Hidden symmetry and potential group of the Maxwell fish-eye

Alejandro Frank
Instituto de Ciencias Nucleares, UNAM, México City, Mexico

François Leyvraz
Instituto de Física, UNAM, Cuernavaca, Mexico

Kurt Bernardo Wolf
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, Cuernavaca, Mexico

(Received 26 March 1990; accepted for publication 16 May 1990)

The Maxwell fish-eye is an exceptional optical system that shares with the Kepler problem and the point rotor (mass point on a sphere) a hidden, higher rotation symmetry. The Hamiltonian is proportional to the Casimir invariant. The well-known stereographic map is extended to *canonical* transformations between of the phase spaces of the constrained rotor and the fish-eye. Their *dynamical* group is a pseudoorthogonal one that permits a succint "4π" wavization of the constrained system. The fish-eye exhibits, unavoidably, chromatic dispersion. Further, a larger conformal dynamical group contains the *potential* group, that relates the closed, inhomogeneous fish-eye system to similar, scaled ones. Asymptotically, it is related to free propagation in homogenous media.

I. INTRODUCTION

The Maxwell fish-eye is an optical medium, in principle, in any number of dimensions, whose refractive index $n(\vec{q})$ is a function $\sim(1+q^2)^{-1}$ of the distance q to the center. It is a spherically symmetric inhomogeneous system that is a *pièce de resistance* in optics textbooks and treatises[1-3] because it is very illustrative to test solution methods since the system possesses exact, closed solutions. The system was originally proposed as a problem by the Irish Academy; it asked for the refractive index of a medium that could conceivably form images in the least depth (fish eyes are notoriously flat) and Maxwell's solution was published in 1854.[4] The medium is ideal, of course, because of light injection and attenuation problems, and because of size restrictions by the physical requirement $n \geqslant 1$. Yet, this system is truly the hydrogen atom of optics, as we shall see: it possesses a manifest SO$(N-1)$ and hidden SO(N) rotation symmetry groups, and SO$(N,1)$, ISO(N), and SO$(N,2)$ dynamical groups.

The paths of light rays in a Maxwell fish-eye medium are closed: They are circles on planes that contain the origin, and whose points form conjugate pairs with respect to the origin. (For fixed ρ and origin, vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are conjugate when they are antiparallel and their magnitudes relate through $q_1 q_2 = \rho^2$.) In the posthumous work of Luneburg, *Mathematical Theory of Optics*,[3] a section titled "The surprising properties of an optical medium of refractive index..." shows that the circles in the Maxwell fish-eye are the *stereographic projection* of great circles on a sphere in one higher dimension. Group theory had not yet come into much vogue before 1949, when the book manuscript was assembled out of lecture notes, and contains no mention of the work of Fock[5] and Bargmann[6] on the hidden rotation symmetry of the hydrogen atom. The statement that a higher rotation symmetry is at play in the Maxwell fish-eye was made in the work by Buchdahl,[7] who mapped the constants of the fish-eye circles (plane orientation and vector to the center) onto the constants of the Kepler orbits (plane orientation and Runge–Lenz vector). These are generators of an SO(4) group under the Poisson bracket.

The Maxwell fish-eye is usually given as an example of a geometric-optics *perfect imaging instrument* because all light rays issuing from any one point in the medium will follow circle arcs that intersect at the point conjugate to the first, and the optical length of all these circle arcs between the two conjugate points is the same.[2] (Nevertheless, chromatic dispersion is not discussed in the standard texts.) Generalizations of the fish-eye, such as the Luneburg lens used in microwave antenna design[8] give the fish-eye a nontrivial practical interest.

The fish-eye is a rare instance of a "4π" optical instrument; as the hydrogen atom, it is a system worth studying for its own, group-theoretical sake. We regard it as a prime example to calibrate the Lie–Hamilton formulation of geometric and wave optics, previously used only for homogeneous optical media;[9,10] the description here includes time evolution. Section II reports a succint derivation of the Hamilton equations of motion of optics in time from the assumption only of the local validity of Snell's law. We find no extra effort for working in $N-1$ dimensions. These equations lead to the classical phase space formulation of geometrical optics, where the momentum vector is constrained to a sphere, rather than a plane as in mechanics.[11]

Section III studies a system in mechanics that is also constrained to a sphere, albeit in configuration space: the point rotor. We believe that the rotor system, rather than the Kepler problem, is the simplest mechanical SO(N) model analog to the fish-eye. We write the rotation generators of the symmetry group, the SO(N) Casimir invariant, and the Hamiltonian of the system as functions on phase space under Lie–Poisson brackets[12] and constrain them to their projection on the equatorial plane of the sphere. In Sec. IV we

 0022-2488/90/122757-12$03.00

introduce an extended *canonical* stereographic map of the phase spaces, the configuration part of which is the familiar sphere-on-plane 1:1 map (excluding the north pole, that maps to the point at infinity). This is an optical aberration map of the generic form of *distortion*;[13] concomitant to this, the momentum spaces of the two systems map mantaining the canonicity of the transformation. The projected rotor Hamiltonian becomes the Hamiltonian of the Maxwell fish-eye, the Casimir invariant.[14,15] We build the SO(N,1) and SO(N,2) dynamical algebra generators in Sec. V, adapting previous results for the hydrogen atom.[16] The exponentiated SO(N,2) group action is given explicitly in the geometric optics representation. This dynamical algebra includes a new *number* generator, square root of the SO(N) Hamiltonian function.

In Sec. VI we follow the same program for wave optics at a lighter pace, because spherical harmonics are well known as solutions to wave motion on a sphere.[17,18] The frequencies are discrete and given by the square roots of the Casimir operator eigenvalues. The medium can thus only sustain a discrete set of colors; sharp wave fronts will undergo chromatic dispersion. Among the concluding remarks in Sec. VII, we point out that SO(N,1) is the *potential group*,[19,20] that bridges between the fish-eye and the homogeneous medium. Finally, an SO(N,2) bundle over configuration space is suggested to describe a more general class of inhomogeneous media.

II. THE HAMILTONIAN-TIME FORMULATION OF OPTICS

We model geometric optical rays as the paths taken by points indicated by $\mathbf{q}(t)\in\Re^D$, at a time parameter t, whose velocity vector may have arbitrary direction but must be of fixed magnitude at each point of the medium,

$$\mathbf{v}(t)=\frac{d\mathbf{q}}{dt},\quad |\mathbf{v}|=\frac{c}{n(\mathbf{q})}. \tag{2.1}$$

Here, c stands for the light velocity in vacuum and $n(\mathbf{q})$ is the refractive index of the medium at the point $\mathbf{q}$($n=1$ characterizes vacuum). We assume that this index is a scalar function of the space coordinates only, and not of time, ray direction, or any other ray descriptor. The time needed to traverse a vanishing distance ds is $dt=n/c\,ds$. Physics asserts that c is a universal constant and that $n<1$ is unphysical.

We shall now build a vector $\mathbf{p}(t)$ tangent to the path $\mathbf{q}(t)$, i.e., parallel to $\mathbf{v}$. Snell's law is particularly transparent in suggesting the right *length* for this tangent vector $\mathbf{p}$. It is *not* $v=c/n$, as could wrongly be inferred from (2.1), but such that at any surface σ possessing a normal vector $\mathbf{\Sigma}$ and separating two otherwise homogeneous media n and n' (constant), there holds the well-known sine law:

$$n\sin\theta=n'\sin\theta', \tag{2.2}$$

with the three vectors $\mathbf{p}$, $\mathbf{p}'$, and $\mathbf{\Sigma}$ coplanar, and where as usual we denote by θ and θ' the angles between $\mathbf{\Sigma}$ and the directions $\mathbf{p}$ and $\mathbf{p}'$ of the ray before and after refraction.

Equation (2.2) may be seen as a *conservation* statement when we write each member as the magnitude of a cross product $|\mathbf{\Sigma}\times\mathbf{p}|=|\mathbf{\Sigma}||\mathbf{p}|\sin\theta$ between the (common) surface normal $\mathbf{\Sigma}$ and the vector $\mathbf{p}$ *constrained* to have length $|\mathbf{p}|=n$, or

$$\mathbf{p}\cdot\mathbf{p}=n(\mathbf{q})^2, \tag{2.3}$$

and similarly for the primed quantities. The requirement of coplanarity is the linear dependence of the three vectors: $\mathbf{p}=\alpha\mathbf{p}'+\beta\,\mathbf{\Sigma}$, for some real α and β. The magnitude of the cross product of $\mathbf{p}$ with $\mathbf{\Sigma}$ is consistent with (2.2) only for $\alpha=1$. A vector statement equivalent to the sine law (2.2) plus coplanarity is therefore that

$$\mathbf{p}-\mathbf{p}'=\beta\,\mathbf{\Sigma}. \tag{2.4}$$

Here, β is a scalar function of the vectors. If we assume $|\mathbf{\Sigma}|=1$ and decompose $\mathbf{p}=\mathbf{p}_\perp+p_\Sigma\mathbf{\Sigma}$, where $p_\Sigma=\mathbf{p}\cdot\mathbf{\Sigma}$ is the coordinate of $\mathbf{p}$ along $\mathbf{\Sigma}$, and $\mathbf{p}_\perp$ is the *conserved* component of $\mathbf{p}$ in the plane tangent to the surface. The picture we obtain of the optical medium is that for every point $\mathbf{q}\in\Re^D$ we have an $\mathscr{S}_{D-1}$ *sphere* in $\mathbf{p}$ space, that will be called the *Descartes* sphere of ray directions, whose radius depends on the point.

Indeed, Snell's law should be called Descartes' if the French philosopher, besides finding Eq. (2.2) and the requirement of coplanarity,[1] had only realized that the appropriate tangent vector $\mathbf{p}$ is *not* the velocity vector of the light corpuscule, but

$$\mathbf{v}=c/n^2\mathbf{p}, \tag{2.5}$$

so that its magnitude be consistent with (2.1) and (2.3). In the denser of two media, the ray approaches the surface normal as a particle falling in a potential step well, but actually travels slower.

We shall now derive from (2.3) and (2.4) the two Hamilton equations of motion for the light points of geometrical optics moving through inhomogeneous media, and find the function $\mathscr{H}^{\rm opt}(\mathbf{q},\mathbf{p})$ that serves as optical Hamiltonian.[11] In fact, we have done so already: Eqs. (2.1) and (2.5) compose to the first equality in

$$\frac{d\mathbf{q}}{dt}=\frac{c}{n^2}\,\mathbf{p}=\frac{\partial\mathscr{H}^{\rm opt}}{\partial\mathbf{p}},\ \text{where}\ \mathscr{H}^{\rm opt}=c\,\frac{\mathbf{p}\cdot\mathbf{p}}{2n^2}+\phi(\mathbf{q}), \tag{2.6}$$

where the second equality defines $\mathscr{H}^{\rm opt}$ up to an arbitrary additive function $\phi(\mathbf{q})$. The equality between the first and third terms is Hamilton's first equation. This equation, we saw, follows from the geometry of tangent vectors and the definition of $\mathbf{p}$ in (2.5).

