The relativistic coma aberration. I. Geometrical optics

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It is shown, in the framework of Hamilton–Lie geometrical optics, that the image on a moving screen undergoes comatic aberration as the conjugate sphere of ray directions distorts under Lorentz boosts.

I. INTRODUCTION

Stellar aberration is a phenomenon known for centuries in positional astronomy: As a result of the Earth's motion in orbit, the directions to stars on the celestial sphere suffer distortion toward the direction of motion. In relativity we know that, corresponding to a ray with angle θ measured from the motion vector, and a velocity $v = c \tanh \alpha$, the distortion is given by the transformation of the *circle*[1]

$$\tan \tfrac{1}{2}\theta \to \tan \tfrac{1}{2}\theta' = e^{-\alpha} \tan \tfrac{1}{2}\theta. \tag{1.1}$$

Hamilton–Lie geometrical optics[2,3] works with phase-space observables on plane screens. It is usually natural to distinguish an *optical axis* when working with optical image-forming systems or optical fibers. In that case it is convenient to perform the aberration expansions of classical geometrical optics.[4] In this paper we treat the aberration phenomenon *globally*, i.e., through exact (closed) expressions valid on the whole optical phase-space manifold: optical momentum is directly related to points on the direction sphere, and this is a *compact* manifold (*unlike* the phase space of point particles).

Distortion of the sphere of directions entails a corresponding *comatic* aberration of ray position at the screen, if the relativistic transformation is to be canonical on optical phase space.

In Sec. II we assemble the basic facts of the Hamilton–Lie account of local and global properties of the phase space of geometric optics. In Sec. III we use this formalism in the framework of Euclidean and special relativity: screens may be translated to new origins, rotated to new optical axes, or boosted to motion. This last transformation is performed by group *deformation*[1,5,6] of the Euclidean to the Lorentz algebra and group, in Sec. IV. In Sec. V the specific aberration due to screen motion along the optical axis is studied as are some of its basic geometric properties for all velocities. Caustic phenomena are highly visible and could be observable in appropriate experimental situations. In the concluding section (VI) some considerations of a mathematical nature are added.

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II. OPTICAL PHASE SPACE

Optical phase space in the three-dimensional world of geometric optics is referred to a two-dimentional screen of positions and a sphere of ray directions. It is a four-dimensional manifold where it is convenient to introduce Cartesian coordinates[7] and write its points as $w = (\mathbf{p};\mathbf{q})$, $\mathbf{p} = (p_x,p_y)$, $\mathbf{q} = (q_x,q_y)$, with $\mathbf{q}\in\mathscr{R}^2$ (the real plane) the *position* vector of the ray's intersection, and $\mathbf{p}$ the *momentum* coordinate. The latter is the projection on the plane of the screen of a three-vector $\vec{n} = (p_x,p_y,h)$ along the ray whose length is n, the refractive index of the medium (constant in this paper, corresponding to a homogeneous optical medium). The two coordinate sets are canonically conjugate, i.e., the Poisson bracket[8] relations hold:

$$\{q_i,p_j\} = \delta_{ij}, \quad \{q_i,q_j\} = 0, \quad \{p_i,p_j\} = 0. \tag{2.1}$$

The origin of phase space is $\mathbf{q} = \mathbf{0}$ (the optical center), and $\mathbf{p} = \mathbf{0}$ (the optical axis).

We note that the *range* of the momentum coordinates is limited by $p^2 \leqslant n^2$, and is the projection of the sphere S^2 of ray directions on the screen plane. It consists of the disk $p^2 < n^2$ counted once for $h > 0$ ("forward" rays), and once for $h < 0$ ("backward" rays); the two disks are at the boundary $p^2 = n^2$ when $h = 0$. We may assume the sign of the z component of $\vec{n}$, i.e., h, is always available to distinguish between the two disks, and we may freely revert to the direction sphere coordinates. (The range of $\mathbf{p}$ in two-dimensional mechanics, in contrast, is the full $\mathscr{R}^2$ plane.)

