Interpolation for Solutions of the Helmholtz Equation

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We study the interpolation problem for solutions of the two-dimensional Helmholtz equation, which are sampled along a line. The data are the function values and the normal derivatives at a discrete set of point sensors. A *wave transform* is used, analogous to the common Fourier transform. The *inverse* wave transform defines the Hilbert space for oscillatory Helmholtz solutions. We thereby introduce an interpolant that has some advantages over the usual sinc x in the Whittaker–Shannon sampling in one dimension; in particular, coefficients of the two-dimensional solution are invariant under translations and rotations of the sampling line. The analysis is relevant for the optical sampling problem by sensors on a screen.

I. INTRODUCTION

Two complementary problems of wave optics and holographic design are: to measure a scalar wavefield at points on a screen to extract information about the field in all of space, and to produce such a field through a phase-and-amplitude controlled source on the screen. In both problems, the approximation of the wavefield by data on a finite number of point sensors is of practical interest [1]. We consider here the case when the wavefield is standing and monochromatic, so it obeys the Helmholtz equation.

In two space dimensions $\mathbf{r} = (x, y)$ the wave equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(x, y; t) = 0, \tag{1.1}$$

where t is time and c is the speed of light. Monochromatic solutions of a fixed real wavenumber $k \in \mathcal{R}$ (in units of inverse length) are of the form

$$\Psi(x, y; t) = \psi(x, y)\ \exp(-ickt), \tag{1.2}$$

where the function $\psi(x, y)$ is a solution of the *Helmholtz* equation

$$\mathcal{H}\,\psi(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\psi(x, y) = 0. \tag{1.3}$$

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The Helmholtz equation is a well-known elliptic differential equation whose plane-wave solutions are

$$\phi_{\mathbf{k}}(x,y) = \frac{1}{2\pi}\exp i(k_x x + k_y y) = \frac{1}{2\pi}\exp i\mathbf{k}\cdot\mathbf{r}, \qquad k_x^2 + k_y^2 = k^2. \tag{1.4}$$

These solutions represent plane waves propagating in the direction of the two-vector $\mathbf{k} = (k_x, k_y)$ that has length $|\mathbf{k}| = k$. The degree of freedom of this vector is in the angle φ with the y-axis, so we may use polar coordinates and write

$$k_x = k\sin\varphi, \qquad k_y = k\cos\varphi, \qquad \phi \in (-\pi, \pi]. \tag{1.5}$$

We remark that the Helmholtz equation, (1.3), may also exhibit unbounded solutions, such as $\exp(\kappa_x x + ik_y y)$ with κ_x and k_y real and related by $k_y^2 - \kappa_x^2 = k^2$. In what follows we shall consider only solutions of the Helmholtz equation that are bounded and have a well-defined Fourier transform, since we are interested in expressing the solution as a generalized linear combination of plane waves (1.4). Henceforth, whenever we refer to the solutions of the Helmholtz equation we imply they are *oscillatory* solutions of this kind. Their values and normal derivatives on a line are *initial conditions* for such solutions.

In Section II we introduce the *wave transform* [2], [3], closely related to the Fourier transform, which maps the field functions and its normal derivative at a linear screen onto the Hilbert space of square-integrable functions on the circle. Section III uses the inverse transform to define a Hilbert space of functions on the screen that has a nonlocal measure, and whose norm has the form of energy [4]. Having a Hilbert space, Section IV applies a classical theorem to approximate and interpolate solutions when the data are on a finite set of points with the analogue of the criterion of minimal energy. In the concluding Section V we discuss the improvement this approximation strategy has over the well-known Nyquist sampling by $\operatorname{sinc} x = x^{-1}\sin x$ functions, and claim the usefulness of the Helmholtz interpolant to reconstruct the full field.

II. HELMHOLTZ WAVE SYNTHESIS

Since the Helmholtz equation is linear, we may write solutions as generalized linear combinations of plane waves. We first give a relation between this generalized plane-wave expansion and its generalized two-dimensional Fourier transform. When the latter exists, Fourier analysis of $\psi(\mathbf{r})$ is

$$\tilde{\psi}(\mathbf{g}) = (\mathcal{F}^{-1}\psi)(\mathbf{g}) = \frac{1}{2\pi}\int_{\Re^2} d^2\mathbf{r}\,\psi(\mathbf{r})\exp(-i\mathbf{g}\cdot\mathbf{r}), \tag{2.1}$$

with $\mathbf{g} = (g_x, g_y) \in \Re^2$. The direct transform, Fourier synthesis of the wave field, is

$$\psi(\mathbf{r}) = (\mathcal{F}\tilde{\psi})(\mathbf{r}) = \frac{1}{2\pi}\int_{\Re^2} d^2\mathbf{g}\,\tilde{\psi}(\mathbf{g})\exp(i\mathbf{g}\cdot\mathbf{r}). \tag{2.2}$$

Thus we state the following theorem.

Theorem 2.1. *If $\psi(\mathbf{r}), \mathbf{r} = (x,y) \in \Re^2$ is a solution of the two-dimensional Helmholtz equation, which has a Fourier transform $\tilde{\psi}(\mathbf{g})$, then, for $\psi^\circ(\varphi)$, a function on the circle S,*

$$\psi(x,y) = \sqrt{\frac{k}{2\pi}}\int_S d\varphi\,\psi^\circ(\varphi)\exp ik(x\sin\varphi + y\cos\varphi). \tag{2.3}$$

Proof. Acting with $\mathcal{H}$ in (1.3) on (2.2) yields zero. Commuting $\mathcal{H}$ into the integral thus leads to

$$\int_{\Re^2} d^2\mathbf{g}\left(k^2 - g_x^2 - g_y^2\right)\tilde{\psi}(\mathbf{g})\exp(i\mathbf{g}\cdot\mathbf{r}) = 0. \tag{2.4}$$

Changing to polar coordinates $(g_x, g_y) = [g, \vartheta]$, with $g_x = g\sin\vartheta$, $g_y = g\cos\vartheta$, this is

$$\int_0^\infty g\,dg \int_S d\vartheta(k^2 - g^2)\tilde{\psi}[g,\vartheta]\exp[ig(x\sin\vartheta + y\cos\vartheta)] = 0. \tag{2.5}$$

If $\psi(\mathbf{r})$ is not zero, the generalized solution to this equation has the generic form

$$\tilde{\psi}[g,\vartheta] = \frac{\delta(k-g)}{k}\psi^\circ(\vartheta), \tag{2.6}$$

where $\psi^\circ(\vartheta)$ is a function over the circle S_1. Replacement of (2.6) into (2.2) leads to (2.3). Q.E.D.

