nu(\tilde{x};\phi_g)\Gamma(\nu+i\tilde{x})\left(\frac{\hbar\omega}{mc^2}e^{i(\pi/2-\phi_g)}\right)^{i\tilde{x}},$$

$$\tilde{x} = \frac{mc}{\hbar}x, \qquad \tilde{c}_n^g = \frac{mc}{\sqrt{2\pi\hbar}\; i^\nu e^{-i(n+\nu)\phi_g}\, c_n^g}. \tag{3.7}$$

4 The states of the continuous spectrum

In the case when $|g| \geq mc\omega$, we can choose $\tanh\theta_g = -mc\omega/g$; then $a = 0$ and $b = (\hbar/mc)\times\sqrt{g^2-(mc\omega)^2}$, reducing (2.14) to the equation

$$\frac{\hbar}{mc}\sqrt{g^2-(mc\omega)^2}\, K_1\widetilde{\psi}^g(x) = E\widetilde{\psi}^g(x). \tag{4.1}$$

Since the noncompact $Sp(2,\Re)$ generator K_1 has the continuous real spectrum $\kappa\in\Re$, from (4.1) it follows that the corresponding wavefunction $\widetilde{\psi}^g(x)$ belongs to the continuous energy spectrum

$$E = (\kappa\hbar/mc)\sqrt{g^2-(mc\omega)^2} = \kappa\hbar\omega\sinh\varphi_g, \tag{4.2}$$

where $|g| = mc\omega\cosh\varphi_g$ and $0\leq\varphi_g<\infty$. Thus the eigenfunctions of the operator (2.12a) in this case satisfy the difference equation

$$\left([\tilde{x}(\tilde{x}+i)+\nu(\nu-1)]\,e^{i\partial_{\tilde{x}}} + \nu(\nu-1)\,e^{-i\partial_{\tilde{x}}} \pm 2\sqrt{\nu(\nu-1)}\,\tilde{x}\cosh\varphi_g\right)\psi^g(x)$$

$$= 2\kappa\sqrt{\nu(\nu-1)}\;\sinh\varphi_g\;\psi^g(x), \tag{4.3}$$

in which $\tilde{x} = mcx/\hbar$ is the dimensionless variable in configuration x-space, and the $\pm$ signs correspond to the positive and negative values of g, respectively. To symmetrize the coefficients of the difference operators $\exp(\pm i\partial_{\tilde{x}})$ in the equation (4.3) let us make the substitution

$$\psi^g(x) = [\nu(\nu-1)]^{-i\tilde{x}/2}\Gamma(\nu+i\tilde{x})\,f(x;g), \tag{4.4}$$

that converts (4.3) into

$$[(\nu-i\tilde{x})\,e^{i\partial_{\tilde{x}}} + (\nu+i\tilde{x})\,e^{-i\partial_{\tilde{x}}} \pm 2\tilde{x}\cosh\varphi_g]\,f(x;g) = 2\kappa\sinh\varphi_g\,f(x;g). \tag{4.5}$$

We note firstly that in the case of the discrete spectrum, this substitution separates out the factor that determines the asymptotic behavior of $\psi_n^g(x)$. Secondly, since the energy E does not depend on the sign of g [see (4.2)], from (4.5) it follows that $f(x,-g) = f(-x;g)$. Hence, the negative sign in (4.5) may be supressed without affecting the generality of its solution.