To introduce the optical *dynamics* contained in Snell's law, we must generalize equation (2.4) for refractive indices $n(\mathbf{q})$ that possess a gradient field ∇n acting as surface normal for infinitesimal refraction $\mathbf{p}'=\mathbf{p}+d\mathbf{p}$ in a time interval dt. Equation (2.4) then becomes

$$\frac{d\mathbf{p}}{dt}=\gamma\nabla n, \tag{2.7a}$$

where we are left to determine the scalar function $\gamma(\mathbf{p},n(\mathbf{q}))$. This we do differentiating Eq. (2.3) in two different ways:

$$\frac{dn^2}{dt}=2n\nabla n\cdot\frac{d\mathbf{q}}{dt}=2n\,\frac{c}{n^2}\,\nabla n\cdot\mathbf{p} \tag{2.7b$_1$}$$

$$=2\mathbf{p}\cdot\frac{d\mathbf{p}}{dt}=2\gamma\mathbf{p}\cdot\nabla n, \tag{2.7b$_2$}$$

whence,

$$\gamma = c/n(\mathbf{q}). \tag{2.7c}$$

From (2.3) and this follows the second Hamilton vector equation of motion

$$\frac{d\mathbf{p}}{dt} = \frac{c}{n}\nabla n = -\frac{\partial \mathscr{H}^{\rm opt}}{\partial \mathbf{q}}, \tag{2.8}$$

where the Hamiltonian function in (2.6) is forced to have $\phi(\mathbf{q}) = \text{const}$, and thus determined as

$$\mathscr{H}^{\rm opt}(\mathbf{p},\mathbf{q}) = c[\mathbf{p}\cdot\mathbf{p}/2[n(\mathbf{q})]^2] + \text{const}, \tag{2.9}$$

is now determined up to an arbitrary additive constant. In fact, the Hamiltonian *is* constant along ray trajectories when the momentum vector $\mathbf{p}$ is everywhere on its Descartes sphere[21,22] of radius $n(\mathbf{q})$. Most important, observe that geometric *optical* Hamiltonians are constrained to have the form

$$\mathscr{H}^{\rm opt} \sim [\text{momentum}]^2 \times [\text{scalar function of position}]. \tag{2.10}$$

The Hamilton equations of the motion are usually derived in a roundabout way from Fermat's global principle of least action, through the variational argument of the Euler–Lagrange equations. Canonical momentum is then defined as the velocity gradient of the Lagrangian and shown to participate in a condensed set of equations that are Hamilton's.[13] It is surprising that the above short derivation seems not to be known. Indeed, these arguments may be repeated *mutatis mutandis* to find the Hamiltonian evolution under translations along the optical axis [involving d/dq_i (Refs. 10,21,22)] or along the ray length [involving d/ds (Ref. 23)]. Here, we take the time dt as the infinitesimal "measuring rod." The form chosen here displays best the group-theoretical properties of the Maxwell fish-eye.

III. THE ISOTROPIC POINT ROTOR

The phase space of a nonrelativistic point mass in N dimensions is the ensemble of position coordinates $\vec{Q}\mathrm{Q} = \{Q_i\}_{i=1}^N \in \mathfrak{R}^N$ and their conjugate momenta $\vec{P} = \{P_i\}_{i=1}^N \in \mathfrak{R}^N$. This is a $2N$-dimensional space where we can introduce an antisymmetric form $\{\cdot,\cdot\}$ between pairs of coordinates given by

$$\{Q_i,P_j\} = \delta_{i,j} = -\{P_j,Q_i\}, \tag{3.1}$$

$$\{Q_i,Q_j\} = 0, \quad \{P_i,P_j\} = 0. \tag{3.2}$$

This can be extended to all formal power series functions f, g, h of phase space through asking the form to be linear, to satisfy Jacobi's identity,[14] and to be a *derivation* ($\{fg,h\} = f\{g,h\} + \{f,h\}g, \{f,\text{const}\} = 0$). These properties are those of a Lie *bracket*,[12,14,15] so the coordinates of phase space serve as the basis for a Lie algebra provided we recognize the "1" in (3.1) as the central element in that algebra, with null brackets $\{1,Q_i\} = 0 = \{1,P_i\}$. This is the *Heisenberg–Weyl* algebra.[24] Classical mechanics (and geometric optics) work with the realization provided by the Poisson bracket[11]

$$\{f,g\} = \sum_{i=1}^N \left(\frac{\partial f}{\partial Q_i}\frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i}\frac{\partial g}{\partial Q_i}\right). \tag{3.3}$$

Under ordinary multiplication these functions of course commute: $fg = gf$.

The Hamilton equations for geometric optics, (2.6) and (2.8), are expressible in Poisson brackets:

$$\frac{dQ_i}{dt} = -\{\mathscr{H},Q_i\}, \quad \frac{dP_i}{dt} = -\{\mathscr{H},P_i\}. \tag{3.4a}$$

We denote $\{\mathscr{H},\cdot\}$ the Lie operator[12] associated to the observable $\mathscr{H}$. The evolution of any function $f(Q_i,P_j)$ with time t is

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial \vec{Q}}\cdot\frac{d\vec{Q}}{dt} + \frac{\partial f}{\partial \vec{P}}\cdot\frac{d\vec{P}}{dt}\\ &= -\frac{\partial f}{\partial \vec{Q}}\cdot\frac{\partial \mathscr{H}}{\partial \vec{Q}} + \frac{\partial f}{\partial \vec{P}}\cdot\frac{d\mathscr{H}}{d\vec{P}} = -\{\mathscr{H}, f\}.\end{aligned} \tag{3.4b}$$

This is the Hamiltonian flow of phase space generated by $\mathscr{H}$.

Since $d\mathscr{H}/dt = \{\mathscr{H},\mathscr{H}\} = 0$, the trajectories in phase space $\vec{Q}(t)$, $\vec{P}(t)$, are flows along surfaces $\mathscr{H} = \text{const}$. We may use functions h other than the Hamiltonian as generators of flows: $df/ds = -\{h,f\}$, with s a length parameter along the flow lines generated by h. In particular, the flow generated by $\{Q_i,\circ\}$ is translation of phase space in the P_i direction. Similarly, the flow generated by $\{P_i,\circ\}$ is translation in the $-Q_i$ direction. Any observable f such that $\{\mathscr{H},f\} = 0$ defines surfaces $f = \text{const}$ that the flow of the Hamiltonian must respect. Finally, note that the commutator of two Lie operators $\{f,\circ\}$, $\{g,\circ\}$ is generally nonzero; in fact, it is the Lie operator of the Poisson bracket of the two functions: From the Jacobi identity we find

$$[\{f,\circ\},\{g,\circ\}] = \{f,\circ\}\{g,\circ\} - \{g,\circ\}\{f,\circ\} = \{\{f,g\}\circ\}. \tag{3.5}$$

When this quantity is zero, the Lie operators commute, and the flow generated by one function is invariant under parallel transport by the other. The Poisson bracket of the two generators is then a constant.

Linear functions of phase space close into the N-dimensional Heisenberg–Weyl algebra w_N. The independent quadratic functions are

$$A_{i,j} = P_iP_j, \tag{3.6a}$$

$$B_{i,j} = Q_iP_j + Q_jP_i, \tag{3.6b}$$

$$C_{i,j} = Q_iQ_j, \tag{3.6c}$$

$$R_{i,j} = Q_iP_j - Q_jP_i, \tag{3.7}$$

and close into the real *symplectic* algebra $\mathrm{sp}(2N,\mathfrak{R})$. The linear *plus* quadratic functions also close, the algebra is $w_N \,\mathrm{sp}(2N,\mathscr{R})$. In particular, the $R_{i,j}$ close into the N-dimensional rotation algebra $\mathrm{so}(N)$ that generates[14] a joint rotation of the $\vec{Q}$ and $\vec{P}$ subspaces in their i-j planes; the flow generated by $\vec{Q}\cdot\vec{P} = \frac{1}{2}\Sigma_i B_{i,j}^2$ is a radially inward flow in the $\vec{Q}$ coordinates and radially outward in the $\vec{P}$ coordinates, and leads to reciprocal scaling of the two subspaces. Flows can of course also mix the position and conjugate momentum subspaces, as those generated by P_iP_j and Q_iQ_j. Among all functions of phase space, there exist subsets that also close into Lie algebras whose vector dimension may be finite or infinite; some of them will come up for scrutiny below.

A *point rotor* is a mass point *constrained* to move on a *sphere* in configuration space

$$\vec{Q}^2=\vec{Q}\cdot\vec{Q}=\sum_{i=1}^{N} Q_i^2=\mathbf{Q}\cdot\mathbf{Q}+Q_N^2=\rho^2, \tag{3.8}$$

where ρ is an arbitrary but fixed radius, and we indicate by $\mathbf{Q}=(Q_1,Q_2,\ldots,Q_{N-1})$ the first $N-1$ components of $\vec{Q}=(\mathbf{Q},Q_N)$. On the sphere, thus,

$$Q_N^{(\sigma)}=\sigma\sqrt{\rho^2-|\mathbf{Q}|^2},\quad \sigma\in\{+1,0,-1\}. \tag{3.9}$$

The *sign* σ of $Q_N^{(\pm)}$ keeps track of the two hemispheres and the $Q_N^{(0)}=0$ equator. We shall not insist on considering separately the $\sigma=0$ lower-dimensional manifold, but we keep in mind the natural continuity conditions between the two hemispheres. The Hamiltonian flow must leave that sphere invariant.

Those functions F of the full $\Re^{2N}$ phase space that have zero Poisson bracket with $\vec{Q}^2$ preserve the sphere where the point moves; among them, our purported *rotor* Hamiltonian $\mathscr{H}^{\rm rot}$. They will generate the *symmetry* and *dynamical* group(s) of the point rotor (the latter contains the former). Functions F of phase space that have zero Poisson bracket with $\vec{Q}^2$ satisfy

$$\{F,\vec{Q}^2\}=0,\quad \text{i.e., } 0=\sum_{i=1}^{N} 2Q_i\{F,Q_i\}=-2\vec{Q}\cdot\frac{\partial F}{\partial\vec{P}}. \tag{3.10}$$

Among the linear and quadratic functions in (3.1), (3.6), and (3.7), only 1, Q_i, Q_iQ_j, and $R_{i,j}=Q_iP_j-Q_jP_i$ have this property, while P_i, P_iP_j, and $B_{i,j}=Q_iP_j+Q_jP_i$ do not. This property yields a Lie algebra of functions under the Poisson bracket,[12] and its universal covering algebra (obtained by ordinary multiplication of the algebra elements) has the same property.