We shall denote by $\mathcal{W}$ the *wave synthesis* given by (2.3). It is an integral transform from functions on the circle $\psi^\circ(\vartheta) \in F_S$ onto solutions of the Helmholtz equation $\psi(x,y) \in F_{\mathcal{H}}$,

$$\mathcal{W}: F_S \mapsto F_{\mathcal{H}}, \tag{2.7}$$

$$\psi(x,y) = (\mathcal{W}\psi^\circ)(x,y) = \sqrt{\frac{k}{2\pi}}\int_S d\vartheta\psi^\circ(\vartheta)\exp ik(x\sin\vartheta + y\cos\vartheta). \tag{2.8}$$

This transform can be inverted, and the richness of known results for the Hilbert space $F_S = \mathcal{L}^2(S)$ applied to the image space of solutions $F_{\mathcal{H}}$. This is achieved through the following theorem.

Theorem 2.2. *If $\psi = \mathcal{W}\psi^\circ$ is as given by (2.8), then*

$$\psi^\circ(\vartheta) = \frac{\sigma}{2}\sqrt{\frac{k}{2\pi}}\int_{\Re} dx\left[\psi(x)\cos\vartheta + \frac{1}{ik}\psi'(x)\right]\exp(-ikx\sin\vartheta),$$

$$\textit{for } \sigma = \text{sign}\cos\vartheta, \tag{2.9}$$

where $\psi(x) = \psi(x,y)|_{y=0}$ is the Helmholtz solution at the line $y = 0$ and $\psi'(x,0) = \partial\psi(x,y)/\partial y|_{y=0}$ is its normal derivative across that line. When $\vartheta = \pm\frac{1}{2}\pi$, this holds in the sense of the average limit.

Proof. From the wave synthesis (2.3)–(2.8) we have

$$\psi(x) = \psi(x,y)|_{y=0} = \sqrt{\frac{k}{2\pi}}\int_S d\vartheta\psi^\circ(\vartheta)\exp(ikx\sin\vartheta), \tag{2.10a}$$

$$\psi'(x) = \frac{\partial\psi(x,y)}{\partial y}\bigg|_{y=0} = ik\sqrt{\frac{k}{2\pi}}\int_S d\vartheta\cos\vartheta\psi^\circ(\vartheta)\exp(ikx\sin\vartheta). \tag{2.10b}$$

Now change variables to two charts of $p = \sin\vartheta$ distinguished by $\sigma = $ sign $\cos\vartheta$, so $d\vartheta = \sigma dp/\sqrt{1-p^2}$ for $\vartheta \neq \pm\frac{1}{2}\pi$, and indicate $\psi^{\sigma=\pm}(p) = \psi^\circ(\vartheta)$ for "forward" ($\sigma = +$, i.e., $|\vartheta| < \frac{1}{2}\pi$) and 'backward' ($\sigma = -$, i.e., $|\pi - \vartheta| < \frac{1}{2}\pi$) chart components, respectively. The integral over the circle thus decomposes into $\int_S d\vartheta\, \psi^\circ(\theta) = \int_{-1}^{1} dp(1-p^2)^{-1/2}\psi^+(p) - \int_1^{-1} dp(1-p^2)^{-1/2}\psi^-(p)$, and we should take care to reverse the integration range in the last integral. Then we extend the integral to the whole real line stating explicitly that the integrand function $(1-p^2)^{-1/2}$ has support on the interval $(-1, 1)$, i.e., is zero for $|p| \geq 1$. When $\psi^\circ(\vartheta)$ is continuous at $\vartheta = \pm\frac{1}{2}\pi$, the singularity due to the factor $(1-p^2)^{-1/2}$ at the endpoints is integrable. The two integral transforms we set out to solve above thus become

$$\psi(x) = \sqrt{\frac{k}{2\pi}} \int_{\Re} \frac{dp}{\sqrt{1-p^2}} [\psi^+(p) + \psi^-(p)] \exp(ikpx), \tag{2.11a}$$

$$\psi'(x) = ik\sqrt{\frac{k}{2\pi}} \int_{\Re} dp[\psi^+(p) - \psi^-(p)] \exp(ikpx). \tag{2.11b}$$

We now apply the Fourier integral theorem to solve for the integrands,

$$\psi^+(p) = \frac{1}{2}\sqrt{\frac{k}{2\pi}} \int_{\Re} dx\left[\psi(x)\sqrt{1-p^2} + \frac{1}{ik}\psi'(x)\right]\exp(-ikxp), \tag{2.12a}$$

$$\psi^-(p) = \frac{1}{2}\sqrt{\frac{k}{2\pi}} \int_{\Re} dx\left[\psi(x)\sqrt{1-p^2} - \frac{1}{ik}\psi'(x)\right]\exp(-ikxp). \tag{2.12b}$$

The result given in (2.9) follows when we replace $\sqrt{1-p^2} = \sigma \cos\vartheta$ to reconstruct the single-chart functions $\psi^\circ(\vartheta)$. Q.E.D.

The wave synthesis operator $\mathcal{W}$ in (2.8) is thus an integral transform between functions $\psi^\circ \in F_S$ on the circle, and initial values $\{\psi(x), \psi'(x)\} \in F_{\mathcal{H}}$ of solutions of the Helmholtz equation on the x-axis. The essential support of the latter is the full real line. We have thus shown that the wave transform can be inverted into an operator $\mathcal{W}^\dagger$ that is represented by the integral (2.9). (We do not call it the *inverse* yet because the domain and range of this operator will be studied in the following section.)

Remark 2.1. The wavenumber parameter k is fixed and related to the wavelength through $\lambda = 2\pi/k$. A dimensional check on the preceding equations is useful: the units of λ are length (say, microns). Indicating $[\lambda] = L$, we have $[k] = L^{-1}$; we may select the function on the circle to be dimensionless, $[\psi^\circ] = 1$; since $[p] = 1$ and $[x] = L$, it follows from (2.3) and (2.10) that

$$[\psi] = L^{-1/2} \quad \text{and} \quad [\psi'] = L^{-3/2}. \tag{2.13}$$

Recall that also in quantum mechanics, the physical wavefunctions are not without units: in one dimension they have units of $L^{-1/2}$ so that inner products among them, with integration over the space variable, are pure numbers.

