The relativistic coma aberration. II. Helmholtz wave optics

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The relativistic coma aberration of geometric optics was examined in the first paper [J. Math. Phys. 30, 2457 (1989)]. Here is a study of a unitary realization of the Euclidean group, the dynamic group for global optics, on the space of solutions of the Helmholtz equation. Deformation to the Lorentz group of relativity yields the self-adjoint generators of boost transformations on that space. Graphic results for the action of a boost normal to the screen on an off-axis Gaussian beam, that may be compared with classical wave optics results on diffraction in aberration, are presented.

I. INTRODUCTION

In the first part of this work[1] we posed the well-known phenomenon of stellar aberration in the context of geometrical Lie optics in phase space.[2] Given a relativistic distortion of ray directions on the sphere, we showed that the canonically conjugate ray positions undergo magnification and a circular comatic aberration. This we called relativistic coma. The phenomenon was analyzed globally, i.e., on the full, proper phase space manifold of rays, and given in closed, explicit formulas, and also as an expansion in aberration up to seventh order. We pointed out that the basic dynamical group of optics is the Euclidean group,[3] rather than the Heisenberg–Weyl,[4] diamond,[5] or Weyl-symplectic[6] groups used in nonrelativistic quantum mechanics. The latter group appears as a *contraction* of the first in the paraxial approximation.[7]

The Euclidean ISO(3) group of rigid motions of three-space was *deformed*[8,9] to relativistic SO(3,1) transformations. This provided stellar aberration for boosts on the ray directions of geometric optics. The spot diagrams were obtained for the full direction sphere, with the standard comatic appearance for boosts normal to the screen plane. For boosts in the screen plane, the aberration had the characteristics of an astigmatism and curvature of field, along and normal to the boost.[1] The group realization we used was that of geometrical optics phase space.

In this paper we apply the same construction to the same groups, but in the realization on the space of solutions of the Helmholtz equation studied in Ref. 10. Geometric optics has no time variable; the Helmholtz equation does not contain it either. The space of solutions f of

$$(\partial_x^2+\partial_y^2+\partial_z^2+k^2)f(x,y,z)=0 \tag{1.1}$$

that are square integrable over any plane *screen* and whose spectrum there is bounded by k, is invariant under the Euclidean group ISO(3) of screen translations and rotations, and we call it $\mathscr{H}_k$. For $\mathbf{f}\in\mathscr{H}_k$, written as a two-function column $\mathbf{f}(\mathbf{q})=(f(\mathbf{q}),f_z(\mathbf{q}))$ on the plane $\mathbf{q}=(x,y)$, we can present the Helmholtz equation in the form

$$\mathbf{H}\mathbf{f}(\mathbf{q},z)=\begin{pmatrix}0 & 1\\ -\Delta_k & 0\end{pmatrix}\begin{pmatrix}f(\mathbf{q},z)\\ f_z(\mathbf{q},z)\end{pmatrix}=\frac{\partial}{\partial z}\begin{pmatrix}f(\mathbf{q},z)\\ f_z(\mathbf{q},z)\end{pmatrix}, \tag{1.2}$$

where $\Delta_k=\partial_x^2+\partial_y^2+k^2$. The formal solution may be written as an evolution of initial conditions:

$$\mathbf{f}(\mathbf{q},z)=\exp(z\mathbf{H})\mathbf{f}(\mathbf{q}), \tag{1.3}$$

for $f(\mathbf{q})$ and its normal derivative $f_z(\mathbf{q})$ at the reference screen $z=0$ to all of $\mathfrak{R}^3$. In $\mathscr{H}_k$, the initial value problem for the system is well posed.[10] As in geometrical optics, we work with observables and wavefunctions in the plane of a screen.

We present the realization of the Euclidean group ISO(3) on $\mathscr{H}_k$ in Sec. II, completed with respect to an inner product that is conserved under Euclidean transformations of the screen. We regard the ensuing unitary representation of the Euclidean group as defining the Helmholtz wavization of geometrical optics. In Sec. III we proceed to deform ISO(3) to the Lorentz group SO(3,1) on $\mathscr{H}_k$. In Sec. IV we present the results for the z boost studied in Ref. 1 on an off-center forward Gaussian beam. The explicit computation is done to fifth order in the relativistic boost parameter. The "isophotes" of $|f(\mathbf{q})|^2$ are comparable to those seen and calculated[11] for diffraction in third-order pure Seidel coma. The closing section presents some conclusions and open comments on nonlocality, observability, and the role of the normal derivative $f_z(\mathbf{q})$ of the field at the screen.