The *symmetry* algebra of the system will be the subset of those functions that have zero Poisson bracket also with $\mathscr{H}^{\rm rot}$. The rotor point mass is on a sphere, with no preferred origin or direction. The set of functions $R_{i,j}$ forms a vector basis for the so(N) algebra and generates the N-dimensional rotation group SO(N). Special consideration is thus due to the SO(N) *Casimir* function of second degree in the generators (and of *fourth* degree in Q_i and P_j):

$$\Phi=\frac{1}{2}\sum_{i,j}R_{i,j}R_{i,j}=\vec{Q}^2\vec{P}^2-(\vec{Q}\cdot\vec{P})^2. \tag{3.11}$$

In $N=3$ dimensions, this is the squared norm of the cross product $\vec{Q}\times\vec{P}$, the angular momentum antisymmetric tensor.

Under Poisson brackets, the functions 1, Q_i, plus $C_{i,j}=Q_iQ_j$ belong to an Abelian ideal of dimension D given by 1, N, plus $\frac{1}{2}N(N+1)$ that, together with the $R_{i,j}$, form a larger "inhomogeneous" algebra $i_D$so(N) and generate a corresponding group. The Lie transformations from these functions do not affect configuration space at all: they translate *momentum* space and mix it with position. The *isotropic* point rotor needs thus one further specification: its *dynamics* must be rotation *invariant*. This means that any and only rotations of a Hamiltonian flow can be Hamiltonian flows. Hence, $\mathscr{H}^{\rm rot}$ must be a *scalar* under rotations and so the indices in the arguments must balance. Exit the single Q_i's from consideration therefore, but retain pairs $Q_iQ_j=C_{i,j}$. Since $C_{i,j}=C_{j,i}$ but $R_{i,j}=-R_{j,i}$, a series expansion of $\mathscr{H}^{\rm rot}$ with balanced indices can contain any number of C's, but only terms with pairs of R's.

On the sphere $|\mathbf{Q}|^2+Q_N^2=\rho^2$, $\Sigma_iC_{i,i}=\Sigma_iQ_iQ_i=\rho^2$ is a constant, and so are $\Sigma_{i,j}C_{i,j}C_{i,j}=\rho^4$ and $\Sigma_{i,j}C_{i,j}R_{i,j}=0$. Higher degree polynomials of the C's and R's do not yield further independent invariants because any number of contracted factors of C, yield a single C (times a constant, since $\Sigma_jC_{i,j}C_{j,k}=\rho^2C_{i,k}$) and any odd number of contracted R's yield a single R times a power of Φ (since $\Sigma_{j,k}R_{i,j}R_{j,k}R_{k,l}=-\Phi R_{i,l}$). The fourth-order invariant is $\Sigma R_{i,j}R_{j,k}R_{k,l}R_{l,i}=2\Phi^2$. This reduces all higher invariants to contracted products of $\cdots RCRC\cdots$ that vanish, since for any i and m, it holds that $\Sigma_{j,k,l}R_{i,j}C_{j,k}R_{k,l}C_{l,m}=0$. The conclusion is therefore that the point rotor Hamiltonian $\mathscr{H}^{\rm rot}$ can be a function of Φ only, the rotation Casimir given in (3.11). Since Φ is quadratic in momentum $\vec{P}$ we may take

$$\mathscr{H}^{\rm rot}=\omega\Phi=\omega[\vec{Q}^2\vec{P}^2-(\vec{Q}\cdot\vec{P})^2]=E, \tag{3.12}$$

for some constant ω, E is a constant of the motion.

The constraint to the sphere $\vec{Q}^2=\rho^2$ leaves us with a $2N$-dimensional phase space $(\vec{Q},\vec{P})$ that is too large, because there is the redundant coordinate Q_N in (3.9). Since Hamiltonian phase spaces come in even dimensions only, we should expect another constraint to be at hand. Indeed, among the quadratic functions (3.6), the rotor evolution Hamiltonian (3.12) leaves invariant the traces $\vec{Q}\cdot\vec{P}=\Sigma_iB_{i,i}=\delta$ and $\vec{P}^2=\Sigma_iC_{i,i}=\gamma^2$ of (3.6); since there is a relation between the constants, $E=\omega[\rho^2\gamma^2-\delta^2]$, we can choose the gauge $\delta=0$ leaving $\gamma^2=E/\omega\rho^2$. We denoted $\vec{Q}=(\mathbf{Q},Q_N)$; let us similarly denote $\vec{P}=(\mathbf{P}^*,P_N)$ where $\mathbf{P}^*$ are the first $N-1$ components, so that $\vec{Q}\cdot\vec{P}=\mathbf{Q}\cdot\mathbf{P}^*+P_NQ_N=0$ fixes

$$P_N^{(\sigma)}=-\mathbf{Q}\cdot\mathbf{P}^*/\sigma\sqrt{\rho^2-|\mathbf{Q}|^2}. \tag{3.13}$$

What happens to the Poisson bracket structure? Disregarding σ, the position space differential under constraint is

$$d\vec{Q}=(d\,\mathbf{Q},dQ_N^{(\sigma)})=\left(d\,\mathbf{Q},\frac{-\mathbf{Q}\cdot d\,\mathbf{Q}}{\sigma\sqrt{\rho^2-|\mathbf{Q}|^2}}\right). \tag{3.14}$$

The *Pfaffian form*[25] (or first integral invariant of Poincaré [16(a)]) is

$$\vec{P}\cdot d\vec{Q}=\mathbf{P}^*\cdot d\,\mathbf{Q}+P_N\,dQ_N \tag{3.15a}$$

$$=\mathbf{P}^*\cdot d\,\mathbf{Q}+\frac{\mathbf{P}^*\cdot\mathbf{Q}\mathbf{Q}\cdot d\,\mathbf{Q}}{\rho^2-|\mathbf{Q}|^2} \tag{3.15b}$$

$$=\mathbf{P}^*\left(\cdot 1\cdot+\frac{\cdot\mathbf{Q}\mathbf{Q}\cdot}{\rho^2-|\mathbf{Q}|^2}\right)d\,\mathbf{Q} \tag{3.15c}$$

$$=\mathbf{P}\cdot d\,\mathbf{Q}\ \text{ for }\ \mathbf{P}=\mathbf{P}^*+\frac{\mathbf{Q}\cdot\mathbf{P}^*}{\rho^2-|\mathbf{Q}|^2}\mathbf{Q}. \tag{3.15d}$$

The last line defines a new set of $N-1$ coordinates $\mathbf{P}$, so that the standard Pfaffian form in those $N-1$ coordinates equals the Pfaffian in the old N coordinates with the constraints. The transformation from $(\vec{Q},\vec{P})$ [restricted by (3.9) and (3.13)] to $(\mathbf{Q},\sigma,\mathbf{P})$ spaces preserves the Pfaffian and, from it, the Poisson and Lie bracket structure [(16a)],[25] *in the new variables* $(\mathbf{Q},\mathbf{P})$. [The sign σ distinguishes between two copies of $\mathbf{Q}$ space; the $\mathbf{P}^* \mapsto \mathbf{P}$ transformation in (3.15d) is 1:1 for all points on the $|\vec{P}|$ sphere (except when $|\mathbf{Q}|^2 = \rho^2$, on the $\sigma = 0$ submanifold).] Note that it would be incorrect to simply "leave out" the Nth coordinate in the reduction from N to $(N-1)$-dimensional Poisson brackets; the projection $\vec{Q} = (\mathbf{Q}, Q_N^{(\sigma)}(\mathbf{Q})) \mapsto \mathbf{Q}$ must be accompanied by the nontrivial momentum map $\vec{P} = (\mathbf{P}^*, P_N(\mathbf{Q},\mathbf{P}^*)) \mapsto \mathbf{P}(\mathbf{Q},\mathbf{P}^*)$. We write this map as $(\mathbf{Q},\sigma,\mathbf{P}) = (\vec{Q},\vec{P})|_{\text{rot}}$.

The Hamiltonian function (3.12) after this substitution becomes

$$\mathscr{H} = \omega \mathscr{C}, \tag{3.16a}$$

$$\mathscr{C} = \Phi|_{\text{rot}} = \rho^2 |\mathbf{P}|^2 - (\mathbf{Q}\cdot\mathbf{P})^2. \tag{3.16b}$$

Under the $|_{\text{rot}}$ map, the so(N) symmetry subalgebra generators become

$$L_{i,j} = R_{i,j}|_{\text{rot}} = Q_i P_j - Q_j P_i, \quad i,j = 1,\ldots,N-1, \tag{3.17}$$

$$M_i = R_{i,N}|_{\text{rot}} = -\sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}\,P_i, \quad i = 1,\ldots,N-1. \tag{3.18}$$

We may verify that under the rotor map their Poisson bracket relations remain the same in the reduced $2(N-1)$-dimensional phase space $(\mathbf{Q},\mathbf{P})$, as if we had simply replaced Q_N and set $P_N = 0$.

There is a well-developed theory of constrained Hamiltonian systems of the *second class* (i.e., when the Poisson brackets of the constraints do not vanish). One may calculate the Dirac bracket [(11b,c)] between two functions of the rotor space constrained by $\xi_1 = \vec{Q}^2 - \rho^2$ and $\xi_2 = P_N$ and find the same formal result when $\xi_i = 0$. The old Poisson bracket is the replaced by the Dirac bracket, and this *equals* the Poisson bracket in the *reduced* subspace of the first $N-1$ components $\mathbf{Q}$ and $\mathbf{P}$.

In conclusion, on a homogeneous, isotropic sphere, the free motion of a point rotor in $(\vec{Q},\vec{P})$ is on sphere geodesics: arcs of great circles. The reduction to $(\mathbf{Q},\sigma,\mathbf{P})$ in effect "projects" the position coordinate of the point rotor on two copies of its equatorial plane, distinguished by the hemisphere sign σ, and with a new canonically conjugate momentum $\mathbf{P}$. In this phase space, the reduced Hamiltonian (3.16) has a natural "kinetic energy" term $|\mathbf{P}|^2$ and obeys a $(\mathbf{Q}\cdot\mathbf{P})^2$ "potential"; the trajectory jumps between the two values of the sign σ (through $\sigma = 0$) when it crosses the $|\mathbf{Q}| = \rho$ equator. We see this motion as spherical rotor motion projected on the equatorial plane.