Let us represent the Helmholtz solution initial values by a column two-vector

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}. \tag{2.14}$$

The operator $\mathcal{W}$ in (2.8) is then represented by a column 2-vector, and $\mathcal{W}^\dagger$ in (2.9) by a row 2-vector of integral kernels. Out of a function on the circle, $\mathcal{W}$ yields its *wave* *synthesis:*

$$\Psi(x) = (\mathcal{W}\psi^\circ)(x) = \int_S d\vartheta\ \mathbf{W}(x,\vartheta)\psi^\circ(\vartheta), \tag{2.15a}$$

$$\mathbf{W}(x,\vartheta) = \sqrt{\frac{k}{2\pi}}\begin{pmatrix} 1 \\ ik\cos\vartheta \end{pmatrix}\exp(ikx\sin\vartheta). \tag{2.15b}$$

Wave analysis of the Helmholtz solution is provided by $\mathcal{W}^\dagger$,

$$\psi^\circ(\vartheta) = (\mathcal{W}^\dagger\Psi)(\vartheta) = \int_{\Re} dx\ \mathbf{W}^{(\dagger)}(\vartheta,x)\Psi(x), \tag{2.16a}$$

$$\mathbf{W}^{(\dagger)}(\vartheta,x) = \frac{\sigma}{2}\sqrt{\frac{k}{2\pi}}\left(\cos\vartheta, \quad \frac{1}{ik}\right)\exp(-ikx\sin\vartheta). \tag{2.16b}$$

Wave analysis and synthesis are the Helmholtz analogues of the direct and inverse common Fourier transform.

Remark 2.2. We use the literal $x \in \Re$ for the *position coordinate* one a one-dimensional optical *screen.* In geometric optics for a homogeneous medium of unit refractive index, the variable $p = \sin\vartheta \in [-1,1]$ that appeared in the proof of Theorem 2.2 is the canonically conjugate *momentum* coordinate.

Remark 2.3. It may seem that one can choose the two initial functions, $\psi(x)$ and $\psi'(x)$, freely. But this is not so, because these are functions built out of Fourier synthesis of waves whose wavenumbers measured on the screen (x-axis) are $|k\sin\vartheta| \le k$, i.e., their spectrum is compact and the component wavelengths are bounded from below by $2\pi/k$. Gaussian beams are very much used in Fourier optics; yet an initial field profile $\simeq \exp(-x^2/2w)$ of width w on the screen does **not** stem from a solution of the Helmholtz equation, because the Fourier transform of such a function is another Gaussian in wavenumber k of width $1/w$, whose support is **not** bounded.

Example 2.1. When the function on the circle S is the Dirac δ distribution,

$$\delta^\circ_\alpha(\vartheta) = \delta(\vartheta - \alpha), \tag{2.17a}$$

then the corresponding Helmholtz solutions is a plane wave whose normal forms an angle α with the y-axis

$$\Delta_\alpha(x) = \begin{pmatrix} \Delta\alpha(x) \\ \Delta'_\alpha(x) \end{pmatrix} = \mathbf{W}(x,\alpha) = \sqrt{\frac{k}{2\pi}}\begin{pmatrix} \exp(ikx\sin\alpha) \\ ik\cos\alpha\exp(ikx\sin\alpha) \end{pmatrix}. \tag{2.17b}$$

Notice the obliquity factor $\cos\alpha$ of the normal derivative; it changes sign beyond $\alpha = \pm\frac{1}{2}\pi$, i.e., for 'backward' rays.

Example 2.2. When the functions on the circle S are the circular harmonics

$$v^\circ_m(\vartheta) = \frac{1}{\sqrt{2\pi}}\exp im\vartheta, \qquad m = 0, \pm 1, \pm 2, \ldots, \tag{2.18a}$$

the corresponding Helmholtz solutions are

$$\mathbf{Y}_m(x) = \begin{pmatrix} \mathrm{Y}_m(x) \\ \mathrm{Y}'_m(x) \end{pmatrix} = \begin{pmatrix} k^{1/2}(-1)^m J_m(kx) \\ ik^{3/2}m(-1)^{m+1}\dfrac{J_m(kx)}{kx} \end{pmatrix}, \tag{2.18b}$$

where $J_m(z) = (-1)^m J_{-m}(z)$ is the Bessel function of integer order m.

The correspondence between circular harmonics and Bessel functions under the Helmholtz wave transform means that if we know the Fourier synthesis coefficients $\{\psi_m\}_{m\in Z}$ (Z is the set of integers) of the function $\psi^\circ \in F_S$,

$$\psi^\circ(\vartheta) = \frac{1}{\sqrt{2\pi}} \sum_{m\in Z} \psi_m e^{im\vartheta}, \tag{2.19a}$$

then the corresponding Helmholtz initial functions will be given by the series

$$\Psi(x) = \sqrt{\frac{k}{2\pi}} \sum_{m\in Z} \psi_m (-1)^m \begin{pmatrix} J_m(kx) \\ -ikm\dfrac{J_m(kx)}{kx} \end{pmatrix}. \tag{2.19b}$$

Example 2.3. The wave transforms of the two basic trigonometric functions are

$$\cos m\vartheta = \frac{1}{2}(v_m^\circ + v_{-m}^\circ)(\vartheta) \overset{W}{\mapsto} \sqrt{2\pi k} \begin{pmatrix} \dfrac{1}{2}[1 + (-1)^m] J_m(kx) \\ ikm\dfrac{1}{2}[1 - (-1)^m]\dfrac{J_m(kx)}{kx} \end{pmatrix}, \tag{2.20a}$$

$$\sin m\vartheta = \frac{1}{2i}(v_m^\circ - v_{-m}^\circ)(\vartheta) \overset{W}{\mapsto} \sqrt{2\pi k} \begin{pmatrix} i\dfrac{1}{2}[1 - (-1)^m] J_m(kx) \\ -km\dfrac{1}{2}[1 + (-1)^m]\dfrac{J_m(kx)}{kx} \end{pmatrix}. \tag{2.20b}$$

In particular,

$$1 \overset{W}{\mapsto} \sqrt{2\pi k}\begin{pmatrix} J_0(kx) \\ 0 \end{pmatrix}, \qquad \cos\vartheta \overset{W}{\mapsto} ik\sqrt{2\pi k}\begin{pmatrix} 0 \\ \dfrac{J_1(kx)}{kx} \end{pmatrix}. \tag{2.20c, d}$$

That is, the constant function over the circle corresponds to a superposition of waves from all directions with the same amplitude and phase that add up to a J_0 Bessel function on the screen and have zero normal derivative; if a superposition of waves is made with a cosine amplitude distribution (subtracting backward waves from forward ones) and a $\frac{1}{2}\pi$ phase, we obtain the zero Helmholtz solution whose normal derivative is a $J_1(\xi)/\xi$-function.

Example 2.4. The inverse wave transforms of the *sinus cardinalis* function $\operatorname{sinc} z = z^{-1} \sin z$, wisely used in sampling theory is, for each component,

$$\zeta_k^0(\vartheta) = \frac{1}{2}\sqrt{\frac{\pi}{2k}}\,|\cos\vartheta| \overset{W}{\mapsto} \begin{pmatrix} \operatorname{sinc} kx \\ 0 \end{pmatrix}, \tag{2.21a}$$

$$\zeta_k^1(\vartheta) = \frac{1}{2}\sqrt{\frac{\pi}{2k}}\,\frac{\sigma}{ik} \overset{W}{\mapsto} \begin{pmatrix} 0 \\ \operatorname{sinc} kx \end{pmatrix}, \tag{2.21b}$$

where we recall that $\sigma = \operatorname{sign}\cos\vartheta$.