II. THE EUCLIDEAN GROUP OF HELMHOLTZ WAVE OPTICS

A well known realization of the generators of the Euclidean algebra of translations and rotations on smooth functions of $\mathfrak{R}^3$, is given by

$$P'_x=-i\text{ƛ}\partial_x,\quad P'_y=-i\text{ƛ}\partial_y,\quad P'_z=-i\text{ƛ}\partial_z, \tag{2.1a}$$

$$R'_x=i(y\partial_z-z\partial_y),\quad R'_y=i(z\partial_x-x\partial_z),\quad R'_z=i(x\partial_y-y\partial_x), \tag{2.1b}$$

where ƛ is a consant with units of length, to render the operators dimensionless. The action of the corresponding Lie exponential group ISO(3) is that of ordinary, rigid transformations of $\mathfrak{R}^3$. These operators are self-adjoint in $\mathscr{L}^2(\mathfrak{R}^3)$

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and their (i) exponential, i.e., $\exp(i\alpha_j P_j')$, $\exp(i\beta_j R_j')$, $j = x,y,z$, $\alpha \in \Re$, is unitary.

For functions in the solution space $\mathscr{H}_k$ of the Helmholtz equation (1.1), we may replace[10] ∂_z by the matrix operator in (1.2), acting on two-functions $\mathbf{f}(\mathbf{q}) = (f(\mathbf{q}), f_z(\mathbf{q}))$ at the reference $z = 0$ plane, and identify $\lambda\!\!\!^{-}$ with the reduced wavelength of (1.1), namely, $\lambda\!\!\!^{-} = \lambda/2\pi = 1/k$. This takes the place of $\hbar$ in quantum mechanics. In the Helmholtz realization, the generators of translations become

$$P_x = -i\lambda\!\!\!^{-}\begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}, \quad P_y = -i\lambda\!\!\!^{-}\begin{pmatrix} \partial_y & 0 \\ 0 & \partial_y \end{pmatrix}, \quad P_z = -i\lambda\!\!\!^{-}\begin{pmatrix} 0 & 1 \\ -\Delta_k & 0 \end{pmatrix}. \tag{2.2a}$$

As in geometrical optics, P_z takes the role of the Hamiltonian operator, generating *z evolution* in the system. The generators of rotations become

$$R_x = -i\begin{pmatrix} 0 & y \\ -y\,\Delta_k - \partial_y & 0 \end{pmatrix}, \quad R_y = -i\begin{pmatrix} 0 & -x \\ x\,\Delta_k + \partial_x & 0 \end{pmatrix}, \quad R_z = -i\begin{pmatrix} x\,\partial_y - y\,\partial_x & 0 \\ 0 & x\,\partial_y - y\,\partial_x \end{pmatrix}. \tag{2.2b}$$

Note that in the 2-1 elements of R_x and R_y there could be an ordering ambiguity between x, y, and Δ_k. These elements are given by the anticommutator, $\frac{1}{2}(x\,\Delta_k + \Delta_k x)$; that is the only possibility when we demand closure under commutation

$$[R_x, R_y] = iR_z, \quad [R_x, P_y] = iP_z, \quad [P_x, P_y] = 0, \quad \text{and cyclically}. \tag{2.3}$$

The i's above fit generators that are self-adjoint under an inner product. It is easy to see that an $\mathscr{L}^2(\Re^2)$ product allows for the self-adjointness of the diagonal matrix operators P_x, P_y, and R_z, generators of the ISO(2) symmetry group of screen motions in its plane, but not for the z evolution P_z and the out-of-screen rotations R_x and R_y.

In Ref. 10, Steinberg and Wolf found the (unique) Euclidean-invariant inner product of solutions of the (well-posed) Helmholtz equations in two dimensions, through proposing a nonlocal sesquilinear form. In the case of *three* dimensions, we have

$$(\mathbf{f},\mathbf{g})_{\mathscr{H}_k} = \int_{\Re^2} d^2\mathbf{q} \int_{\Re^2} d^2\mathbf{q}'\,\mathbf{f}(\mathbf{q})^\dagger \mathbf{M}^k(\mathbf{q},\mathbf{q}')\mathbf{g}(\mathbf{q}'). \tag{2.4a}$$

The 2×2 matrix $\mathbf{M}^k(\mathbf{q},\mathbf{q}') = \|M^k_{j,j'}(x,y,x',y')\|$ is determined by the requirement that the algebra operators P_j, R_j, satisfy $(\mathbf{f}, P_j\mathbf{g})_{\mathscr{H}_k} = (P_j\mathbf{f}, \mathbf{g})_{\mathscr{H}_k}$, etc. We find here

$$\mathbf{M}^k(\mathbf{q},\mathbf{q}') = \begin{pmatrix} kj_1(k\,|\mathbf{q}-\mathbf{q}'|)/|\mathbf{q}-\mathbf{q}'| & 0 \\ 0 & j_0(k\,|\mathbf{q}-\mathbf{q}'|) \end{pmatrix}, \tag{2.4b}$$

where j_0 and j_1 are the spherical Bessel functions. The form is positive definite on $\mathscr{H}_k$. Completion with respect to this inner product turns $\mathscr{H}_k$ into a Hilbert space where the Euclidean transformations are unitary. This realization we call the Helmholtz representation of ISO(3). The Casimir invariants are $P^2 = P_x^2 + P_y^2 + P_z^2 = 1$, and $\mathbf{P}\cdot\mathbf{R} = 0$. This means that we have a sphere for homogeneous space under the Euclidean algebra, its enveloping algebra, and the group; physically, it is a homogeneous space of unit refractive index. The SO(3) subgroup Casimir operator is diagonal (but not simply a multiple of the unit operator):