IV. THE STEREOGRAPHIC MAP

The stereographic map is a bijection between the manifolds of the sphere $S_{N-1} \subset \Re^N$ and $\Re^{N-1} \cup \{\infty\}$ (the projection pole maps on ∞). This map was applied *deus ex machina* by Fock[5] in 1935 to the hydrogen atom Schrödinger equation in momentum representation, to obtain the hydrogenic wave functions in terms of the four-dimensional spherical harmonics. Figure 1 shows the geometry of the map for $N = 2$, between the circle S_1 and the line $\Re$, and the essentials of the general N-dimensional case. The surface of the sphere $\Sigma_{i=1}^N Q_i^2 = |\mathbf{Q}|^2 + Q_N^2 = \rho^2$ maps on the $(N-1)$-dimensional optical position space, of vectors $\mathbf{q} = \{q_i\}_{i=1}^{N-1}$. A point on the sphere that subtends the angle χ between the projecting pole and the $-Q_N$ axis, will measure an angle 2χ from the center of the sphere. The first

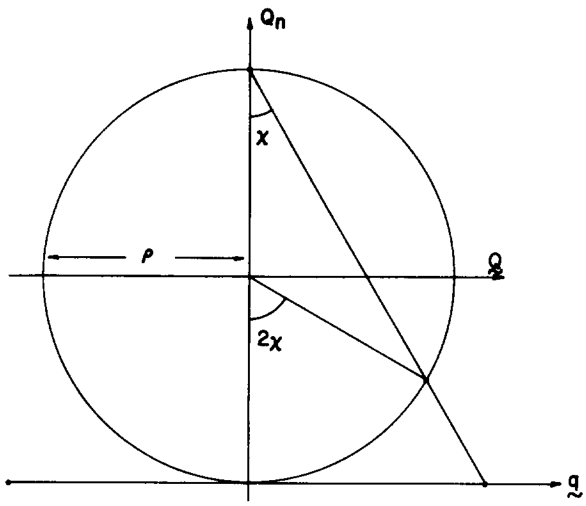

FIG. 1. The stereographic projection maps the circle on the line in $N = 2$ dimensions. The two regions $|\mathbf{Q}| < \rho, \sigma = \pm 1$, and their boundary $|\mathbf{Q}| = \rho$, $\sigma = 0$, map onto the full line $\mathbf{q}$; the boundary maps on $|\mathbf{q}| = 2\rho$.

appears in right triangle with sides 2ρ, $|\mathbf{q}|$, and $\sqrt{|\mathbf{q}|^2 + 4\rho^2}$, and its double in another right triangle of sides $-Q_N$, $|\mathbf{Q}|$, and ρ.

Trigonometric functions of χ and 2χ are

$$\sin\chi = \mathbf{q}/\sqrt{|\mathbf{q}|^2 + 4\rho^2}, \quad \sin 2\chi = \mathbf{Q}/\rho, \tag{4.1a}$$

$$\cos\chi = 2\rho/\sqrt{|\mathbf{q}|^2 + 4\rho^2}, \quad \cos 2\chi = -Q_N/\rho, \tag{4.1b}$$

$$\tan\chi = \mathbf{q}/2\rho, \quad \tan 2\chi = -\mathbf{Q}/Q_N. \tag{4.1c}$$

From common identities, we find

$$\mathbf{q} = \frac{2\rho\mathbf{Q}}{\rho - \sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}} \tag{4.2a}$$

and

$$|\mathbf{q}|^2 = 16\rho^2 \frac{\rho + \sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}}{\rho - \sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}}. \tag{4.2b}$$

This is the transformation that "opens" the sphere to the plane.[10] Similarly, we find the inverse transformation

$$\mathbf{Q} = \frac{4\rho^2\mathbf{q}}{|\mathbf{q}|^2 + 4\rho^2} \tag{4.3a}$$

and

$$Q_N^{(\sigma)} = \sigma\sqrt{\rho^2 - |\mathbf{Q}|^2} = \rho\,\frac{|\mathbf{q}|^2 - 4\rho^2}{|\mathbf{q}|^2 + 4\rho^2}. \tag{4.3b}$$

A given point $|\mathbf{q}| < 2\rho$ $(\chi < \frac{1}{4}\pi)$ is mapped by (4.3) on $|\mathbf{Q}| < \rho$, $\sigma = -1$ (since $Q_N < 0$); the equator $|\mathbf{q}| = 2\rho$ $(\chi < \frac{1}{4}\pi)$ maps on $|\mathbf{Q}| = \rho$ and $\sigma = 0$ (with $Q_N = 0$). As $|\mathbf{q}|$ increases beyond 2ρ $(\frac{1}{4}\pi < \chi < \frac{1}{2}\pi)$, the range sweeps again through $|\mathbf{Q}| < \rho$ with $\sigma = +1$ (i.e., $Q_N > 0$). The points at zero and infinity in the plane correspond to the center of the balls $\sigma = -1$ and $\sigma = +1$ (i.e., $|\mathbf{Q}| = 0$, $Q_N = \mp\rho$), respectively. The stereographic projection is therefore a map between $\mathbf{q} \in \Re^{N-1}$ and *two open balls* $\mathbf{Q} \in \Re^{N-1}$, $|\mathbf{Q}| < \rho$, labeled by the sign σ of Q_N, whose boundaries $|\mathbf{Q}| = \rho(Q_N = 0)$ are identified.

Functions $F(\vec{Q})$ on the original sphere thus become functions $F(\vec{Q}(\mathbf{Q},\sigma))$ upon reduction of variables, and $F(\vec{Q}(\mathbf{q}))$ upon stereographic projection. In Fig. 1, where $N=2$, the $\mathbf{q}$ space is the horizontal line and $\mathbf{Q}$ space is the segment $-\rho<Q_1<\rho$ *twice*. Figure 2 displays the $N=3$ situation showing a great circle of the S_2-sphere mapping onto a fish-eye orbit in a two-dimensional optical world.

Great circles on the $\vec{Q}$-sphere project by the stereographic map onto circles in the $\mathbf{q}$ plane.[3] Only the Nth axis is distinguished, so we may rotate the first $N-1$ coordinates such that the great circle lies in the $1-2-N$ submanifold, reducing the construction to that of Fig. 2. Further, the vector normal to the circle plane may be made to lie on the $Q_2=0$ plane, tilted by β in the Q_1 direction.

To parametrize explicitly, let us use Euler angles (β,γ) for the points of the great circle in $\vec{Q}$ space,

$$\begin{pmatrix} Q_1(\gamma) \\ Q_2(\gamma) \\ Q_N(\gamma) \end{pmatrix} = \rho \begin{pmatrix} \cos\beta\cos\gamma \\ \sin\gamma \\ -\sin\beta\cos\gamma \end{pmatrix}, \quad \begin{array}{l} \beta\in[0,\pi], \\ \gamma\in S_1 = \Re \bmod 2\pi, \end{array} \tag{4.4}$$

where the components $3,\ldots,N-1$ are zero and omitted. In $\mathbf{Q}$ space, i.e., the equatorial plane, this draws out an ellipse of semimajor axis ρ and semiminor axis $\rho\cos\beta$. Actually, due to the twice changing sign of $Q_N^{(\sigma)}(\gamma)$, half the ellipse lies on the $\sigma=+1$ chart and half on the $\sigma=-1$ one. Through (4.2a) we find the stereographic projection of the great circle (4.4). It is

$$\begin{aligned} q_1(\gamma) &= 2\rho\frac{\cos\beta\cos\gamma}{1+\sin\beta\cos\gamma}, \\ q_2(\gamma) &= 2\rho\frac{\sin\gamma}{1+\sin\beta\cos\gamma}, \\ \gamma&\in S_1. \end{aligned} \tag{4.5}$$

Analytic geometry verifies that this is a circle of radius $2\rho\sec\beta$ with center on $-2\rho\tan\beta$. For every point $\mathbf{q}(\mathbf{Q},\sigma)\in\Re^{N-1}$ we may define its *conjugate* point[3] $\mathbf{q}^*=\mathbf{q}(-\mathbf{Q},-\sigma)$, stemming from the *antipodal* point $-\vec{Q}$ on the sphere. The vectors $\mathbf{q}$ and $\mathbf{q}^*$ are antiparallel and satisfy $|\mathbf{q}||\mathbf{q}^*|=\rho^2$.

A spherical coordinate grid with colatitude circles such as (4.5) will map onto families of *bipolar* coordinates on the plane with respect to the two poles.[17] Letting γ stand for time, it is clear that any great circle arc followed between two conjugate points will be traversed in the same time interval.[1-3] It is also clear that rotations of the sphere in the $1,\ldots,N-1$ subspace will rotate the $\mathbf{q}$ plane simultaneously; rotations into the Q_N direction produce the full family of different-sized circles that join any two fixed conjugate points in the optical $\mathbf{q}$ space. The former is the *manifest symmetry* while the latter is the *hidden symmetry* of the system.