Remark 2.4. If we are required to work with real Helmholtz solutions $\Psi(x)$, from the above equations we can see the restrictions imposed on the corresponding wave spectral function $\psi^\circ(\vartheta)$. Indeed, consider the trigonometric Fourier expansion of the latter, in sine and cosine functions, with coefficients that have a real and imaginary part. Consider first real coefficients, from (2.20) we see that for the Helmholtz solution $\psi(x)$ to be real, no $\sin m\vartheta$ terms with m odd should be present—for m even, the factor $\frac{1}{2}[1 - (-1)^m]$ is zero, and for the normal derivative $\psi'(x)$ to be real, no $\cos m\vartheta$ terms with m odd should appear. That is, for $\Psi(x)$ to be real, the real part of $\psi^\circ(\vartheta)$ should contain no odd harmonics. A similar argument shows that the imaginary part of $\psi^\circ(\vartheta)$ must not contain the even harmonics. A real $\psi^\circ(\vartheta)$ that is of even parity under the reflection $\vartheta \leftrightarrow -\vartheta$ of the circle will synthesize an acceptably real Helmholtz initial function $\Psi(x)$ that is even under $x \leftrightarrow -x$ screen reflection. The initial conditions $\psi(x)$ real and $\psi'(x) = 0$ correspond to $\Re\psi^\circ(\vartheta) = 0$ and $\Im\psi^\circ(\vartheta)$ odd. Finally, $\psi(x) = 0$ and $\psi'(x)$ real imply that $\Re\psi^\circ(\vartheta)$ is even and $\Im\psi^\circ(\vartheta = 0)$.

From (2.3) comes the following corollary.

Corollary 2.1. *When a solution $\psi(x,y)$ of the Helmholtz equation corresponds to a function over the circle $\psi^\circ(\vartheta)$, and the space is translated in x by χ and in y by η, the translated solution and initial values will correspond to the same $\psi^\circ(\vartheta)$, only multiplied by phases:*

$$\Psi(x - \chi) \overset{\mathcal{W}^\dagger}{\mapsto} \psi^\circ(\vartheta) \exp(-jk\,\chi \sin\vartheta) \tag{2.22a}$$

$$\begin{pmatrix} \psi(x,y)|_{y=\eta} \\ \left.\dfrac{\partial\psi(x,y)}{\partial y}\right|_{y=\eta} \end{pmatrix} \overset{\mathcal{W}^\dagger}{\mapsto} \psi^\circ(\vartheta)\exp(ik\eta\cos\vartheta). \tag{2.22b}$$

III. HELMHOLTZ HILBERT SPACE

The natural sesquilinear inner product of two Lebesgue square-integrable functions on the circle,

$$(\psi^\circ, \phi^\circ)_S = \int_S d\vartheta\, \psi^\circ(\vartheta)^* \phi^\circ(\vartheta), \tag{3.1}$$

defines the well known Hilbert space $\mathcal{L}^2(S)$. The wave transform $\mathcal{W}$ maps this space of functions on a space of *pairs* of functions (initial values of Helmholtz solutions) that we call Helmholtz space, $\mathcal{H}_k$, for fixed k. Its definition is

$$\mathcal{H}_k = \{\Psi \mid \mathcal{W}^\dagger\Psi \in \mathcal{L}^2(S)\}. \tag{3.2}$$

The inner product in $\mathcal{L}^2(S)$ induces an inner product in $\mathcal{H}_k$ with the following interesting property.

Theorem 3.1. *The Helmholtz space $\mathcal{H}_k$ is a Hilbert space of functions with the nonlocal inner product*

$$(\Psi, \Phi)_{\mathcal{H}_k} = \int_\Re dx \int_\Re dx'\, \Psi^\dagger(x)\mathbf{H}_k(x - x')\Phi(x'), \tag{3.3a}$$

$$\mathbf{H}_k(x - x') = \frac{1}{4}\begin{pmatrix} k\frac{J_1(k(x-x'))}{k(x-x')} & 0 \\ 0 & k^{-1}J_0(k(x - x')) \end{pmatrix}, \tag{3.3b}$$

where $\Psi^\dagger(x)$ is the row vector of two-functions that is transpose conjugate to $\Psi(x)$. This inner product is definite, i.e., *for the induced norm $|\Psi|_{\mathcal{H}_k} = (\Psi, \Psi)_{\mathcal{H}_k}$, $|\Psi|_{\mathcal{H}_k} \geq 0$ and $|\Psi|_{\mathcal{H}_k} = 0 \Longleftrightarrow \Psi = 0$.*

Proof. To find the integral measure in (3.3), we replace the integral vector forms of the wave transform in (2.16) into the $\mathcal{L}^2(S)$ inner product (3.1),

$$\begin{aligned}(\psi^\circ, \phi^\circ)_S &= \int_S d\vartheta \left[\int_{\mathfrak{R}} dx\, \mathbf{W}^{(\dagger)}(\vartheta, x)\Psi\right]^\dagger \left[\int_{\mathfrak{R}} dx'\, \mathbf{W}^{(\dagger)}(\vartheta, x')\Phi(x')\right] \\ &= \int_{\mathfrak{R}} dx \int_{\mathfrak{R}} dx'\, \Psi(x)^\dagger \left[\int_S d\vartheta\, \mathbf{W}^{(\dagger)}(\vartheta, x)^\dagger \mathbf{W}^{(\dagger)}(\vartheta, x')\right]\Phi(x'). \end{aligned} \tag{3.4a}$$

The sought measure will be thus

$$\mathbf{H}_k(x - x') = \frac{k}{8\pi}\int_S d\vartheta \begin{pmatrix} \cos^2\vartheta & \dfrac{1}{ik}\cos\vartheta \\ \dfrac{-1}{ik}\cos\vartheta & \dfrac{1}{k^2} \end{pmatrix} \exp(ik(x - x')\sin\vartheta). \tag{3.4b}$$

Due to the parity properties commented in Remark 2.3, we can use the standard integral

$$\int_S d\vartheta \cos^n\vartheta \exp(i\xi\sin\vartheta) = \begin{cases} 2\pi(n-1)!!\dfrac{J_{n/2}(\xi)}{\xi^{n/2}}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{3.4c}$$

where $m!! = (m-2)!!$, $1!! = 0!!, = 1$, and $(-1)!! = -1$. When $n = 0, 1, 2$, the expression (3.3b) for the measure follows. Since the Helmholtz wave transform is based on the Fourier transform, it is a unitary transformation between $\mathcal{L}^2(S)$ and the Hilbert space $\mathcal{H}_k$. This is their Parseval relation. Q.E.D.