Now it remains only to compare (4.5) with (3.5) in order to find that

$$f(x;g) = e^{-\tilde{x}(\pi/2+i\varphi_g)}\Gamma(\nu+i\tilde{x})\,F(\nu+i\tilde{x},\nu-i\kappa;2\nu;1-e^{-2\varphi_g}). \tag{4.6}$$

On combining the formulas (4.4) and (4.6), we get the explicit form of the wavefunctions for the continuous spectrum as

$$\psi^g(x) = \left(\frac{\hbar\omega}{mc^2}e^{i\pi/2-\varphi_g}\right)^{i\tilde{x}}\Gamma(\nu+i\tilde{x})\,F(\nu+i\tilde{x},\nu-i\kappa;2\nu;1-e^{-2\varphi_g}). \tag{4.7}$$

Knowing this, it is easy to define the wave functions in the momentum realization by using the transformation inverse to (2.1), which can be written in terms of the dimensionless light-front variable η in (3.6), as

$$\psi^g(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \left(\frac{\hbar\omega}{mc^2}\,\eta\right)^{-i\bar{x}} \psi^g(x). \tag{4.8}$$

Indeed, substituting (4.7) into (4.8) and integrating the hypergeometric series termwise with the aid of

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, \Gamma(\mu + it)\, z^{-it} = z^{\mu} e^{-z}, \tag{4.9}$$

yields the explicit form of the wavefunctions in the momentum realization as

$$\psi^g(p) = \frac{\sqrt{2\pi\hbar}}{mc}(-i\eta e^{\varphi_g})^{\nu} \exp(i\eta e^{\varphi_g})\, {}_1F_1(\nu - i\kappa; 2\nu; -2i\eta \sinh\varphi_g), \tag{4.10}$$

where ${}_1F_1(\alpha;\beta;z)$ is the confluent hypergeometric function.

Before closing this section we emphasize that the convenience of using the parametrizations $g = \pm mc\omega \cosh\varphi_g$ and $g = mc\omega\cos\phi_g$ for the continuous and discrete cases of the previous and this Section, respectively, is that they can be transformed into each other by the formal relation $\varphi_g = i\phi_g$. Under this substitution, the formulas (4.7) and (4.10) yield, up to a common normalization constant, the explicit form of the wavefunctions for the discrete spectrum (3.7) and (3.6) respectively, after the corresponding replacement $\kappa = -i(n+\nu)$, $n = 0, 1, 2, \ldots$. To verify this one needs the explicit formulas for the generalized Laguerre and Meixner-Pollaczek polynomials [24]

$$L_n^{\mu-1}(z) = \frac{(\mu)_n}{n!}\, {}_1F_1(-n;\mu;z), \tag{4.11a}$$

$$P_n^{\lambda}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} F(-n, \lambda + ix; 2\lambda; 1 - e^{-2i\phi}). \tag{4.11b}$$

In the special case when $|g| = mc\omega$, the initial equation (2.12*b*) has the form $\hbar\omega(K_0 \pm K_1)\psi^g(x) = E\psi^g(x)$, and the energy spectrum is again continuous and positive, *i.e.*, $E > 0$ (*cf.* [25], [26]).

5 Appendix

In this Appendix we present some formulas for Hermite, Meixner-Pollaczek, Meixner, and Charlier polynomials which are used in the body of the text, and some interesting relations that follow from them.

1. **The Meixner-Pollaczek polynomials** $P_n^{\nu}(x;\varphi)$ [24] are defined on the full real line $-\infty < x < \infty$ by recurrence relation

$$(n+1)P_{n+1}^{\nu}(x;\varphi) = 2[(n+\nu)\cos\varphi + x\sin\varphi]P_n^{\nu}(x;\varphi) - (n-1+2\nu)P_{n-1}^{\nu}(x;\varphi),$$

$$P_0^{\nu}(x;\varphi) = 1, \qquad P_{-1}^{\nu}(x;\varphi) = 0, \qquad n = 0, 1, 2, \ldots \tag{A.1}$$

with the parameters $\nu > 0$ and $0 < \varphi < \pi$.