The stereographic map and its inverse have been considered thus far as transformations of position space. They are called point transformations because the canonically conjugate momentum does not enter. In optics, such a map is called a pure distorsion. Indeed, what happens in momentum space? The coordinates $\mathbf{p}=\{p_i(\mathbf{Q},\mathbf{P})\}_{i=1}^{N-1}$ that are canonically conjugate to the components of

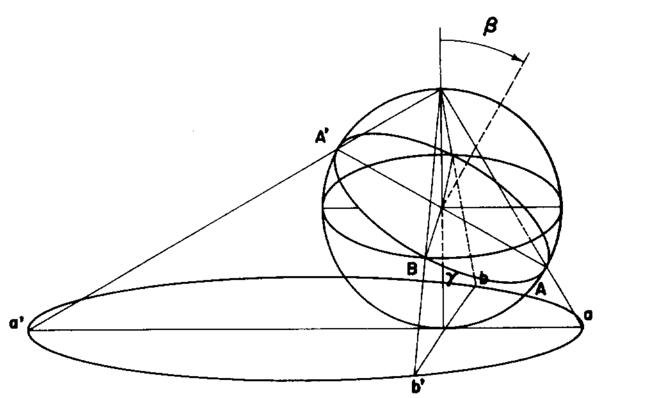

FIG. 2. The stereographic projection mapping a great circle on the $\vec{Q}$ sphere tilted by an angle β onto a circle in the $\mathbf{q}$ plane. Two pairs of antipodal-conjugate points, A-A' and B-B', are shown to map on a-a' and b-b'.

$$\mathbf{q}_i(\mathbf{Q})=\phi(|\mathbf{Q}|)Q_i, \quad i=1,2,\ldots,N-1, \tag{4.6a}$$

may be found again from the conservation of the Pfaffian form, written $\mathbf{p}\cdot d\mathbf{q}=\mathbf{P}\cdot d\mathbf{Q}$ (Ref. 25), where now

$$\begin{aligned} d\mathbf{q}_i &= \sum_j \frac{\partial q_i}{\partial Q_j}dQ_j = \phi\, dQ_i + Q_i\sum_j\frac{\partial\phi}{\partial Q_j}dQ_j \\ &= \sum_j\left(\phi\delta_{i,j}+\phi'\frac{Q_iQ_j}{|\mathbf{Q}|}\right)dQ_j = \sum_j J_{i,j}\,dQ_j. \end{aligned} \tag{4.6b}$$

Once the Jacobian matrix $\mathbf{J}(Q)=\{J_{i,j}\}$ is known for $d\mathbf{q}=\mathbf{J}\,d\mathbf{Q}$, then $\mathbf{p}\cdot d\mathbf{q}=\mathbf{p}\cdot\mathbf{J}\,d\mathbf{Q}=\mathbf{P}\cdot\mathbf{Q}$ solves as $\mathbf{p}=\mathbf{J}^\tau(\mathbf{Q})^{-1}\mathbf{P}$ and $\mathbf{P}=\mathbf{J}^\tau(Q(q))\mathbf{p}$, where $^\tau$ means matrix transposition. In our case the $\{Q_iQ_j/|\mathbf{Q}|^2\}$ are idempotent matrices and we can find the canonically conjugate momentum map to be

$$\mathbf{p}=\frac{\rho-\sigma\sqrt{\rho^2-|\mathbf{Q}|^2}}{2\rho}\mathbf{P}-\frac{\mathbf{Q}\cdot\mathbf{P}}{2\rho^2}\mathbf{Q}. \tag{4.7}$$

Notice that $Q_N^{(\sigma)}=\sigma\sqrt{\rho^2-|\mathbf{Q}|^2}$ appears with its sign, so that $\mathbf{p}$ is a single-valued function over the rotor sphere. Similarly, we find the inverse transformation

$$\mathbf{P}=\frac{|\mathbf{q}|^2+4\rho^2}{4\rho^2}\left(\mathbf{p}+\frac{2\mathbf{q}\cdot\mathbf{p}}{4\rho^2-|\mathbf{q}|^2}\mathbf{q}\right). \tag{4.8}$$

We are working here only with maps between $2(N-1)$-dimensional phase spaces; the $(\vec{Q},\vec{P})$ space where the rotor motion is embedded and constrained need not be used.

Assured that the transformation between $(\mathbf{Q},\sigma,\mathbf{P})$ and $(\mathbf{q},\mathbf{p})$ spaces is canonical, we may write the so(N) functions (3.17) and (3.18) in $(\mathbf{q},\mathbf{p})$. They are

$$L_{i,j}=q_ip_j-q_jp_i, \quad i,j=1,\ldots,N-1, \tag{4.9}$$

$$M_i=\rho\left[\left(1-\frac{|\mathbf{q}|^2}{4\rho^2}\right)\mathbf{p}+\frac{\mathbf{q}\cdot\mathbf{p}}{2\rho^2}\mathbf{q}\right], \quad i=1,\ldots,N-1, \tag{4.10}$$

and will close into the same algebras under the Poisson bracket in $(\mathbf{q},\mathbf{p})$ space as they did before under the Poisson bracket in $(\vec{Q},\vec{P})$ and in $(\mathbf{Q},\mathbf{P})$.

Finally, from the discussion in this section, the Casimir and Hamiltonian functions in (3.16) can be calculated replacing $\mathbf{q}(\mathbf{Q},\mathbf{P})$ and $\mathbf{p}(\mathbf{Q},\mathbf{P})$. They are

$$\mathscr{C} = \left(\sum_{i<j=2}^{N-1} L_{i,j}^2 + \sum_{i=1}^{N-1} M_i^2\right)$$
$$= \rho^2\left(1 + \frac{|q|^2}{4\rho^2}\right)^2 |\mathbf{p}|^2 = \frac{1}{\omega}\mathscr{H}(\mathbf{q},\mathbf{p}) = \frac{\mathrm{E}}{\omega}. \qquad (4.11)$$

Now we compare this "stereographically projected" rotor Hamiltonian $\mathscr{H}$ with the generic *optical* Hamiltonian $\mathscr{H}^{\mathrm{opt}} = c|\mathbf{p}|^2/n(\mathbf{q})|^2$ in Eq. (2.9). Their dependence on the squared momentum $|\mathbf{p}|^2$ is the same and they coincide only when the refractive index of the optical medium is

$$n(\mathbf{q}) = \frac{n_0}{1 + |\mathbf{q}|^2/4\rho^2}, \quad n_0 = n(\mathbf{0}) = \frac{1}{\rho}\sqrt{\frac{c}{2\omega}}. \qquad (4.12)$$

This is the refractive index that characterizes the Maxwell fish-eye.

V. THE FISH-EYE so(N,2) DYNAMICAL ALGEBRA

The realization of the so(N) algebra in (4.9) and (4.10) is well known from the theory of the hydrogen atom.[5,6,15,16] The momentum space of that system is the stereographic projection of the Fock sphere, where rotor momentum moves. Here, it is configuration space that maps under the stereographic projection. The earliest reference to (the Fourier transform $[\mathbf{q}\mapsto\mathbf{p},\mathbf{p}\mapsto-\mathbf{q}]$ of) the so(N,2) algebra written here seems to be Ref. 16. It is perhaps best known from the book by Wybourne,[26] who quotes the result of Barut and Bornzin.[27] Here, we shall use these results to examine the exponentiation to the SO(N,2) *group* of transformations of the Maxwell fish-eye optical phase space. This phase space is, we recall, $\mathbf{q}\in\Re^{N-1}$ and $\mathbf{p}$ constrained, for each point in $\mathbf{q}$ space, to lie on the Descartes ray-direction sphere (2.3), i.e., a sphere $S_{N-2}\subset\Re^{N-1}$ of radius

$$p(\mathbf{q}) = |\mathbf{p}| = \sqrt{\mathscr{C}}/(\rho[1 + |\mathbf{q}|^2/4\rho^2]),$$

as demanded by the constancy of (4.11).

The set of functions $L_{i,j}$ in (4,7), for $i,j = 1,2,...,N-1$, that generate ordinary joint rotations of $\mathbf{q}$ and $\mathbf{p}$ spaces, close into an so($N-1$) algebra that integrates to an SO($N-1$) group. These transformations map fish-eye orbits onto similar fish-eye orbits, rotated around the origin. This is the *manifest symmetry* group of the Maxwell fish-eye.

The N functions M_i, $i = 1,2,...,N-1$ transform as a vector under SO($N-1$) and, together with the $L_{i,j}$'s, are generators of an so(N) algebra. The corresponding group is SO(N); it maps any given fish-eye orbit onto all other possible orbits in the same medium. Fish-eye orbits can thus be made to change their radius and center. This is the hidden symmetry group of the fish-eye. Still a symmetry, though.

Enter dynamics. We may calculate that

$$\{M_i,q_j\} = -\delta_{i,j}\rho(1 - |\mathbf{q}|^2/4\rho^2) - q_iq_j/2\rho, \qquad (5.1a)$$
$$\{M_i,p_j\} = L_{i,j}/2\rho + \delta_{i,j}\mathbf{q}\cdot\mathbf{p}/2\rho. \qquad (5.1b)$$

The first expression shows that under the integrated action of $\{\mathbf{a}\cdot\mathbf{M},\circ\}$, $\mathbf{q}$ will map (nonlinearly) into $\mathbf{q}'(\mathbf{q},\mathbf{a})$. The second expression shows that if we consider p_j, $\mathbf{q}\cdot\mathbf{p}$, and

$$\{M_i,\mathbf{q}\cdot\mathbf{p}\} = \rho\left[-\left(1 + \frac{|\mathbf{q}|^2}{4\rho^2}\right)p_i + \frac{\mathbf{q}\cdot\mathbf{p}}{2\rho^2}q_i\right], \qquad (5.2)$$

then further Poisson brackets of these functions will close into an algebra larger than so(N). To identify this algebra we recall that the Cartesian basis of so(N,M) generators satisfy

$$\{\Lambda_{i,j},\Lambda_{k,l}\} = g_{j,k}\Lambda_{l,i} + g_{i,l}\Lambda_{k,j} + g_{j,l}\Lambda_{i,k} + g_{i,k}\Lambda_{j,l}, \qquad (5.3a)$$

where

$$g_{j,k} = \begin{cases} 1, & j = k \leqslant N, \\ -1, & N+1 \leqslant j = k \leqslant N+M, \\ 0, & \text{otherwise.} \end{cases} \qquad (5.3b)$$

We take $\Lambda_{i,j} = -\Lambda_{j,i}$ for i,j both in the range $(1,...,N)$ or both in $(N+1,...,N+M)$, and $+\Lambda_{j,i}$ otherwise.

We write the so(N,1) generators in the following way:

$$L_{i,j} = \Lambda_{i,j}, \qquad (5.4a)$$
$$M_i = \Lambda_{i,N}, \quad i,j = 1,2,...,N-1, \qquad (5.4b)$$
$$K_i = \Lambda_{i,N+1} = M_i - 2\rho p_i$$
$$= \rho\left[-\left(1 + \frac{|\mathbf{q}|^2}{4\rho^2}\right)p_i + \frac{\mathbf{q}\cdot\mathbf{p}}{2\rho^2}q_i\right], \qquad (5.4c)$$
$$K_N = \Lambda_{N,N+1} = -\mathbf{q}\cdot\mathbf{p}. \qquad (5.4d)$$

We recognize the "noncompact" generators to be the K's, formally because of the minus sign in $\{K_i,K_j\} = -L_{i,j}$, and manifestly because $-\mathbf{q}\cdot\mathbf{p}$ generates unbounded magnifications of configuration space. On smooth functions f of phase space,

$$\exp\beta\{K_N,\circ\}:f(\mathbf{q},\mathbf{p})\mapsto f(e^{-\beta}\mathbf{q},e^{\beta}\mathbf{p}), \quad \beta\in\Re. \qquad (5.5)$$

We note that this action is no longer an invariance transformation of the fish-eye Hamiltonian (4.9) but, for $\mathscr{H}^{\text{fish-eye}}$ as a function of rotor radius ρ,

$$\exp -\beta\{K_N,\circ\}:\mathscr{H}(\rho)^{\text{fish-eye}}$$
$$\mapsto\omega\rho^2(1 + e^{2\beta}|\mathbf{q}|^2/4\rho^2)^2e^{-2\beta}|\mathbf{p}|^2 = \mathscr{H}(e^{-\beta}\rho)^{\text{fish-eye}}. \qquad (5.6)$$

In particular, we note that the radius ρ of the rotor sphere dilates to infinity for $\beta\to-\infty$. If we set $\omega = \frac{1}{2}cn_o^{-2}\rho^{-2}$, then by (4.10) we map the Maxwell fish-eye Hamiltonian $\mathscr{H}(\rho)^{\text{fish-eye}}$ onto $\mathscr{H}(\infty)^{\text{fish-eye}} = n_o$, the optical Hamiltonian of an infinite homogeneous medium.