The inner product (3.3) is *nonlocal* but is *homogeneous* and *isotropic,* i.e., it integrates over the range of the two functions, of x and x', but weights the product by functions of their absolute mutual distance, $\xi = k|x - x'|$. These functions are the Bessel functions $J_1(\xi)/\xi$ and $J_0(\xi)$, shown in Fig. 1. They are real, even, and centrally peaked; the second

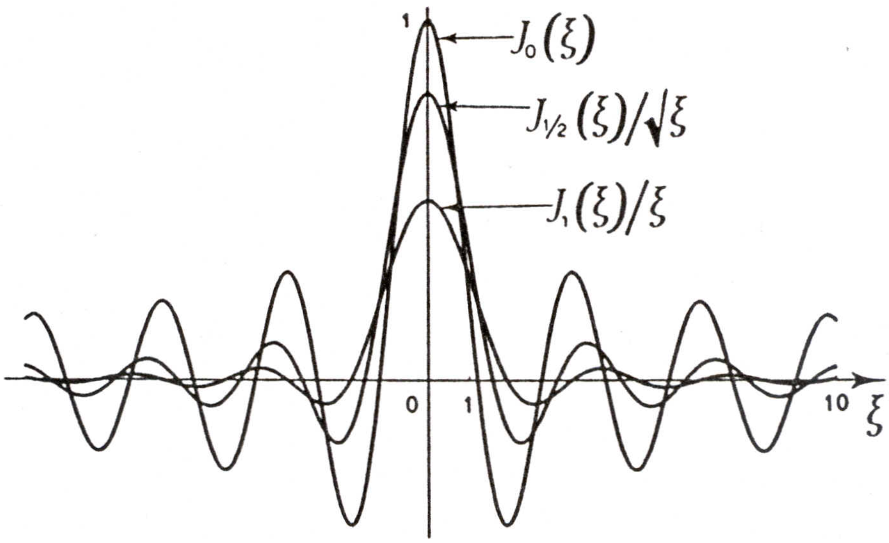

FIG. 1. Bessel functions $J_0(\xi)$, $J_1(\xi)/\xi$. and $\sqrt{\pi}$ sinc ξ.

is narrower than the first (see Remark 3.3, below). Their first simple zero appears at $\xi = j_{1,1} = 3.83171\ldots$ and $j_{0,1} = 2.40482\ldots$; although the functions change sign at their simple zeros, having negative values every other interval, the integral over growing $|\xi|$-intervals remains positive; as a *measure*, $\mathbf{H}_k(|x = x'|)\,dx\,dx'$ is positive definite. Finally, their asymptotic behavior at $|\xi| \to \infty$ is $\sim \xi^{-3/2}\cos(\xi - \frac{1}{4}\pi)$ and $\sim \xi^{-1/2}\cos(\xi - \frac{3}{4}\pi)$, respectively, with a common factor of $\sqrt{2/\pi}$.

The Helmholtz Hilbert space excludes functions that are not initial values of oscillatory Helmholtz solutions. If we propose a "bad" function *outside* $\mathcal{H}_k$, say $\exp(iKx)$ with $K > k$ for the first and/or second components, and place it in the integral (3.3a) in company with any other function, then performing the first integration with the measure functions yields zero. This is so because the integral is the Fourier transform of a function of compact support valuated where it is zero [cf. (3.4c)]. Therefore, "bad" functions are automatically orthogonal to the "good" solutions of the Helmholtz equation when we use the Helmholtz inner product, and their norm is zero. A complete orthonormal basis for the Hilbert space $\mathcal{H}_k$ is the set of harmonic functions $\{\Upsilon_m(x)\}_{m\in Z}$ in (2.18b), because they are the wave transforms of the well-known circular function basis of $\mathcal{L}^2(S)$ given in (2.18a); a Dirac basis is provided by the plane waves (2.17). The application of the wave analysis *and* synthesis operators $\mathcal{W}\mathcal{W}^\dagger$ to arbitrary (integrable) functions, projects to zero the complement subspace whose wavenumber exceeds k. Only on the space $\mathcal{H}_k$ of solutions of the Helmholtz equation, $\mathcal{W}^\dagger$ is the right inverse of the wave transform $\mathcal{W}$.

The action of operator $\mathcal{W} \circ \mathcal{W}^\dagger$ as projector is shown in Fig. 2, applied on a discontinuous function which is not symmetric about the origin. Its wave analysis finds the part of the spectrum with wavenumbers less than k; subsequent wave synthesis produces a solution of the Helmholtz equation, a low-pass filtered and aliased version of the original, its $\mathcal{H}_k$-content. See Appendix A for the discretized algorithm that generated the figures.

The Helmholtz norm $|\Psi|_{\mathcal{H}_k} = (\Psi, \Psi)_{\mathcal{H}_k}$ measures an intrinsic quantity of the solution field $\psi(x, y)$ of the Helmholtz equation because it is independent of the placement of the measurement screen—up to now the $y = 0$ line: if we translate or rotate the line in the x–y plane we get the same numerical result. This is contained in the following theorem.

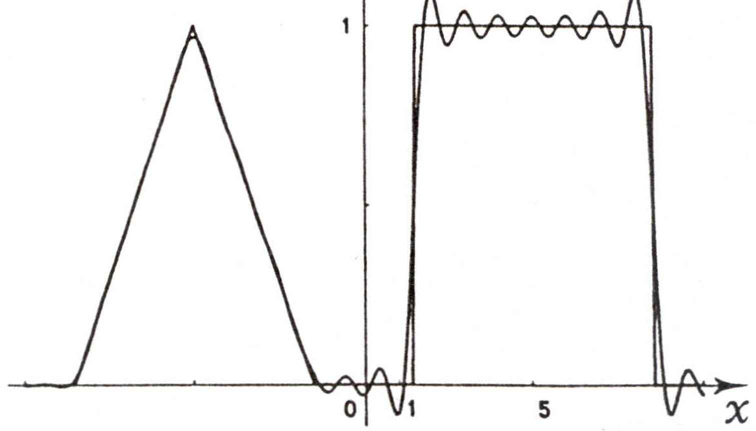

FIG. 2. An "arbitrary" function (composed of straight segments) under the action of the projection operator $\mathcal{W}\mathcal{W}^\dagger$ that turns it into the closest initial value for a Helmholtz solution corresponding to the wavenumber $k = 2\pi$, or wavelength $\lambda = 1$. Notice the Gibbs-like oscillations (*aliasing*) in the neighborhood of the discontinuities with the characteristic wavelength $\lambda = 1$.