The z component of the direction vector $\vec{n}$ is

$$h = (n^2 - p^2)^{1/2} \tag{2.2}$$

and serves as (minus) the optical Hamiltonian.[3] (The series expansion $h = n - p^2/2n \cdots$ suggests giving n the analog role of a potential, notwithstanding that it also appears in the denominator, where mass ought to be in mechanics.)

Lie optics uses the symplectic structure (2.1) to build *Lie operators* $\hat{f} = \{f,\cdot\}$ associated to differentiable functions $f(\mathbf{p},\mathbf{q})$. Their action on phase space is

$$\hat{f}\mathbf{p} = \{f,\mathbf{p}\} = \frac{\partial f}{\partial \mathbf{q}}, \quad \hat{f}\mathbf{q} = \{f,\mathbf{q}\} = -\frac{\partial f}{\partial \mathbf{p}}. \tag{2.3}$$

Various properties follow,[9] such as

$$\hat{f}g(\mathbf{p},\mathbf{q}) = \{f,g\}(\mathbf{p},\mathbf{q}) = g(\hat{f}\mathbf{p},\hat{f}\mathbf{q}), \tag{2.4a}$$

$$(\{f,g\})\hat{=}[\hat{f},\hat{g}], \tag{2.4b}$$

where $[\cdot,\cdot]$ is the *commutator* of operators. They allow us to work with the enveloping algebra of (2.1), and exponentiate to the corresponding *Lie transformation* generated by f:

$$F_\alpha = \exp(\alpha \hat{f}) = 1 + \alpha \hat{f} + 1/2! \alpha^2 \hat{f}^2 + \cdots, \tag{2.5}$$

so that acting on suitably smooth functions $g(\mathbf{p},\mathbf{q})$ (such as $\mathbf{p}$ and $\mathbf{q}$ themselves),

$$F_\alpha g = g + \alpha\{f,g\} + 1/2! \alpha^2\{f,\{f,g\}\} + \cdots . \tag{2.6}$$

Also, Lie transformations are *canonical*,[9] i.e., for arbitrary $g_1(\mathbf{p},\mathbf{q})$ and $g_2(\mathbf{p},\mathbf{q})$,

$$\{F_\alpha g_1, F_\alpha g_2\} = \{g_1,g_2\}. \tag{2.7}$$

As a first (counter-) example, consider f to be a function *quadratic* in the components of $\mathbf{p}$ and $\mathbf{q}$. Then[2,9] F_α will map the components of $\mathbf{p}$ and $\mathbf{q}$ linearly among themselves, thus generating Sp $(4,R)$, the group of linear symplectomorphisms of phase space. In this example, however, the natural range of optical momentum $p^2 \leqslant n^2$ is *not* preserved. [In spite of not globally respecting optics, Sp $(4,R)$ has been extremely useful in treating aberration expansions by order around an optical center and axis.[10]]

The position coordinates (q_x,q_y), in particular, are also *not* good functions to generate Lie transformations for global optics, since they translate the $\mathbf{p}$ plane, as in mechanics, and do not respect the natural range $p^2 \leqslant n^2$ of optical momentum.

In fact, it seems rather difficult to write Lie transformations that *do* not preserve the optical momentum range, except for one very obvious class: point-to-point mappings of the sphere, i.e., *rotations* and *distortions* $S^2 \to S^2$ so that $\mathbf{p} \to \mathbf{p}' = \mathbf{p}'(\mathbf{p}, \operatorname{sgn} h)$. These are distortions in the sense that $\mathbf{p}'$ is *not* a function of $\mathbf{q}$. [In the optical distortion aberration,[4] $\mathbf{q}'(\mathbf{q})$ is independent of $\mathbf{p}$; the latter is the Fourier conjugate variable except for ranges.]