$$\mathbf{R}^2 = R_x^2 + R_y^2 + R_z^2 = \begin{pmatrix} D(D+1) + q^2k^2 & 0 \\ 0 & (D+1)(D+2) + q^2k^2 \end{pmatrix}, \tag{2.5}$$

where

$$D = x\,\partial_x + y\,\partial_y. \tag{2.6}$$

The Helmholtz representation space $\mathscr{H}_k$ has an inner product (2.4) that is nonlocal in the wave function $f(\mathbf{q})$ to the extent of $j_1(|\mathbf{q}|/\lambda\!\!\!^{-})/(|\mathbf{q}|/\lambda\!\!\!^{-})$, and in the normal derivative function $f_z(\mathbf{q})$ to the extent of $j_0(|\mathbf{q}|/\lambda\!\!\!^{-})$, of the order of $\lambda\!\!\!^{-}$. Both f and f_z contribute to the energy of an elastic medium[12] so we may identify $(\mathbf{f},\mathbf{f})_{\mathscr{H}_k}$ with total field energy on the screen. As we shall show below, this inner product may be brought to local form in an appropriate transform space. Finally, it seems we should identify $|f(\mathbf{q})|^2$ with the visible image illumination.

The Euclidean algebra and its covering have been used by Vilenkin[13] and Miller[14] to find all separable solutions of the Helmholtz equation (1.1). The algebra itself yields the three subalgebra bases of Cartesian, cylindrical, and spherical coordinates; the corresponding separated functions are plane waves, the nondiffracting J_m beams of Durnin *et al.*,[15] and multipole solutions. The covering algebra provides the rest of the 11 coordinate system where the equation separates. Let us draw attention here to the plane-wave generalized eigenbasis of the translation subalgebra P_x, P_y, and a sign (of P_z). Up to an arbitrary normalization constant κ, with units of (illumination/area)$^{1/2}$,

$$\Phi_{k_x,k_y,\sigma} = \frac{\kappa}{2\pi}\begin{pmatrix} 1 \\ ik_z \end{pmatrix} e^{i(xk_x + yk_y)} \tag{2.7}$$

is a plane wave on the screen, labeled by the respective dimensionless eigenvalues, $\lambda\!\!\!^{-} k_x$, $\lambda\!\!\!^{-} k_y$, and $\sigma \in \{-1,0,+1\}$ $(k_z = \sigma\sqrt{k^2 - k_x^2 - k_y^2})$. The manifold of plane waves (k_x,k_y,σ) is that of two disks δ_k^σ, of radius k, sown at the edge of δ_k^0. This is the sphere of ordinary plane-wave direction three-vectors, projected on its equatorial screen plane. Solutions in $\mathscr{H}_k$ may be written as a generalized linear combination of these basis functions,

$$\mathbf{f}(\mathbf{q}) = \lambda^2 \sum_\sigma \int_{\delta_k^\sigma} d^2\mathbf{k}\,\tilde{f}_\sigma(\mathbf{k})\Phi_{\mathbf{k},\sigma}(\mathbf{q}). \tag{2.8}$$

We refer to the ordinary two-dimensional Fourier transform to write explicitly

$$f(\mathbf{q}) = \kappa\lambda\!\!\!^{-2} \int_{\delta_k^+} d^2\mathbf{k}\,[\tilde{f}_+(\mathbf{k}) + \tilde{f}_-(\mathbf{k})]e^{i\mathbf{k}\cdot\mathbf{q}}, \tag{2.9a}$$

$$f_z(\mathbf{q}) = i\kappa\lambda\!\!\!^{-2} \int_{\delta_k^+} d^2\mathbf{k}\,k_z[\tilde{f}_+(\mathbf{k}) - \tilde{f}_-(\mathbf{k})]e^{i\mathbf{k}\cdot\mathbf{q}}. \tag{2.9b}$$

Now, replacing (2.9) in the convoluted inner product (2.4), we exchange integrals and note that the $\mathbf{q}$-Fourier transform of $kj_1(k|\mathbf{q}|)/|\mathbf{q}|$ is k_z, and that of $j_0(k|\mathbf{q}|)$ is $1/k_z$ on $\delta_k^\pm$, and zero outside. We may thus write the Parseval relation between (2.4) and the local form on one disk δ_k,

$$(\mathbf{f},\mathbf{g})_{\mathscr{H}_k} = 2\pi|\kappa\hbar|^2 \int_{\delta_k} d^2\mathbf{k}\,\frac{k}{k_z}\,[\tilde{f}_+(\mathbf{k})^*\tilde{g}_+(\mathbf{k}) + \tilde{f}_-(\mathbf{k})^*\tilde{g}_-(\mathbf{k})]. \quad (2.10)$$

This is a local integration over the wave-vector sphere projected on the screen plane, including both forward (+) and backward (−) waves, with the obliquity factor $k/k_z = \sec\theta$, where θ is the angle between the wave three-vector and the normal to the screen. The $\mathscr{L}^2(\Re^2)$ norm majorizes the $\mathscr{H}_k$ norm.