Theorem 3.2. *The Helmholtz inner product (3.3) is invariant under Euclidean transformations of the plane.*

Proof. From (2.21), translations of $\psi(x,y)$ correspond to multiplication of $\psi^\circ(\vartheta)$ by phases. Rotations by τ correspond to a translation in the argument to $\psi^\circ(\vartheta - \tau)$. These are invariances of the $\mathcal{L}^2(S)$ inner product (3.1). The result follows for (3.3) from the Parseval relation. Q.E.D.

Remark 3.1. The inner product (3.3) was found by Steinberg and Wolf [4] searching for Euclidean invariants among general sesquilinear products. That construction process shows it is the *only* Euclidean invariant inner product.

Remark 3.2. The Helmholtz norm given by the sesquilinear form (3.3) is similar to the energy in a homogenous isotropic vibrating lattice or medium, of a disturbance $f(x,t)$ and its time derivative $\dot{f}(x,y)$. In this mechanical case, the energy has the form $E = \frac{1}{2}(f, Vf) + \frac{1}{2}(\dot{f}, M\dot{f})$, with a nondiagonal interaction operator V, a (diagonal) mass operator M, and (f,f) a local inner product (with single integration over x). The difference with the Helmholtz case is, of course, that the latter is invariant under two space translations and rotations. Moreover, the normal derivative of a Helmholtz solution is in a space direction, rather than in the time direction, where the M-term is kinetic energy. It is, thus, quite natural to have in the Helmholtz case a "mass" term that is not diagonal.

We can construct the *reproducing kernel* for the space $\mathcal{H}_k$ out of the complete orthonormal basis $\{\mathbf{Y}_m(x)\}_{m\in Z}$ in (2.18b). It is a 2×2 matrix function $K_\chi(x) = K(\chi, x)$ such that for every $\mathbf{\Psi}(x) \in \mathcal{H}_k$,

$$(\mathbf{K}_\chi, \mathbf{\Psi})_{\mathcal{H}_k} = \mathbf{\Psi}(\chi). \tag{3.5}$$

The reproducing kernel is the limit of the sum

$$K(\chi,x) = \sum_{m\in Z} \mathbf{Y}_m(x)\mathbf{Y}_m(\chi)^\dagger$$

$$= \sum_{m\in Z} \begin{pmatrix} kJ_m(k_\chi)J_m(kx) & ik^2 m \dfrac{J_m(k_\chi)}{k_\chi} J_m(kx) \\ -ik^2 m J_m(k_\chi)\dfrac{J_m(kx)}{kx} & k^3 m^2 \dfrac{J_m(k_\chi)}{k_\chi}\dfrac{J_m(kx)}{kx} \end{pmatrix}. \tag{3.6a}$$

Bessel functions satisfy convolution sums such as [5]

$$\sum_{m\in Z} J_{n+m}(\xi_1)J_m(\xi_2) = J_n(\xi_1 - \xi_2), \tag{3.6b}$$

and the recursion relation $mJ_m(\xi)/\xi = \frac{1}{2}[J_{m+1}(\xi) + J_{m-1}(\xi)]$. From here follows

$$\mathbf{K}(\chi,x) = \begin{pmatrix} kJ_0(k(\chi - x)) & 0 \\ 0 & k^3 \dfrac{J_1(k(\chi - x))}{k(\chi - x)} \end{pmatrix}. \tag{3.7}$$

Thus, in $\mathcal{H}_k$ the reproducing kernel is a well-behaved analytic diagonal matrix function.

To verify that (3.7) has indeed the stated reproducing kernel property (3.5), we note that the inner product with integral on $\xi = kx$ is

$$(\mathbf{K}_\chi, \mathbf{\Psi})_{\mathcal{H}_k} = \frac{1}{k^2}\int_{\Re} d\xi \int_{\Re} d\xi' \, \mathbf{K}_\chi(\xi)\mathbf{H}_k(\xi - \xi')\mathbf{\Psi}(\xi'). \tag{3.8a}$$

This contains, in each of the two components, the integral

$$\frac{1}{4}\int_{\Re} d\bar{\xi}\, J_0(\bar{\xi} - \bar{\chi})\,\frac{J_1(\bar{\xi})}{\bar{\xi}} = \frac{1}{2\pi}\int_{-1}^{1} dp \exp i\bar{\chi}p = \frac{1}{\pi}\frac{\sin\bar{\chi}}{\bar{\chi}}, \tag{3.8b}$$

for $\bar{\xi} = \chi - \xi$ and $\bar{\chi} = \chi - \xi'$. To prove the first equality, we may use (3.4c) for both Bessel functions to find, after exchanging the integrals, the standard Fourier representation of $\delta(\sin\vartheta - \sin\vartheta') = |\cos\vartheta|^{-1}[\delta(\vartheta - \vartheta') + \delta(\vartheta - (\pi - \vartheta))]$ inside the double circle integral. This eliminates one of the integrals; in the remaining integration we change variables to $p = \sin\vartheta$. In both the components of $(\mathbf{K}_\chi, \Psi)_{\mathcal{H}_k}$ we are left with an integral of the form

$$\frac{1}{2\pi}\int_{\Re} d\xi' \int_{-1}^{1} dp\, F(\xi')\exp(i(\chi - \xi')p) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} dp\, \tilde{F}(p)\exp i\chi p = F(\chi), \tag{3.8c}$$

with $F = \psi$ or ψ', because the Fourier transform $\tilde{F}(p)$ of initial values of solutions of the Helmholtz equation has support only on $\xi \in [-1, 1]$.

Remark 3.3. The reproducing kernel in the Hilbert space of functions that are Fourier transforms of $\mathcal{L}^2(-k, k)$ functions, is the well-known *sinus cardinalis* function

$$L_\chi(x) = \frac{1}{\pi}\frac{\sin k(x - \chi)}{k(x - \chi)} = \frac{1}{\pi}\operatorname{sinc} k(x - \chi) = \frac{J_{1/2}(k(x - \chi))}{\sqrt{2\pi k(k - \chi)}}. \tag{3.9}$$

This is an even function with a positive central peak and its two first zeros at $x = \pm\pi/k = \pm\frac{1}{2}\lambda$. Compare this with the Helmholtz space $\mathcal{H}_k$, whose reproducing kernel is a diagonal 2×2 matrix with functions $J_0(kx)$ and $(kx)^{-1}J_1(kx)$, and whose first simple zeros appear at $x \approx \pm 2.40482/k \approx \pm 0.38274\lambda$ and $x \approx \pm 3.83171/k \approx \pm 0.60983\lambda$, respectively. The widths of the Helmholtz reproducing kernel functions thus bracket the width of the Fourier sinc reproducing kernel. Applied to monochromatic light waves, this statement is germane to the relation between Helmholtz and Fourier optics [6]; in this and other properties [2], the value of a quantity in Fourier optics falls within the corresponding pair of values in Helmholtz optics.