To avoid uncomfortable formulas at the joining of the two momentum disks, let us use explicit spherical coordinates for the three-vector of ray directions:

$$p_x = n \sin\theta \sin\phi, \tag{2.8a}$$

$$p_y = n \sin\theta \cos\phi, \tag{2.8b}$$

$$h = n \cos\theta. \tag{2.8c}$$

We now define the Lorentz group action of special relativity on the phase space of geometrical optics through binding (2.8) to be the three-vector parallel to the three-vector part of a lightlike four-vector (k_x,k_y,k_z,k_0) undergoing such tranformations. Thus setting $\mathbf{k} = (k_x,k_y)$, $k = (k_x^2 + k_y^2)^{1/2}$, $\mathbf{p} = n\mathbf{k}/k_0$, and $h = nk_z/k_0$, we obtain the following relations:

$$p/n = \sin\theta = k/k_0, \tag{2.9a}$$

$$h/n = \cos\theta = k_z/k_0, \tag{2.9b}$$

$$p/(n+h) = \tan\tfrac{1}{2}\theta = k/(k_z + k_0). \tag{2.9c}$$

Hence when the lightlike vector $(\mathbf{k},k_z,k_0)$ undergoes a boost in the z direction, the transformation of (2.9c) yields (1.1).

III. EUCLIDEAN TRANSFORMATIONS

Three functions that generate Lie transformations that map optical phase space onto itself properly are the components of $\vec{n} = (p_x,p_y,h)$. The first two generate translations in $\mathbf{q} \in \mathscr{R}^2$,

$$\exp(\mathbf{a}\cdot\hat{\mathbf{p}}) g(\mathbf{p},\mathbf{q}) = g(\mathbf{p},\mathbf{q} - \mathbf{a}); \tag{3.1}$$

while the last one generates translations along the optical axis normal to the screen,

$$\exp(z\hat{h}) g(\mathbf{p},\mathbf{q}) = g(\mathbf{p},\mathbf{q} + z\,\mathbf{p}/h). \tag{3.2}$$

The transformation of $\mathbf{q}$ in the last argument reads $q + z\tan\theta$ in the direction of $\mathbf{p}$, as is clear from simple geometry. The three generating functions commute under the Lie bracket: $\{p_i,p_j\} = 0$, $\{p_i,h\} = 0$.

Another set of S^2-preserving Lie transformations is the group of rotations of the screen in three-space.[11] To simplify arguments, consider the *two*-dimensional optics case depicted in Fig. 1, where a ray is seen in two different frames rotated by γ, as θ and as $\theta' = \theta + \gamma$, or, in two-dimensional phase space,

$$p \to p' = p\cos\gamma + h\sin\gamma, \tag{3.3a}$$

$$h \to h' = -p\sin\gamma + h\cos\gamma, \tag{3.3b}$$

$$q \to q' = q/(\cos\gamma - p/h\,\sin\gamma). \tag{3.3c}$$

The last relation is obtained from the law of sines in the triangle of the figure. From here, the generator of two-dimensional rotations $\exp(\gamma\hat{m})$ may be found through

$$\frac{\partial m}{\partial q} = \hat{m}p = \frac{\partial p'}{\partial \gamma}\bigg|_{\gamma=0}, \quad \frac{\partial m}{\partial p} = -\hat{m}q = \frac{\partial q'}{\partial \gamma}\bigg|_{\gamma=0},$$

and is[11] $m = qh$. In three-dimensional optics, if Fig. 1 is the x-z plane, the generator will be that of rotation r_y around the y axis, and if it is the y-z plane the generator will be $-r_x$. Hence the generators of rotations of the direction sphere are [cf. Ref. 11, Eq. (2.9)]

$$r_x = q_y h, \tag{3.4a}$$

$$r_y = -q_x h, \tag{3.4b}$$

$$r_z = q_x p_y - q_y p_x = \mathbf{q} \times \mathbf{p}, \tag{3.4c}$$

and are easily checked to close into an so (3) algebra under the Lie bracket of geometrical optics:

$$\{r_x,r_y\} = r_z, \quad \{r_y,r_z\} = r_x, \quad \{r_z,r_x\} = r_y. \tag{3.5}$$

The first of Eqs. (3.5) may be used to define r_z in (3.4c); this quantity generates rotations in the plane of the screen around the optical center, together with rotations of the direction sphere around the optical axis. Its square is the Petzval invariant of optics.[10]

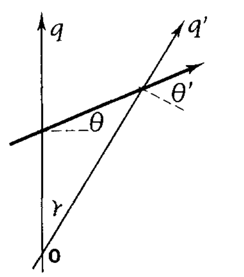

FIG. 1. The transformation of optical phase space due to rotation of the screen about the origin by γ. The ray $(p = n\sin\theta, q)$ transforms to $(p' = n\sin\theta', q')$.