The Euclidean group has thus a geometrical optics model and a Helmholtz optics model. The generators of the Abelian translation ideal of the abstract Lie algebra, P_x, P_y, and P_z, are in geometrical optics the optical momenta and the Hamiltonian p_x, p_y, and $p_z = h = \sqrt{1-|\mathbf{p}|^2}$ (for unit refractive index); the Lie bracket is the Poisson bracket. In Helmholtz wave optics, the homomorphic realization of this subalgebra is given by (2.3a) acting on $\mathscr{H}_k$ described above, and the Lie bracket is the commutator.

III. THE DEFORMATION OF THE EUCLIDEAN TO THE LORENTZ GROUP

Out of the Helmholtz representation of the Euclidean algebra (2.2)–(2.4) we may construct a representation of the Lorentz algebra through deformation. The deformation extends to the corresponding groups. The generators of the SO(3,1) Lorentz group are the following: the SO(3) generators are those of the Euclidean group R_x, R_y, and R_z in (2.2b); the boosts are built as $B_j = [R^2, P_j] + (\tau + i)P_j$, $j = x,y,z$. This formula is the heart of the deformation process.[16] For real τ, the boost generators are self-adjoint in $\mathscr{H}_k$, and belong to the nonexceptional continuous representation series. As in Ref. 1, we set $\tau = 0$ on the grounds that this parameter only reflects an admixture of translations to the basic boost transformation.

We build the following explicit boost matrix operators:

$$B_x = R_zP_y - R_yP_z + iP_x = \hbar\begin{pmatrix} (D+1)\partial_x + k^2x & 0 \\ 0 & (D+2)\partial_x + k^2x \end{pmatrix}, \quad (3.1a)$$

$$B_y = R_xP_z - R_zP_x + iP_y = \hbar\begin{pmatrix} (D+1)\partial_y + k^2y & 0 \\ 0 & (D+2)\partial_y + k^2y \end{pmatrix}, \quad (3.1b)$$

$$B_z = R_yP_x - R_xP_y + iP_z = \hbar\begin{pmatrix} 0 & D+1 \\ -(D+2)\Delta_k + k^2 & 0 \end{pmatrix}. \quad (3.1c)$$

We may verify that these indeed close into the Lie algebra of the SO(3,1) group:

$$[R_x,R_y] = iR_z, \quad [R_x,B_y] = iB_z, \quad [B_x,B_y] = -iR_z, \quad \text{and cyclically}. \quad (3.2)$$

The Lorentz Casimir invariants are $B^2 - R^2 = \mathbf{1}$ [for arbitrary τ it would be $(\tau+1)^2\mathbf{1}$], and $\mathbf{B}\cdot\mathbf{R} = 0$.

In most applications of Lie theory—including geometric optics[1]—the algebra generators are functions of some variable(s) ξ, times a first-order differential operator in ξ, that we write $\partial^{(1)}$ for short. These lead to point-to-point transformations $\exp(i\alpha\,\partial^{(1)})\,f(\xi) = f(\xi'(\xi;\alpha))$. Bargmann[8] deformed the Euclidean algebra with $\partial^{(1)}$ plus $\partial^{(0)}$—functions of ξ; this leads to multiplier representations of the group as $\exp(i\alpha[\partial^{(1)} + \partial^{(0)}])\,f(\xi) = \mu(\xi;\alpha)\,f(\xi'(\xi;\alpha))$. The multiplier function μ just offsets the change in the measure $\partial\xi/\partial\xi'$ so that the generators be Hermitian; when technical conditions on the domain hold, they are self-adjoint.[8] The exponentiation of second-[17,18] and higher-order[19] differential operators leads in general to integral transforms: $\exp(i\alpha\partial^{(\geqslant 2)})\,f(\xi) = \int d\xi'\,K(\xi,\xi';\alpha)\,f(\xi')$. The particular case of paraxial scalar wave optics[20] is well studied and requires of $\partial^{(0)}$, $\partial^{(1)}$, and $\partial^{(2)}$; it leads to the canonical integral transforms.[6,18]

Of the boost generators (3.1), B_x and B_y are matrices with elements $\partial^{(0)} + \partial^{(1)} + \partial^{(2)}$ on the diagonal, while B_z has $\partial^{(0)} + \partial^{(1)}$ and $\partial^{(0)} + \partial^{(1)} + \partial^{(2)} + \partial^{(3)}$ on the antidiagonal. They seem to be more difficult to exponentiate in closed form than they were in Ref. 1, and here we do not attempt to do so. We can state, however, that because of the homomorphism of transformations of ray momenta in geometric optics and plane-wave vectors under SO(3,1), the latter will behave in the same way as the former [viz., Ref. 1, Eq. (4.6)], conforming to the distortion of directions characteristics of stellar aberration, namely,

$$\tan\tfrac{1}{2}\theta \mapsto \tan\tfrac{1}{2}\theta' = e^{-\alpha}\tan\tfrac{1}{2}\theta, \quad (3.3)$$

where $\theta,\theta' \in [0,\pi]$ are the angles between the rays or wave vectors and the direction of the boost to velocity $v = c\tanh\alpha$. We now examine the effect of these unitary transformations on the screen images for the case of boosts in the z direction.