The other K_i, $i = 1,2,...,N-1$ will produce SO(N)-rotated versions of this action. Of these we wish to remark a direct physical interpretation. The SO(N,1) generators in (5.4c) may *also* be obtained through the standard *deformation* formula[14,28] as $K_i = \{\mathscr{C},p_i\}$, in general with a summand τp_i, $\tau = $ const. This suggests $\omega^{-1}\{\mathscr{H}^{\text{fish-eye}},p_i\} = \omega^{-1}dp_i/dt$, to be a ray "acceleration" vector.

We observe that the functions p_i are linear combinations of so(N,1) generators, viz., $p_i = (M_i - K_i)/2\rho$. Thus a second visible noninvariance transformation of the optical fish-eye Hamiltonian is

$$\exp\Sigma_ia_i\{M_i - K_i,\circ\}:\mathscr{H}(\mathbf{q},\mathbf{p})^{\text{fish-eye}}\mapsto\mathscr{H}(\mathbf{q} - 2\rho\mathbf{a},\mathbf{p})^{\text{fish-eye}}, \qquad (5.7)$$

i.e., the map is to *another* fish-eye whose origin is at $\mathbf{a}$ instead of the origin. The algebra generated by $L_{i,j}$'s and p_i's is the

Euclidean algebra $\mathrm{iso}(N-1)\subset \mathrm{so}(N,1)$.

The group $\mathrm{SO}(N,1)$ thus obtained therefore contains not only the symmetry group of the Maxwell fish-eye, but also the transformations between all possible such fish-eyes, translated and dilated, up to and including asymptotically the homogeneous medium. This is the *potential group* of the Maxwell fish-eye. Potential *algebras* of the family $\mathrm{so}(M,N)$ were used in Refs. 19 and 20, to relate the quantum Pöschl–Teller and other mostly one-dimensional potentials to the free particle. That this approach also serves optical systems, is in principle remarkable.

A search for further functions closing under Poisson brackets with the generators of $\mathrm{so}(N,1)$ is rewarded when we introduce the function $p=\sqrt{\mathbf{p}\cdot\mathbf{p}}=|\mathbf{p}|$. We may then verify that an algebra is formed by the previous $\mathrm{so}(N,1)$ generators, plus

$$H_i=\Lambda_{i,N+2}=q_i p,\quad i=1,2,\dots,N-1, \tag{5.8a}$$

$$H_N=\Lambda_{N,N+2}=H_{N+1}-2\rho p=-\rho(1-|\mathbf{q}|^2/4\rho^2)p, \tag{5.8b}$$

$$\mathcal{N}=H_{N+1}=\Lambda_{N+1,N+2}=\rho(1+|\mathbf{q}|^2/4\rho^2)p=+\sqrt{\mathscr{C}}. \tag{5.8c}$$

We note prominently that $\mathcal{N}$ in (5.8c) is the square root of the $\mathrm{so}(N)$ Casimir function $\mathscr{C}$ in (4.9b). It is a compact generator of an $\mathrm{so}(N,2)$ algebra. In the hydrogen-atom system,[16] this 'last' generator is the *number* operator.

Let us examine finally the integrated group action generated by $\mathcal{N}=H_{N+1}=\Lambda_{n+1,N+2}$. This is a *sui generis* evolution of the fish-eye system. From (5.3),

$$\exp s\{\mathcal{N},\circ\}:\Lambda_{i,j}$$
$$=\begin{cases}\Lambda_{i,j}, & 1\leqslant i,j\leqslant N,\\ K_i\cos s-H_i\sin s, & 1\leqslant i\leqslant N,\ j=N+1,\\ K_i\sin s+H_i\cos s, & 1\leqslant i\leqslant N,\ j=N+2.\end{cases} \tag{5.9}$$

Hence, for $p_i=(M_i-K_i)/2\rho$ and $p=(H_{N+1}-H_N)/2\rho$,

$$\exp s\{\mathcal{N},\circ\}:p_i=\left[1-\frac{1-\cos s}{2}\left(1+\frac{|\mathbf{q}|^2}{4\rho^2}\right)\right]p_i + \left[p\sin s+\frac{1-\cos s}{4\rho^2}\,\mathbf{q}\cdot\mathbf{p}\right]q_i, \tag{5.10a}$$

$$\exp s\{\mathcal{N},\circ\}:p=\left[\cos s+\frac{1-\cos s}{2}\left(1+\frac{|\mathbf{q}|^2}{4\rho^2}\right)\right]p -\frac{1}{2\rho}\,\mathbf{q}\cdot\mathbf{p}\sin s. \tag{5.10b}$$

Finally, for $q_i=H_i/p$,

$$\exp s\{\mathcal{N},\circ\}:q_i=\frac{[p\cos s+\mathbf{q}\cdot\mathbf{p}/2\rho\sin s]q_i-\rho(1+|\mathbf{q}|^2/4\rho^2)\sin s\quad p_i}{[\cos s+\frac{1}{2}(1-\cos s)(1+|\mathbf{q}|^2/4\rho^2)]p-\frac{1}{2}\mathbf{q}\cdot\mathbf{p}/\rho\sin s}. \tag{5.11}$$

The evolution parameter s is along the flow lines $v=\rho(1+|\mathbf{q}|^2/4\rho^2)p=\text{const}$, or $H^{\text{fish-eye}}=\text{constant}$. Thus we find the optical evolution generated by the Hamiltonian $H^{\text{fish-eye}}=\omega\mathcal{N}^2$ through observing that

$$\exp t\{H^{\text{fish-eye}},\circ\}=\exp(\omega t/2v)\{\mathcal{N},\circ\}. \tag{5.12a}$$

The result is then given by (5.10) and (5.11), replacing the parameter s by the time t through

$$s=\omega t/2\rho(1+|\mathbf{q}|^2/4\rho^2)p. \tag{5.12b}$$

This is the transformation of phase space along the orbits of the fish-eye system.

VI. WAVIZATION OF THE MAXWELL FISH-EYE

In this section we shall use another well-known realization of the symmetry and potential algebras and groups that describes wave optics. We will "wavize" (or "ondulate"?) the Maxwell fish-eye by a method analogous to the dynamical quantization of mechanical systems.[29]

The scalar wave equation for the field amplitude $\Phi(\vec{Q},t)$ in a homogeneous N-dimensional medium $\vec{Q}\in\Re^N$, of refractive index n_0, is

$$\sum_{i=1}^{N}\frac{\partial^2\Phi(\vec{Q},t)}{\partial Q_i^2}=\left(\frac{n_0}{c}\right)^2\frac{\partial^2\Phi(\vec{Q},t)}{\partial t^2}. \tag{6.1}$$

This equation may be put in evolution form (i.e., with first-order time derivative) through doubling the function space,

$$\begin{pmatrix}0 & 1\\ (c/n_0)^2\Delta & 0\end{pmatrix}\begin{pmatrix}\Phi\\ \dot\Phi\end{pmatrix}=\frac{\partial}{\partial t}\begin{pmatrix}\Phi\\ \dot\Phi\end{pmatrix}, \tag{6.2a}$$

where the space Laplacian Δ is

$$\Delta=\sum_{i=1}^{N}\frac{\partial^2}{\partial Q_i^2}. \tag{6.2b}$$

The first component equation in (6.2a) defines $\dot\Phi(\vec{Q},t)=\partial\Phi(\vec{Q},t)/\partial t$, and the second then reproduces (6.1). The solutions of the wave equation can be expressed in terms of the initial conditions $(\Phi_0,\dot\Phi_0)$ at $t=0$, through integration of the one-parameter evolution group. This is

$$\begin{pmatrix}\Phi(\vec{Q},t)\\ \dot\Phi(\vec{Q},t)\end{pmatrix}=\exp t\begin{pmatrix}0 & 1\\ (c/n_0)^2\Delta & 0\end{pmatrix}\begin{pmatrix}\Phi_0(\vec{Q})\\ \dot\Phi_0(\vec{Q})\end{pmatrix}, \tag{6.3}$$

provided the refractive index n_0 is independent of time—a good general assumption.

When light is of a definite *color*, i.e., when the time behavior of the wave function is that of a single Fourier component ν,

$$\Phi^{(\nu)}(\vec{Q},t)=\Phi_0(\vec{Q})e^{i\nu t}, \tag{6.4}$$

then the time-independent wave equation for a homogeneous medium factorizes as

$$\Delta\Phi_0(\vec{Q}) = \left(\frac{1}{|\vec{Q}|^{N-1}}\frac{\partial}{\partial|\vec{Q}|}|\vec{Q}|^{N-1}\frac{\partial}{\partial|\vec{Q}|}+\frac{\widetilde{\mathscr{C}}}{|\vec{Q}|^2}\right)\Phi_0(\vec{Q}) = -\left(\frac{\nu n_0}{c}\right)^2\Phi_0(\vec{Q}); \tag{6.5}$$

here we have further factorized into radial and angular variables

$$\widetilde{\mathscr{C}} = -\frac{1}{2}\sum_{j,k}^{N}\widetilde{\Lambda}_{j,k}^2 \tag{6.6a}$$

$$\widetilde{\Lambda}_{j,k} = i\left(Q_j\frac{\partial}{\partial Q_k} - Q_k\frac{\partial}{\partial Q_j}\right). \tag{6.6b}$$

Notice that in (6.5), the factor $(\nu n_0/c)^2 \geqslant 0$ appears where the familiar energy eigenvalue $2mE/\hbar^2$ appears in the time-independent Schrödinger equation of quantum mechanics.