The polynomials $P_n^{\nu}(x;\varphi)$ with different values of the parameter φ are related to each other as

$$P_n^{\nu}(x;\varphi_1) = \left(\frac{\sin\varphi_1}{\sin\varphi_2}\right)^n \sum_{k=0}^{n} \binom{n+2\nu-1}{k} \left(\frac{\sin(\varphi_2-\varphi_1)}{\sin\varphi_1}\right)^k P_{n-k}^{\nu}(x;\varphi_2). \tag{A.2}$$

From $(A.2)$ with the aid of the limit formula

$$\lim_{\nu\to\infty} \nu^{-n/2} P_n^{\nu}\left(\sqrt{\nu}\, x; \arccos\frac{y}{\sqrt{\nu}}\right) = \frac{1}{n!} H_n(x+y), \tag{A.3}$$

which can be proven by induction from $(A.1)$, one obtains (*cf.* [27], p. 388, problem 68)

$$H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} (2y)^k H_{n-k}(x). \tag{A.4}$$

This relation for the Hermite polynomials can be verified by means of the generating function

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) = \exp(2xz - z^2)$$

and Cauchy's rule for the multiplication of two infinite series, *i.e.*,

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} b_k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}.$$

Exactly in the same way, by using the generating function

$$\sum_{n=0}^{\infty} P_n^{\nu}(x;\varphi)\, z^n = (1 - ze^{i\varphi})^{ix-\nu}(1 - ze^{-i\varphi})^{-ix-\nu}, \quad |z| < 1,$$

one can prove the addition formula for the Meixner-Pollaczek polynomials

$$\sum_{k=0}^{n} P_k^{\nu}(x;\varphi) P_{n-k}^{\mu}(y;\varphi) = P_n^{\nu+\mu}(x+y;\varphi). \tag{A.5}$$

2. The Meixner and Charlier polynomials.

The unitarity property $S_g\, S_g^{\dagger} = 1$ of the operator $S_g = \exp(i\theta_g K_2)$, $\theta_g = \frac{1}{2}\ln[(mc\omega - g)/(mc\omega + g)]$, in (2.17) leads to the known orthogonality relation for the Meixner polynomials of discrete variable

$$M_n(x;\beta,c) = F(-n,-x;\beta;1-c^{-1}),$$

while the condition $(S_g^{\dagger})_{m,n} = (S_g)_{n,m}$ implies the duality property $M_m(n;\beta,c) = M_n(m;\beta,c)$ of these polynomials. From the equality $\psi_n^{g_2}(x) = S_{g_2} S_{g_1}^{\dagger}\, \psi_n^{g_1}(x)$ follows the addition formula

$$\sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} (\tanh\frac{1}{2}\theta_1 \tanh\frac{1}{2}\theta_2)^k M_k(n;\nu,\tanh^2\frac{1}{2}\theta_1)\, M_k(m;\nu,\tanh^2\frac{1}{2}\theta_2) \tag{A.6}$$

$$= (-1)^n \left(\frac{\cosh\frac{1}{2}\theta_1 \cosh\frac{1}{2}\theta_2}{\cosh\frac{1}{2}(\theta_2-\theta_1)}\right)^{\nu} \frac{(\tanh\frac{1}{2}(\theta_2-\theta_1))^{m+n}}{(\tanh\frac{1}{2}\theta_1)^n (\tanh\frac{1}{2}\theta_2)^m} M_n(m;\nu,\tanh^2\frac{1}{2}(\theta_2-\theta_1)).$$

Since the Charlier polynomials $C_n(x;a)$ can be obtained from the Meixner polynomials by the limit formula

$$\lim_{\beta\to\infty} M_n(x;\beta,a\beta^{-1}) = C_n(x;a), \tag{A.7}$$

then $(A.6)$ yields, in this limit, the addition rule

$$\sum_{k=0}^{\infty} \frac{(ab)^k}{k!} C_k(n;a^2)\, C_k(m;b^2) = (-1)^n e^{ab} \frac{(a-b)^{n+m}}{a^n b^m} C_n(m;(a-b)^2). \tag{A.8}$$

When $n = 0$ (or $m = 0$), $(A.6)$ and $(A.8)$ coincide with the known generating functions for the Meixner and Charlier polynomials, respectively. When $\theta_1 = \theta_2$ and $a = b$, these formulas lead to the orthogonality relations for the same polynomials.

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