IV. THE COMA ABERRATION IN z BOOSTS

We consider boosts in the direction normal to the screen. The antidiagonal matrix elements of B_z in (3.1c) may be written in terms of formal Schrödinger operators [$\mathsf{Q}_jf(\mathbf{q}) = q_jf(\mathbf{q})$, $\mathsf{P}_jf(\mathbf{q}) = -i\hbar\,\partial_{q_j}f(\mathbf{q})$, $j = x,y$, distinguished in sans serif font] in the following way:

$$\mathscr{D} = \hbar(D+1) = i\mathsf{Q}\cdot\mathsf{P} + \hbar = i(\mathbf{p}\cdot\mathbf{q})_Q = 2iX_0^1, \quad (4.1a)$$

$$\begin{aligned}\mathscr{E} &= \hbar[k^2 - (D+2)\Delta_k] \\ &= ik^2[\mathsf{Q}\cdot\mathsf{P}\mathsf{P}^2 - \mathsf{Q}\cdot\mathsf{P} - 2i\hbar\mathsf{P}^2 + i\hbar] \\ &= ik^2[(p^2\mathbf{p}\cdot\mathbf{q})_Q - (\mathbf{p}\cdot\mathbf{q})_Q] = ik^2(X_1^2 - 2X_0^1).\end{aligned} \quad (4.1b)$$

In these expressions, $\mathscr{D}$ and $\mathscr{E}$ are written in terms of standard ordered operators (derivatives to the right), and in terms of the [$\mathscr{L}^2(\Re^2)$-Hermitian] operators that quantize, à la Schrödinger (Q), the classical observables $\mathbf{p}\cdot\mathbf{q}$ and $p^2\mathbf{p}\cdot\mathbf{q}$, namely, $X_0^1 = \frac{1}{2}(\mathbf{p}\cdot\mathbf{q})_Q$ and $X_1^2 = (p^2\mathbf{p}\cdot\mathbf{q})_Q$. We recall that the quantization of functions of the general form $A(p)q + B(p)$ to $\mathscr{L}^2$-Hermitian operators is (modulo weak technical assumptions) independent of the scheme (Weyl, Born–Jordan, symmetrization, etc.).[4] The operator X_0^1 generates $\mathscr{L}^2(\Re^2)$-unitary *dilatations* on functions of position and, since $p^2\mathbf{p}\cdot\mathbf{q}$ generates the geometrical Seidel *coma* aberration, X_1^2 will be the coma operator. The indices reflect the placement of the X's within the symplectic aberration multiplets classified in Ref. 3, and the results on their quantization in Ref. 21. Note that the off-diagonal operator $\mathscr{E}$ is a combination of magnification and coma with the same order in α. [The exponentiation of the boost generators in the plane of the screen, B_x and B_y entails exponentiating the diagonal elements $\mathscr{D}\nabla + k\mathbf{q} = k\mathsf{Q} - (\mathbf{p}\cdot\mathbf{q})_Q\mathsf{P}$ and $\nabla\mathscr{D} + k\mathbf{q} = k\mathsf{Q} - \mathsf{P}(\mathbf{p}\cdot\mathbf{q})_Q$.]

The effect of a finite boost in the z direction on wave functions on the screen and their normal derivatives may be written formally in terms of the operators $\mathscr{F}^2 = \mathscr{D}\mathscr{E}$ and $\mathscr{G}^2 = \mathscr{E}\mathscr{D}$ [see (3.5)], as

$$\begin{pmatrix} f(\alpha) \\ f_z(\alpha) \end{pmatrix} = \exp i\alpha \begin{pmatrix} 0 & \mathscr{D} \\ \mathscr{E} & 0 \end{pmatrix} \begin{pmatrix} f(0) \\ f_z(0) \end{pmatrix} = \begin{pmatrix} \cos\alpha\mathscr{F} & i\alpha\mathscr{D}\,\mathrm{sinc}\,\alpha\mathscr{G} \\ i\alpha\mathscr{E}\,\mathrm{sinc}\,\alpha\mathscr{F} & \cos\alpha\mathscr{G} \end{pmatrix} \begin{pmatrix} f(0) \\ f_z(0) \end{pmatrix} \tag{4.2a}$$

where only even powers of $\mathscr{F}$ and $\mathscr{G}$ appear in the cosine and sinc functions [$\mathrm{sinc}\, x = x^{-1}\sin x = 1 - (1/3!)x^2 + (1/5!)x^4 - \cdots$]. The expansion of the matrix to fifth order in α is