The wave equation on the sphere $S_{N-1}\subset\Re^N$ is obtained from (6.1) after separation in spherical coordinates. This reduces the equation to

$$\widetilde{\mathscr{C}}\Phi(\vec{Q}) = \left(\frac{n_0\nu\rho}{c}\right)^2\Phi(\vec{Q}). \tag{6.7}$$

The operators (6.6b) are well known $\mathscr{L}_2(\Re^N)$ self-adjoint realizations of the rotation algebra and group generators. Only $N-1$ variables among the Q_i are independent because, as in (3.9), $Q_N = \sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}$. Through the stereographic projection (4.2) and (4.3) we may now map the Q_i coordinates of the ρ sphere in (6.6) onto the $\mathbf{q}$ plane $\Re^{N-1}$ where the Maxwell fish-eye lives. The chain rule for (4.2a) would yield

$$\frac{\partial}{\partial Q_i} = \left(1+\frac{|\mathbf{q}|^2}{4\rho^2}\right)\left(\frac{\partial}{\partial q_i}+\frac{2q_i}{4\rho^2-|\mathbf{q}|^2}\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{q}}\right), \tag{6.8}$$

for $i = 1,2,\ldots,N-1$. Compare with the geometrical (classical) expression (4.8) for the canonical conjugate P_j: functions on the ρ sphere no longer have an independent coordinate Q_N, and $\partial/\partial Q_N$ acts as zero and plays no further role.

While in $\Re^N$, the independent formal operators, Q_i and $\partial/\partial Q_j$ close into the N-dimensional Heisenberg–Weyl algebra, after restriction to S_{N-1} and subsequent stereographic projection, they do so only for $i,j = 1,2,\ldots,N-1$. The operators (6.8) generate translations that do not leave the sphere S_{N-1} invariant, so they cannot be exponentiated alone; they are not self-adjoint on the space of functions on the sphere. However, the rotation generators $\widetilde{\Lambda}_{i,j}$ in (6.6b) *are* self-adjoint, and hence valid operators on the sphere. Other valid operators are the Q_i, $i = 1,2,\ldots,N-1$, and products or uniformly convergent series thereof.

The Casimir operator $\mathscr{C}$ appears in (6.7); its eigenvalues on the space of single valued and bounded functions are

$$\lambda = l_N(l_N+N-2),\quad l_N = 0,1,2,\ldots. \tag{6.9a}$$

The light colors ν that the compact space S_{N-1} can sustain are thus limited to the discrete frequencies

$$\nu_N = \frac{c}{n_0\rho}\sqrt{l_N(l_N+N-2)},\quad l_N = 0,1,2,\ldots. \tag{6.9b}$$

We know[18] the Hilbert space $\mathscr{L}^2(S_{N-1})$ of Lebesgue square-integrable functions $\Phi(\vec{Q})$ over the sphere S_{N-1}. These functions are first mapped on the two functions $\overline{\Phi}_{\sigma=\pm1}(\mathbf{Q}) = \Phi(\vec{Q})$ on the balls in $\Re^{N-1}$ where $|\mathbf{Q}|<\rho$, as we saw in Sec. IV. Then we proceed through the stereographic projection on wave functions $\phi(\mathbf{q}) = \overline{\Phi}_\sigma(\mathbf{Q})$, with $\mathbf{q}(\mathbf{Q},\sigma)\in\Re^{N-1}$ as given in (4.2). The $\mathscr{L}^2(S_{N-1})$ inner product of two functions Φ, Ψ is thus

$$(\Phi,\Psi)_{\mathscr{L}^2(S_{N-1})} = \int_{S_{N-1}} d^{N-1}\Omega(\vec{Q})\Phi(\vec{Q})^*\Psi(\vec{Q}) \tag{6.10a}$$

$$= \sum_{\sigma=\pm1}\frac{1}{\rho^{N-2}}\int_{|\mathbf{Q}|<\rho}\frac{d^{N-1}\mathbf{Q}}{\sigma\sqrt{\rho^2-|\mathbf{Q}|^2}}\overline{\Phi}_\sigma(\mathbf{Q})^*\overline{\Psi}_\sigma(\mathbf{Q}) \tag{6.10b}$$

$$= \frac{1}{\rho^{N-1}}\int_{\Re^{N-1}}\frac{d^{N-1}\mathbf{q}}{(1+|\mathbf{q}|^2/4\rho^2)^{N-1}} \times\Phi(\vec{Q}(\mathbf{q}))^*\overline{\Phi}(\vec{Q}(\mathbf{q})) \tag{6.10c}$$

$$= (\phi,\psi)_{\text{fish-eye}} = \int_{\Re^{N-1}} d^{N-1}\mathbf{q}\,\phi(\mathbf{q})^*\psi(\mathbf{q}), \tag{6.10d}$$

where

$$\phi(\mathbf{q}) = \frac{\Phi(\vec{Q}(\mathbf{q}))}{\sqrt{(1+|\mathbf{q}|^2/4\rho^2)^{N-1}}}, \tag{6.11}$$

and Ψ and ψ are bound by a similar relation.

Under this inner product, symmetry transformations are unitary and their infinitesimal generators (6.6b) are self-adjoint. When we use the customary inner product form $(\phi,\psi)_{\text{fish-eye}}$ in (6.10d) for a "flat" space of measure $d^{N-1}\mathbf{q}$, the so(N) generators $\hat{L}_{i,j}$ and $\hat{M}_i$ are the Schrödinger quantization of the "classical functions" in (3.17) and (3.18) and (4.9) and (4.10) [i.e., through the replacements $q_i\mapsto\hat{q}_i\cdot$, (multiplication by q_i), and $p_i\mapsto\hat{p}_i = -i\,\partial/\partial q_i$]. Because the functions involved are linear in the components of p_i, there is no operator-ordering ambiguity in this case: any quantization rule that guarantees self-adjointness under the inner product $\int_{\Re^{N-1}} d^{N-1}\mathbf{q}\cdots$ Ref. 24 yields

$$f(q_i)p_j \mapsto \tfrac{1}{2}\{f(\hat{q}_i),\hat{p}_j\}_+ = \tfrac{1}{2}[f(\hat{q}_i)\hat{p}_j + \hat{p}_j f(\hat{q}_i)] = -if(q_i)\frac{\partial}{\partial q_j}+\frac{1}{2}i\frac{\partial f(q_i)}{\partial q_j}.$$

This allows us to write the Maxwell fish-eye *dynamical* generators $\hat{K}_i$ in (5.4), independent of any ordering rule.

The optical fish-eye Hamiltonian is the Casimir operator of SO(N) built in (3.17) and (4.9). By itself, as a function in $(\mathbf{Q},\mathbf{P})$ or $(\mathbf{q},\mathbf{p})$, the Hamiltonian function would be subject to ordering-rule ambiguities;[24] however, as Casimir operator, $\mathscr{C}$ is the sum of squares of the operators (6.6a); defined thus, the Hamiltonian is unique and independent of the quantization scheme. All higher-order Casimir invariants are zero: the sphere S_{N-1} can only support the *totally symmetric* representations of SO(N). The number of independent Maxwell fish-eye states that are degenerate for some l_N is given by the branching rules of the so(N) representations. The representation *row* indices are provided by the canonical basis,[14] and given as a $(N-1)$-plet $\{l_N, l_{N-1},\ldots,l_2\}$, with the integer labels l_j bound by

$l_N \geqslant l_{N-1} \geqslant \cdots \geqslant l_3 \geqslant |l_2|$. The count is $2l_2 + 1$ for the spherical harmonics on the surface of the ordinary sphere S_2, and $(l_3+1)^2$ for the curved S_3 space that we use in the model of the "physical" Maxwell fish-eye.

In the plane two-dimensional optical world projected in Fig. 2 ($N=3$), the description of the wave patterns in the Maxwell fish-eye is easy: The labels are the usual $\{l,m\}$, and the wave solutions are in the linear span of the ordinary solid spherical harmonic basis $\{\mathscr{Y}_{l,m}(\vec{Q})\}_{m=-l}^{l}$ (Ref. 18), each vibrating with an angular frequency (6.9b), namely

$$\nu(l) = (c/n_0\rho)\sqrt{l(l+1)}, \quad l = 0,1,2,\ldots . \tag{6.12a}$$

The time evolution of a linear combination of harmonics with coefficients $f_{l,m}$ is then

$$\Phi(\vec{Q},t) = \sum_{l=0}^{\infty} e^{i\nu(l)t} \sum_{m=-l}^{l} f_{l,m}\mathscr{Y}_{l,m}(\vec{Q}). \tag{6.12b}$$

These functions can be visualized as a pattern of light with intensity $(\Phi,\Phi)_{\mathscr{L}^2(S_2)}$ on the ordinary S_2 sphere of radius ρ. Light of a given color number l has only $2l+1$ distinct wave patterns labeled by $m = 0, \pm 1,\ldots, \pm l$. Rotations of the sphere will mix m's, maintaining the linear subspaces l invariant. When the stereographic projection (4.3) is applied, with the weight (obliquity) factor given in (6.11), the projections of the spherical harmonics on the optical $\mathbf{q}$ space will provide an $\mathscr{L}^2(\Re^2)$-orthogonal basis for the Maxwell fish-eye solutions. These are

$$\Upsilon_{l,m}(\mathbf{q},t) = \frac{Y_{l,m}(\mathbf{q}/(1+|\mathbf{q}|^2/4\rho^2))}{1+|\mathbf{q}|^2/4\rho^2} \times \exp(ict\sqrt{l(l+1)}/n_0\rho, \tag{6.13}$$

where we have written the solid spherical harmonic constrained to the sphere in Cartesian coordinates as $\mathscr{Y}_{l,m}(\vec{Q})|_{S_2} = \mathscr{Y}_{l,m}(\mathbf{Q},\sigma\sqrt{\rho^2-|\mathbf{Q}|^2}) = \rho^l Y_{l,m}(\mathbf{Q},\sigma)$.

Consider first the "extreme" $m = \pm l$ wave patterns $\Upsilon_{l,\pm l}(\mathbf{q}(\beta,\gamma),t)$, using polar angles (β,γ) on the sphere. The functions behave as $\sin^l\beta \exp i(\pm l\gamma + ct/n_0\rho\sqrt{l(l+1)})$ and will exhibit l moving nodal meridians with a braid of maxima at the equator $\beta = \pi/2$. The pattern is a function of $\gamma + \omega_l t$, where ω_l is the angular velocity of the sphere, $\omega_l = (c/n_0\rho)\sqrt{1+1/l}$, $l = 1,2,\ldots$. That rotating light pattern will project on the optical plane as the circular motion of waves in the fish-eye, with l nodes as spokes in a rigid rotating wheel. The belt of light maxima may also rotate on an inclined axis; it will then project its equatorial braid on an off-center circle on the fish-eye plane, the nodal meridians will project on circular nodes that cross through the two conjugate points that are images of the new rotation poles. These 'circle-of-light' rotating solutions, we surmise, are the best wave analogs of the geometric light orbits, such as that of Fig. 2.