IV. APPROXIMATION OF HELMHOLTZ SOLUTIONS

The classical orthogonal projection theorem [7] for Hilbert spaces allows us to find the unique solution to the *Helmholtz interpolation problem.*

Theorem 4.1. *Consider the Helmholtz Hilbert space $\mathcal{H}_k$ with inner product $(\Psi, \Phi)_{\mathcal{H}_k}$ given by (3.3) and reproducing kernel $\mathbf{K}(\chi, x)$ given by (3.7). Assume that a field $\Psi(x)$ has been sampled at N points on the screen $x_1, x_2, \ldots, x_N$, yielding numerical values $v_1 = \psi(x_1), \ldots, v_N = \psi(x_N)$ for the amplitudes and $v_1' = \psi'(x_1), \ldots, v_N' = \psi'(x_N)$ for the normal derivatives. Then there exists a unique interpolating function $\Pi\Psi \in \mathcal{H}_k$ such that norm is minimal*

$$(\Pi\Psi, \Phi)_{\mathcal{H}_k} = \inf_{\Phi \in \mathcal{H}_k} (\Phi, \Phi)_{\mathcal{H}_k}, \tag{4.1a}$$

and such that it reproduces the data values

$$(\mathbf{K}(x_j, \cdot), \Pi\Psi)_{\mathcal{H}_k} = \begin{pmatrix} v_j \\ v_j' \end{pmatrix}, \quad j = 1, \ldots, N. \tag{4.1b}$$

This $\Pi\Psi(x)$ is an oscillatory Helmholtz solution and is given explicitly by

$$\Pi\Psi(x) = \begin{pmatrix} \Pi\psi(x) \\ \Pi\psi'(x) \end{pmatrix} = \sum_{j=1}^{N} \begin{pmatrix} \gamma_j k J_0(k(x_j - x)) \\ \lambda'_j k^3 \dfrac{J_1(k(x_j - x))}{k(x_j - x)} \end{pmatrix}, \tag{4.2}$$

where the coefficients $\{\lambda_j\}_{j=1}^{N}$ and $\{\lambda'_j\}_{j=1}^{N}$ are the solutions of the following two nonsingular linear $N \times N$ systems of equations:

$$k\begin{pmatrix} 1 & J_0(\xi_1 - \xi_2) & \dots & J_0(\xi_1 - \xi_N) \\ J_0(\xi_2 - \xi_1) & 1 & \dots & J_0(\xi_2 - \xi_N) \\ \vdots & \vdots & \ddots & \vdots \\ J_0(\xi_N - \xi_1) & J_0(\xi_N - \xi_2) & \dots & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} \tag{4.3a}$$

$$k^3\begin{pmatrix} \dfrac{1}{2} & \dfrac{J_1(\xi_1 - \xi_2)}{\xi_1 - \xi_2} & \dots & \dfrac{J_1(\xi_1 - \xi_N)}{\xi_1 - \xi_N} \\ \dfrac{J_1(\xi_2 - \xi_1)}{\xi_2 - \xi_1} & \dfrac{1}{2} & \dots & \dfrac{J_1(\xi_2 - \xi_N)}{\xi_2 - \xi_N} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{J_1(\xi_N - \xi_1)}{\xi_N - \xi_1} & \dfrac{J_1(\xi_N - \xi_2)}{\xi_N - \xi_2} & \dots & \dfrac{1}{2} \end{pmatrix} \begin{pmatrix} \lambda'_1 \\ \lambda'_2 \\ \vdots \\ \lambda'_N \end{pmatrix} = \begin{pmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_N \end{pmatrix} \tag{4.3b}$$

where $\xi_j = kx_j$.

The interpolant obtained by (4.2) will be called the *Helmholtz interpolant*. This is the unique norm-minimizing function that satisfies the interpolation conditions.

The two matrices, obtained from the reproducing kernel diagonal matrix elements, are symmetric, diagonal-peaked matrices that are invertible, and their determinant is nonzero for any finite N. From a computational point of view, however, for a large number of data points, they can be numerically ill-conditioned.

According to Gershgorin's theorem, if a matrix has all diagonal elements of the same order, and these elements are large compared with the off-diagonal ones, then the matrix has a small *condition number* [8]. It is evident that the matrices (4.3a) and (4.3b) satisfy the first condition. Regarding the second condition, we see that the elements of these two matrices satisfy $|M_{i,i}| > |M_{i,j}|$ for $i, j = 1, \dots, N$ and, as we noted before, the asymptotic behavior of the elements is $J_0(\xi) \sim O(\xi^{-1/2})$ and $J_1(\xi)/\xi \sim O(\xi^{-3/2})$. The condition number of Gershgorin's theorem gives an estimate of the numerical accuracy of the matrix inversion by calculating the determinant (indicated by $\|\cdot\|$), $\eta = \|\mathbf{M}\|\,\|\mathbf{M}^{-1}\|$. which according to the above remarks we expect to be good.

As a graphical example, in Fig. 3 we show the Helmholtz interpolant for $N = 11$ sensors on a linear grid of points equally spaced by $\varepsilon = \xi_j - \xi_{j+1} = 1$, i.e., one reduced wavelength measured on the screen: $x_j - x_{j+1} = \varepsilon/k = \varepsilon\lambda/2\pi$, extending between $x_1 = -N\lambda/2\pi$ and $x_{2N+1} = N\lambda/2\pi$. The values were obtained from the projection of the rectangle function on the Helmholtz space through $\mathcal{W} \circ \mathcal{W}^\dagger$ as discussed in Section III. Two lines are shown: one for data representing the field value, and one for data representing the normal derivatives at the screen.

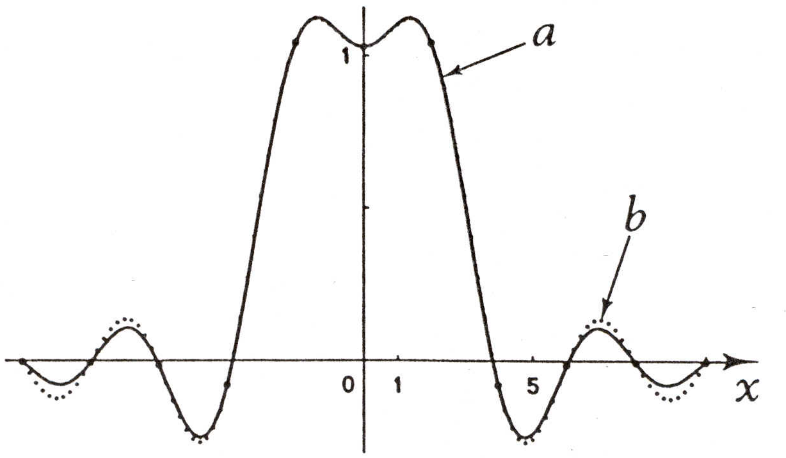

FIG. 3. A sample of 11 data points (taken from a Helmholtz initial value) is used to obtain the Helmholtz interpolant for wavenumber $k = 2\pi$ (wavelength $\lambda = 1$). The continuous line interpolates data as initial values of the Helmholtz field solution, and the dotted line interpolates the data as representing the normal derivative.