$$\begin{pmatrix} 1 - (1/2!)\alpha^2\mathscr{D}\mathscr{E} + (1/4!)\alpha^4\mathscr{D}\mathscr{E}\mathscr{D}\mathscr{E} + \cdots & i\alpha\mathscr{D} - i(1/3!)\alpha^3\mathscr{D}\mathscr{E}\mathscr{D} + i(1/5!)\alpha^5\mathscr{D}\mathscr{E}\mathscr{D}\mathscr{E}\mathscr{D} + \cdots \\ i\alpha\mathscr{E} - i(1/3!)\alpha^3\mathscr{E}\mathscr{D}\mathscr{E} + i(1/5!)\alpha^5\mathscr{E}\mathscr{D}\mathscr{E}\mathscr{D}\mathscr{E} + \cdots & 1 - (1/2!)\alpha^2\mathscr{E}\mathscr{D} + (1/4!)\alpha^4\mathscr{E}\mathscr{D}\mathscr{E}\mathscr{D} + \cdots \end{pmatrix}. \tag{4.2b}$$

In order to present concrete results comparable with other developments in Fourier aberration optics,[22] we apply this expansion to a forward Gaussian beam with waist at the screen plane and centered at $\mathbf{q} = \mathbf{a}$. This we write as

$$\mathbf{G}_w^a(\mathbf{q}) = \begin{pmatrix} 1 \\ ik \end{pmatrix} E_w(\mathbf{q} - \mathbf{a}), \quad E_w(\mathbf{q}) = \exp(-|\mathbf{q}|^2/2w). \tag{4.3}$$

Of course, a Gaussian is not strictly in $\mathscr{H}_k$, since its Fourier transform is a Gaussian (in $\mathbf{k}$) of width $1/w$, that has generally small but nonzero support outside the disk δ_k. We assume that the spread of directions off the $+z$ axis is small, so that the approximation holds good and that we may replace the obliquity factor k_z in the normal derivative by the constant k in (4.3). Gaussians beams in the $-z$ direction reverse the sign of the second component, i.e., complex conjugate (4.3). (A null second component would indicate it is a solution even in z, with maximum amplitude at the screen.)

Equation (4.2b) gives the effect of a z boost as a series of derivative operators that is straightforward to apply to the Gaussian (4.3) algorithmically through symbolic computation, albeit the approximation errors of the assumption that are not easy to estimate except by examining the stability of the main features of the graphic outcome. The series for the amplitude f is

$$f_{w,a}^\alpha(\mathbf{q}) = (1 - \alpha k\mathscr{D} - (1/2!)\alpha^2\mathscr{D}\mathscr{E} + (1/3!)\alpha^3 k\mathscr{D}\mathscr{E}\mathscr{D} + (1/4!)\alpha^4\mathscr{D}\mathscr{E}\mathscr{D}\mathscr{E} - (1/5!)\alpha^5 k\mathscr{D}\mathscr{E}\mathscr{D}\mathscr{E}\mathscr{D} + \cdots)E_w(\mathbf{q} - \mathbf{a}). \tag{4.4}$$

We recall that $\mathscr{D}$ and $\mathscr{E}$ are real derivative operators; when acting on Gaussian functions they yield polynomial factors of degrees 2 and 4, respectively. The summands in (4.4) thus yield polynomials of degrees 0, 2, 6, 8, 12, and 14 in x and y, followed by degrees 18, 20, 24,..., that we disregard on account of the power of α of the approximation.

To first order in α, (4.2b) and (4.4) represent magnification $f \to f + i\alpha f_z + \cdots$ by the normal derivative. The factor polynomial of the Gaussian is $1 - \alpha[1 - \mathbf{q}\cdot(\mathbf{q} - \mathbf{a})]$. For $\alpha > 1/\sqrt{1 + |\mathbf{a}|^2/4w}$ this polynominal vanishes on a circle with center at $\mathbf{q} = \frac{1}{2}\mathbf{a}$, and radius $\sqrt{|\mathbf{a}|^2/4 + w(1 - 1/\alpha)}$; the radius grows with α bounded by $\sqrt{|\mathbf{a}|^2/4 + w}$. To second order in α, the operator $\mathscr{E}$, containing coma, appears in product with a $\mathscr{D}$, acting on the wave function. Two $\mathscr{D}$'s with an $\mathscr{E}$ appear for third order in α acting on the normal derivative, and so forth.

In Fig. 1 we show the squared first component of a forward Gaussian beam. In units of $\lambda\!\!\!^-$ we have placed the center of the Gaussian at the point $\mathbf{a} = (10,0)$. We have set the width of the Gaussian to be $w = 4$, so the squared amplitude drops to $e^{-1} = 0.3679...$ of its maximum value at $|\mathbf{q} - \mathbf{a}| = 2$. The conjugate wavenumber Gaussian has width $\frac{1}{4}$ and is thus comfortably concentrated within the $|\mathbf{k}| = 1$ disk.