We note that, inevitably, *chromatic dispersion* takes place: $\omega_l \sim \sqrt{1+1/l}$ is not independent of l, as it is in a homogeneous optical medium where wave velocity is independent of wave number. For $l=0$, $\Upsilon_{0,0}(\mathbf{q}) = \text{const}/(1+|\mathbf{q}|^2/4\rho^2)$ in (6.13). For growing l, $|\omega_l|$ decreases monotonically from $\omega_1 = c\sqrt{2}/n_0\rho$ down to $\omega_\infty = c/n_0\rho$. Asymptotically with l, we see that the surface of the sphere of radius ρ moves at the equator with velocity $v = \omega_\infty = c/n_0$.

Another set of harmonics that are easy to visualize, are the $m=0$ solutions $\mathscr{Y}_{l,0}(Q_3)$. They contain a Legendre polynomial $P_l(\cos\beta)$, that has l nodal circles (only one can be the equator great circle); they are independent of the longitude angle γ. These linear multipole standing-wave solutions have their global absolute maxima at the two sphere poles, with a relative sign $(-1)^l$ between the two, and their single, *sui generis* vibration frequency $\nu(l)$ in (6.12a). When this vibration mode is tilted by SO(3) to any angle and projected on $\mathbf{q}$ space, the strongly elongating polar regions will map on a conjugate pair of pulsating light zones in the Maxwell fish-eye. They will be in or out of phase according to the parity of l. A Dirac δ flash at some point of the fish-eye, or at a closed wave front line $\mathbf{q}(\beta_0,\gamma)$, $\gamma \in S_1$, will decompose by

$$\delta(\cos\beta - \cos\beta_0) \sim \sum_{l=0}^{\infty} P_l(\cos\beta)P_l(\cos\beta_0)$$

into a weakly convergent series of the above conjugate-pair "standing waves."

There is dispersion again. Under time evolution, the coefficients $P_l(\cos\beta_0)$ in the series will be multiplied by $e^{i\nu(l)t}, \nu(l) \sim \sqrt{l(l+1)}$, whose periods are incommensurable. Thus although the optical path between two conjugate points is equal along any circle arc joining the points, and wave fronts are well defined,[1] we see that the Maxwell fish-eye is *not quite* a perfect imaging device[2] in the sense that it cannot forestall the chromatic dispersion that will smear out any pulse. This is in contrast with optics in a homogeneous space, where spherical δ wave fronts propagate as such in odd dimensions (and develop a trailing wake in even dimensions[17]). Although we can work with Dirac δ's on the S_2 sphere, they are not *eigen*functions of any rotation generators.

In the plane optical world of Fig. 2, the manifest symmetry generator is $\hat{L}_{1,2}$ and the hidden symmetry generators are $\hat{M}_1$ and $\hat{M}_2$. The extra generators of the dynamical algebra so(3,1) are $\hat{K}_1$, $\hat{K}_2$, and $\hat{K}_3$, given by (5.4c) and (5.4d). This enlarged linear space of operators may be used to define other bases for the polychromatic, wavized fish-eye. We refer to Eqs. (5.4c) to choose the two commuting operators

$$\hat{p}_i = \frac{1}{2\rho}(\hat{M}_i - \hat{K}_i) = -i\frac{\partial}{\partial q_i}, \quad i = 1,2, \tag{6.14}$$

that are generators of the Euclidean algebra iso(3) together with the symmetry generators. These define the plane-wave generalized basis of the $\mathscr{L}^2(\Re^2)$ with the inner product $(\cdot,\cdot)_{\text{fish-eye}}$ in (6.10d). Such solutions quickly loose their shape under Maxwell fish-eye evolution because they are eigenfunctions of operators that do not commute with the driving Hamiltonian. Their time evolution may be calculated group theoretically through the transformation (5.10a) or as overlap coefficients between the elliptic (l,m) and parabolic (p_1,p_2) subgroups of the SO(3,1) group that we have realized on the $\Re^2$ plane.

Extending further the dynamical algebra group to the conformal SO(3,2) by the geometric-optics generators $H_i = q_i p$, $i = 1,2$, $H_3 = \mathscr{N} - 2\rho p$, and $H_4 = \mathscr{N}$

$= \rho(1+|q|^2/4\rho^2)p$ [generically SO$(N,2)$ for $(N-1)$-dimensional fish-eyes], we have integral operators: the scalar root of the Laplacian, $p=\sqrt{p_1^2+p_2^2}$. Its action $\hat{p}$ on smooth, properly decreasing functions $\phi(\mathbf{q})$ is through a formally divergent kernel (as a Dirac δ' derivative) given by

$$(\hat{p}\phi)(\mathbf{q}) = \int_{\mathfrak{R}^2} d^2\mathbf{q}'\Pi(\mathbf{q}-\mathbf{q}')\phi(\mathbf{q}'),$$

$$\Pi(\mathbf{q}) = \frac{i}{4\pi^2}\int_{\mathfrak{R}^2} d^2\mathbf{p}|\mathbf{p}|\exp i\mathbf{p}\cdot\mathbf{q}. \tag{6.15}$$

This scalar root operator may enter commutators and symmetrized products. Indeed, the Heisenberg–Weyl commutators of

$$\widehat{H}_i = \tfrac{1}{2}(\hat{q}_i\hat{p}+\hat{p}\hat{q}_i) \text{ and } \widehat{\mathcal{N}} - \rho\hat{p} = \tfrac{1}{2}(|\hat{q}|^2\hat{p}+\hat{p}|\hat{q}|^2)$$

close, with the rest of the SO(3,1) generators, into the Lie algebra of SO(3,2) in Eqs. (5.3). In this realization we have the SO(3) Casimir operator written as $\widehat{\mathcal{N}}(\widehat{\mathcal{N}}+1)$. Generally, for SO$(N,2)$, it is $\widehat{\mathcal{N}}(\widehat{\mathcal{N}}+N-2)$.

VII. CONCLUDING REMARKS

When a "physical" three-dimensional optical medium is homogeneous, the symmetry of the system is the Euclidean algebra iso(3).[9] In this case there are no additional, hidden symmetries. The optical Hamiltonian is then also the Casimir invariant, with eigenvalues $-k^2 \geqslant 0$; the irreducible representation is then labeled by the wave number $k \in \mathfrak{R}$ (the second Casimir, $\vec{L}\cdot\vec{p}$, is zero). We may thus isolate any single "color" k and work with monochromatic optics. On the other hand, the Maxwell fish-eye seen here has the symmetry algebra so(4) and its representation labels are *discrete*, allowed only for discrete colors ν_n, $n=0,1,2,\ldots$. Once one n is chosen, the space of wave functions is of finite dimensions n^2, just as in the hydrogen atom. All these so(4) representations fit into a single degenerate representation of its dynamical algebra so(4,1).

The dynamical algebra so(4,1) also contains the *Euclidean* iso(3) algebra of rotations and space translations of homogeneous media. Homogeneous and fish-eye optical spaces are thus identified as different subalgebra reductions of their common dynamical algebra so(4,1). The iso(3) algebra is a contraction of so(4) by the scaling generator $\Lambda_{N,N+1}$ seen in (5.4d). In this sense, so(4,1) is the *potential* algebra[19] (or group,[20] that binds the fish-eye light orbits to free propagation in a homogeneous optical medium.

The role of the larger dynamical algebra so(4,2) in the Maxwell fish-eye is more subtle: it yields the Hamiltonian time evolution as a *number* generator in the algebra, $\mathcal{N} = \Lambda_{5,6}$, that exponentiates easily to the evolution subgroup. Indeed, the same strategy of finding a larger group, applied to homogeneous space optics, will use the square root p of the Casimir invariant p^2 of iso(3). This quantity is $p=(\Lambda_{4,6}-\Lambda_{5,6})/2\rho$ and commutes with all iso(3) operators. The scaling generator also contracts $\mathcal{N}$ to p.

We have seen here that there is a realization of the algebras that models scalar geometrical optics, and another realization that models *wave* optics, both for the Maxwell fish-eye and for homogeneous media. Evolution along an optical axis has been the primary concern for Euclidean optics,[9] while evolution in time is highlighted here to predict chromatic dispersion.

Small neighborhoods around points q_0 in a smooth radially symmetric inhomogeneous medium may be approximated by neighborhoods of Maxwell fish-eyes that are displaced by $\mathbf{v}$ and/or scaled by ρ and n_0, in such a way as to approximate the refractive index $n(\mathbf{q}_0+\mathbf{q}) = n(\mathbf{q}_0) + \Sigma_i q_i[\partial_{qi} n(\mathbf{q})]_{\mathbf{q}_0} + \cdots$ by the "local" fish-eye shape

$$n_0/(1+|\mathbf{q}-\mathbf{v}|^2/4\rho^2) \sim n_0(1+|\mathbf{v}|^2/4\rho^2) - n_0\mathbf{q}\cdot\mathbf{v}/2\rho^2 + \cdots$$

through their value and gradient, when $n(|\mathbf{q}|)$. This construction is a section in a bundle over configuration space, where for each $\mathbf{q}\in\mathfrak{R}^3$ there is an so(4,2) evolution direction determined by the local Hamiltonian. While the curvature is positive, the "compact" $\Lambda_{5,6}$-generated subgroup is followed, or its translates by SO(2,1) $\subset$ SO(4,2) group transformations. In the regions where n is constant, the direction is along the free-flight Euclidean number operator p. Finally, when the curvature is negative ($\rho \mapsto i\rho$), the "noncompact" generator is $\Lambda_{4,6}$. This 'hyperbolic' Maxwell fish-eye carries its corresponding local so(3,1) symmetry algebra. Work is being done to understand further the Lie algebra and global group properties of particular inhomogeneous optical systems.

ACKNOWLEDGMENTS

For references and useful advice on inhomogeneous media and group theory, we would like to thank Miguel Cervantes (Instituto Nacional de Astrofísica, Optica y Electrónica, Tonantzintla), Bogdan Mielnik (Centro de Investigación y Estudios Avanzados, IPN), Enrique López Moreno (Facultad de Ciencias, UNAM), Thomas H. Seligman (Instituto de Física, UNAM-Cuernavaca), Gerardo Torres del Castillo (Universidad Autónoma de Puebla), and R. López-Peña (ICN-UNAM).

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