The six functions $(p_x, p_y, h; r_x, r_y, r_z)$ close under Poisson brackets into the Lie algebra iso (3) of the Euclidean group of motions: the three translations leaving the direction sphere invariant and the three joint rotations intertwine through

$$\{r_i, p_j\} = p_k, \tag{3.6}$$

with i,j,k cyclic permutations of x,y,z, and $p_z = h$. The Euclidean group ISO (3), containing the Hamiltonian among its generators, is the dynamical group of[11] geometric optics in a homogeneous medium. The two Euclidean invariants are $p^2 + h^2 = n^2$ and $\vec{n}\cdot\vec{p} = 0$.

IV. THE DEFORMATION ISO (3)→SO (3,1)

We recall the classic deformation process[1,5] that builds the Lorentz algebra so (3,1) out of the generators of the Euclidean iso (3), realized on a sphere, and generalizations thereof.[6] Basically, one builds bilinear functions of the generators of iso (3) with the right transformation properties under so (3). These will close into so (3,1) on the sphere. One may also add linearly the generators of the translation subalgebra, thus arriving at all representations of the nonexceptional continuous series. In geometric optics we may propose the three-vector

$$\vec{b} = \vec{r}\times\vec{p} + \sigma\vec{p}. \tag{4.1}$$

As vector functions in the phase-space coordinates, the components are

$$\mathbf{b} = n\mathbf{q} - \mathbf{p}\cdot\mathbf{q}\mathbf{p}/n + \sigma\mathbf{p}, \tag{4.2a}$$

$$b_z = -\mathbf{p}\cdot\mathbf{q}h/n + \sigma h. \tag{4.2b}$$

These three functions transform under the so (3) subalgebra (3.4) as the components of a proper three-vector,

$$\{r_x, b_y\} = b_z \quad \text{(and cyclically)}. \tag{4.3}$$

Finally, they close under the Lie (Poisson) bracket of the algebra, into the Lorentz algebra so (3,1):

$$\{b_x, b_y\} = -r_z \quad \text{(and cyclically)}. \tag{4.4}$$

The constant σ in the boost generators (4.2) is also in the Lorentz invariant $\vec{b}^2 - \vec{r}^2 = n^2\sigma^2$ while $\vec{r}\cdot\vec{b} = 0$.

It is noteworthy that we may express the ray position coordinate $\mathbf{q}$ in terms of the functions generating Lorentz boosts and Euclidean translations:

$$\mathbf{q} = \mathbf{b}/n - b_z\mathbf{p}/nh. \tag{4.5}$$

We shall examine in detail the boosts along the optical axis; these are generated by b_z in (4.2b) as the Lie transformation $\exp(\alpha\hat{b}_z)$ acting on the reference (stationary) screen phase space $(\mathbf{p},\mathbf{q})$, to produce the phase space $(\mathbf{p}'(\mathbf{p},\mathbf{q},\alpha), \mathbf{q}'(\mathbf{p},\mathbf{q},\alpha))$ of a screen in motion with velocity $v = c\tanh\alpha$. On the momentum coordinates, we find the integrated group action to be

$$\mathbf{p}'(\mathbf{p},\alpha) = \exp(\alpha\hat{b}_z)\mathbf{p} = \mathbf{p}/(\cosh\alpha + h/n\sinh\alpha). \tag{4.6}$$

This, we duly note, is a mapping $\mathbf{p}'(\mathbf{p})$ independent of position $\mathbf{q}$ and the "Lorentz constant" σ. The momentum distortion of S^2 is precisely—of course—the stellar aberration (1.1), as may be verified through (2.9).