Figures 2 and 3 show the square of the resulting aberrated function on the screen, $|f_{w,a}^\alpha(\mathbf{q})|^2$ for $\alpha = 0.3$ and $\alpha = -0.3$. We have chosen these values so that the figures will be comparable with those of Ref. 1. The geometric coma caustic angle (60°) is superposed on the figures, with the

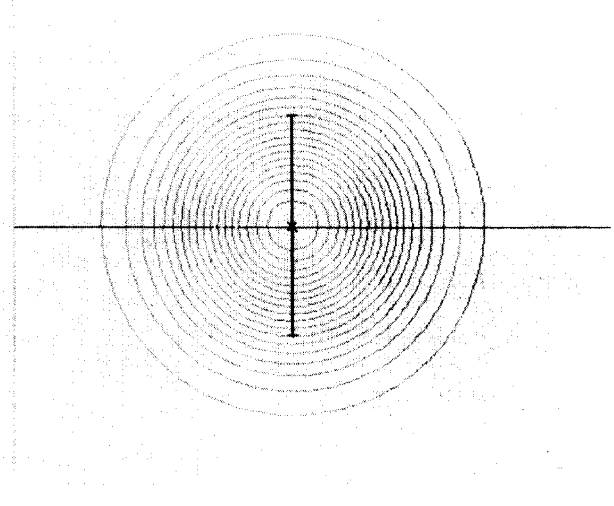

FIG. 1. Contours of the square of the amplitude of a reference Gaussian placed at $\times$. The width is $w = 4\lambda\!\!\!^{-2}$ (we mark the $|\mathbf{q} - \mathbf{a}| = 2$ distance at which the function drops to $e^{-1} = 0.3679...$of its maximum). The optical center is $10\lambda\!\!\!^-$ to the left of $\times$; the vertical line stands at $x = 5\lambda\!\!\!^-$. We have used 20 "isophote" contours spaced by 0.05, from 0.0 (hence not shown in the figure) to 1.0 (marked by $\times$).

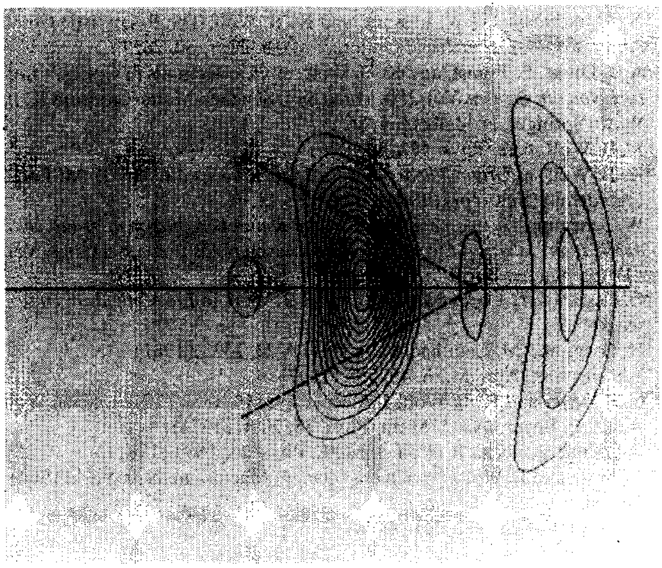

FIG. 2. The relativistic coma-aberrated forward Gaussian of Fig. 1, for $\alpha = +0.3$ ($v = c \tanh \alpha$). We indicate the apex and the opening 60°-caustic angle of the geometric Seidel coma image.

apex at $e^{\alpha}\mathbf{a}$ (13.5 and 7.408 units from the optical center for $\alpha = \pm 0.3$). The figures were drawn after evaluating the polynomial factor in the series (4.4) to fifth degree in α for the above parameter values. They show that the single Gaussian peak unfolds into several local maxima, separated by crescent-shaped "dark fringes," whose number was seen to increase with the truncation degree of α in the series (4.4). The location of the global maxima (0.804 and 0.818 of the reference Gaussian maximum) changed only slowly from first degree on in the direction of magnification. New, smaller local maxima are added with increasing degree.

We should compare these features to those calculated for diffraction in aberration under pure Seidel coma [Ref. 11, Figs. 9.6(a) and 9.6(b)]. Our Fig. 3 seems to conform better than Fig. 2 to the familiar pattern of fringes of pure coma, where crescents bend in the direction of the apex; however, for the parameters of the second figure, $\alpha = +0.3$, the first-degree term is magnification. As we saw above, this will introduce a circular dark fringe of radius nearly 4 with the center at (5,0). This fringe seems to be the dominant feature that keeps the higher-degree crescents bending toward the optical center over the coma bending of the same. We note that the crescents of Fig. 2 are slightly "stiffer" than those of Fig. 3; this may indicate that in the former, the purely comatic bending weakly counteracts the basic magnification bending.