V. CONCLUDING REMARKS

We have presented Helmholtz wave analysis and synthesis, found the associated Helmholtz Hilbert space with the nonlocal inner product of Theorem 3.1 and shown the solution to the corresponding interpolation problem in Theorem 4.1. Since Eq. (4.3) for the expansion coefficients require the inversion of $N \times N$ numerical matrices, we should compare these results with the simpler and common wave synthesis of classical optical sampling interpolation theory [6]; the latter expands in displaced *sinus cardinalis* functions,

$$\begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = \sum_{j \in Z} \begin{pmatrix} u(x_j) \\ u'(x_j) \end{pmatrix} \operatorname{sinc} k(x - x_j), \tag{5.1}$$

treating the solution values on the screen and their normal derivatives on the screen as two totally *independent* functions. In (5.1), the matrices to be inverted in the *finite-N* case (according to the analogue of Theorem 4.1) are diagonal when the x_j's are placed π/k apart, i.e., at a Nyquist sample. Their computation is less expensive.

Our aim is this work has been not only to interpolate the initial values $\psi(x)$ and $\psi'(x)$, but also to approximate in the best possible way the full solution $\psi(x, y)$ of a Helmholtz wavefield thus determined by the initial data at sampling points x_j (that need not be a Nyquist sample). This we do through two operations:

a. Once the interpolated initial two-function $\Pi\Psi(x)$ has been obtained through the formulas (4.1)–(4.3) of Theorem 4.1, we perform the wave analysis (2.9)–(2.16) to find the approximate $\Pi\psi^\circ(\vartheta) = (\mathcal{W}^\dagger \Pi \Psi)(\vartheta)$, this is translated in y by (2.22b) into $\Pi\psi_y^\circ(\vartheta) = \Pi\psi^\circ(\vartheta)\exp(iky\cos\vartheta)$.
b. Then we perform wave synthesis (2.8)–(2.15) to finally obtain the interpolated 2-dimensional wavefield $\Pi\psi(x,y) = (\mathcal{W}\Pi\psi_y^\circ)(x)$.

Although there are many Hilbert spaces of functions whose Fourier transform is of compact support, which can provide interpolant functions for discrete data values, the interpolant (4.2) that we present in this article minimizes the norm (3.3) that we interpret as field energy. As pointed out in Remark 3.1, it is the only sesquilinear form that is invariant under Euclidean transformations. Consequently, any other inner product that we may use to interpolate the initial values will produce different approximations to the two-dimensional Helmholtz solution $\psi(x,y)$ depending on the translation and rotation of the screen. Only $\Pi\psi(x,y)$ is independent of the placement of our line of sensors.

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APPENDIX A. DISCRETIZATION FOR COMPUTATION

The wave analysis and synthesis (2.15)–(2.16) are integral transforms with a kernel $\exp(ik\sin\vartheta)$. There seem to be few elementary functions with exactly integrable wave transforms besides those already given in the text. Numerical integration methods are needed for the task of studying examples and interpreting data.

The graphs to be generated come from a "data" two-function $\Psi^D(x)$ (not necessarily Helmholtz initial values and normal derivatives on the screen), its wave analysis $\psi^\circ(\vartheta)$ (that filters out wavelengths smaller than $2\pi/k$), and the latter's wave synthesis $\Psi(x)$, which *is* an initial value of a solution of the Helmholtz equation for a fixed k, or wavelength $\lambda = 2\pi/k$. Figure 2 was computed through a simple numerical algorithm in FORTRAN, whose limiting assumption is that the "significant" part of the data function is within a symmetric finite interval $[-L, L]$. If so, then the integral over $x \in [-L, L]$ in (2.15) is approximated by a sum over the $2N+1$ equidistant points

$$x_m = \frac{L}{N}(m - N - 1), \quad m = 1,2,\ldots,2N+1, \quad \text{spaced by } \Delta x = \frac{L}{N}, \tag{A.1}$$

which include $x_{N+1} = 0$ and the endpoints. The integral over $\vartheta \in S$ is approximated by a similar sum over $2N+1$ equidistant points on the circle

$$\vartheta_j = \frac{2\pi}{2N+1}(j - N - 1), \quad j = 1,2,\ldots,2N+1, \quad \text{spaced by } \Delta\vartheta = \frac{2\pi}{2N+1}, \tag{A.2}$$

which include $\vartheta_{N+1} = 0$ and end in $\pm(\pi - \frac{1}{2}\Delta\,\vartheta)$. For continuous functions $F(x)$ and $F^\circ(\vartheta)$, the approximations of the integrals are

$$\int_{-\infty}^{\infty} dx\, F(x) \mapsto \sum_{m=1}^{2N+1} \Delta x\, F(x_m), \qquad \int_{-\pi}^{\pi} d\vartheta\, F^\circ(\vartheta) \mapsto \sum_{j=1}^{2N+1} \Delta\vartheta\, F(\vartheta_j). \tag{A.3}$$

For a fixed wavelength $\lambda = 2\pi/k$, the discretized wave analysis projector is, thus, a map from $\Psi^D(x_m) = \begin{pmatrix}\Psi^D(x_m)\\ \Psi^{D'}(x_m)\end{pmatrix} = \begin{pmatrix}\Psi^D_m\\ \Psi^{D'}_m\end{pmatrix}$ to $\psi^\circ(\vartheta_j) = \psi^\circ_j$,

$$\psi^\circ_j = \frac{L}{2N}\sqrt{\frac{k}{2\pi}}\sum_{m=1}^{2N+1}\left(|\cos\vartheta_j|\Psi^D_m + \frac{\operatorname{sign}(\cos\vartheta_j)}{ik}\Psi^{D'}_m\right)\exp(-ikx_m\sin\vartheta_j). \tag{A.4}$$

The discretized wave synthesis is then the map from ψ°_j to a "good" approximation of a Helmholtz solution, with "significant" support in $[-L, L]$, that is a two-function

$$\Psi(x_m) = \begin{pmatrix}\Psi_m\\ \Psi'_m\end{pmatrix} = \frac{\sqrt{2\pi k}}{2N+1}\sum_{j=1}^{2N+1}\psi^\circ_j\begin{pmatrix}1\\ ik\cos\vartheta_j\end{pmatrix}\exp(ik\,x_m\sin\vartheta_j). \tag{A.5}$$

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