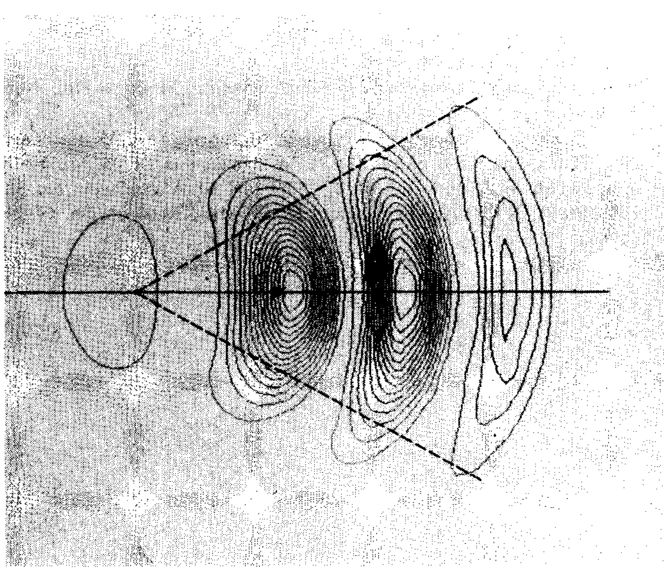

FIG. 3. The relativistic-coma-aberrated Gaussian of Fig. 1, for $\alpha = -0.3$. This is equivalent to a backward-directed Gaussian beam with positive α parameter. We indicate the geometric opening caustic angle for Seidel coma.

V. RECAPITULATION AND CONCLUDING REMARKS

The quantization of a system on the level of its dynamical group has been proposed in Refs. 23, among many others, providing self-adjoint representations on a space with a physical interpretation, such as the $\mathscr{L}^2(\mathfrak{R}^3)$ Hilbert space of quantum wave functions. The dynamical group of optics in homogeneous media is the Euclidean group. The representations we have explored are that of directed lines through a screen in geometrical optics, and that of a two-functions on a reference plane in Helmholtz optics. Both remain homogeneous spaces under the deformation of the Euclidean to the Lorentz group.

In this way, boosted screens are described on par with rotated or translated screens, and special relativity is brought in contact with geometric and Helmholtz optics, that *prima facie* had little to do with motion because they contain no time variable. Solutions of the two-dimensional Helmholtz equation have been subjected to the Lorentz group before,[24] but we did not realize then that the Euclidean group has a transparent optical interpretation. The group action is correct as far as the prediction of the familiar stellar aberration for ray directions and plane waves. The relativistic coma phenomenon is the "Fourier conjugate" of that distortion of the sphere. The quotation marks are to withold a precise definition that encompasses canonical conjugation in geometric optics, and integral transformation into the basis of the plane waves (2.7) by (2.8) in Helmholtz optics.

Even more pressing than the question of a time variable, is the absence of a space variable, $\mathbf{q}$, within the dynamical group. In geometric optics, $\mathbf{q}$ is the canonical conjugate to ray momentum, $\mathbf{p}$, within the Heisenberg–Weyl algebra under Poisson brackets. In Helmholtz wave optics, a position operator $Qf(\mathbf{q}) = \mathbf{q}f(\mathbf{q})$ is not self-adjoint in $\mathscr{H}_k$, and hence does not lead to a standard observable within the framework of wave mechanics. We see this as a welcome feature of our theory, since Dirac δ's on the screen cannot be strictly produced. Sinc-type or J_0 beams[15] may be the best approximations. Here, we have an inner product (2.4) that has a Bessel-function nonlocality. Mathematically, this is Parseval equivalent to the presence of the obliquity factor k_z/k in the plane-wave basis inner product (2.10). The obliquity factor must be there for geometric reasons. Both f and f_z should be present if the squared norm is to mean total field energy of the system.

In fact, in two-dimensional optics,[25,6] where ray directions θ range over a circle S_1, the Fourier conjugate observable will generate rotations of that circle, indicating that $i\hbar d/d\theta$ may be an appropriate position operator (cf. Ref. 4, Sec. V, for the conjugate problem of quantum mechanics on S_1). The spectrum of such an operator in $\mathscr{L}^2(S_1)$ is discrete and equally spaced by $\hbar$, consistent with the sampling theorem of Whittaker and Shannon.[26] In the three-dimensional case we have a direction sphere S_2 (*not* a torus), so the identification of the traditional position operators with our rotation generators $\hbar R_x, \hbar R_y$, could be appropriate in view of the paraxial contraction limit, where they commute. Alternatively, the position observable in geometrical optics was written in Ref. 1, Eq. (4.5), as an algebraic function of the translation and boost operators, but the wave version of this relation is not obvious. The plane q of the figures, however intuitive as the screen where images form, still requires further understanding. This also applies to the role of the normal derivative f_z that does not seem to be directly observable on the screen, but could be inferred from the values of the field amplitude at two different locations near the screen.

The endowment of a physical system with a Lie algebraic structure allows the compact statement of a cornucopia of properties, such as polarization, separation of variables, classification of solutions, and transformations, Clebsch–Gordon coupling, and aberration expansions—to mention the most obvious ones for the Euclidean groups. These will be developed elsewhere for optics and relativistic oscillator mechanics,[27] going beyond the present description of scalar fields in homogeneous, empty space.

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