We may also find the action of this boost on the position coordinate $\mathbf{q}$ with the help of (4.5) and (2.4). The transformed position coordinate is

$$\begin{aligned}\mathbf{q}'(p,q,\alpha) &= \exp(\alpha\hat{b}_z)\mathbf{q}\\ &= (\cosh\alpha + (h/n)\sinh\alpha)\\ &\quad\times\left(\mathbf{q} + \frac{\sinh\alpha}{n\sinh\alpha + h\cosh\alpha}\right.\\ &\quad\left.\times\left[-\frac{\mathbf{p}\cdot\mathbf{q}}{n} + \sigma\right]\mathbf{p}\right).\end{aligned} \tag{4.7}$$

The magnification and aberrations present in (4.7) will be studied in Sec. V. We only point out here that the meaning of the arbitrary "Lorentz" constant σ may be elucidated in the Inönü–Wigner contraction of SO (3,1) to ISO (3), when $\alpha\to 0$, $\sigma\to\infty$, with finite $z = \alpha\sigma$. Then $\mathbf{q}'\to\mathbf{q} + z\mathbf{p}/h$, showing an (arbitrary) amount of z translation (3.2), which will not affect ray direction. We will disregard this (purely spherical[2,4]) aberration and set $\sigma = 0$ henceforth. The transformation (4.6) and (4.7) of phase space may be verified to be canonical.

V. THE RELATIVISTIC COMA

Transformations of a four-dimensional manifold are difficult to visualize. The canonicity of the transformation only assures us that the manifold of rays will move as specks of dust in an incompressible fluid (Louville's theorem). A *section* of much use in optics is to choose a single "object" point $\mathbf{q}_0$, and plot $\mathbf{q}'(\mathbf{p},\mathbf{q}_0)$ as a function of $\mathbf{p}$ on part (or the whole) of its range. This corresponds to a bundle (or all) rays passing through the chosen $\mathbf{q}_0$ (as a point light source) imaged after the transformation. In the figures of this section we let $\mathbf{p}$ draw a polar coordinate grid around the optical axis, and plot the image $\mathbf{q}'\in\mathscr{R}^2$; this is the *spot* diagram of the optical transformation for $\mathbf{q}_0$.

When we take a square lattice of such object points a distance d apart, at $\mathbf{q}_0 + n_x(d,0) + n_y(0,d)$; $n_x n_y$ integers, we obtain the *spots* diagram (as in our figures), usually also called "spot." It depicts what is seen on the screen of an array of luminous points after the transformation to $(\mathbf{p}'(\mathbf{p},\mathbf{q}), \mathbf{q}'(\mathbf{p},\mathbf{q}))$. [The spot diagram *before* the transformation, i.e., $(\mathbf{p},\mathbf{q})$ is simply a square array of points, a perfectly focused 1:1 unit transformation of the object.]

We start the analysis of relativistic coma in the context of *aberration–expansion* optics, and will later consider its *global* characteristics. We must assume p/n to be less than unity so that the expansion of (4.6) and (4.7) may be performed by powers of p^2. This may mean $p^2 < n^2/10$ ($\theta < 18°26'\ldots$) or $p^2 < n^2/2$ ($\theta < 45°$), according to how high the order of aberration we are willing to calculate. To fifth aberration order we have the following fifth-degree approximation of relativistic coma:

$$\mathbf{p}' = e^{-\alpha}\mathbf{p} + \tfrac{1}{2}n^{-2}\sinh\alpha\, e^{-2\alpha}p^2\mathbf{p} + \tfrac{1}{4}n^{-4}\sinh\alpha\, e^{-2\alpha}(1-\tfrac{1}{2}e^{-2\alpha})p^2\mathbf{p} + \cdots, \qquad (5.1a)$$

$$\mathbf{q}' = e^{\alpha}\mathbf{q} - n^{-1}\sinh\alpha\,\mathbf{p}\cdot\mathbf{q}\mathbf{p} - \tfrac{1}{2}n^{-2}\sinh\alpha\, p^2\mathbf{q} - \tfrac{1}{2}n^{-3}\sinh\alpha\, e^{-2\alpha}p^2\mathbf{p}\cdot\mathbf{q}\mathbf{p} - \tfrac{1}{8}n^{-4}\sinh\alpha(p^2)^2\mathbf{q} - \cdots, \qquad (5.1b)$$

with increasingly complicated coefficients for higher $(p^2)^m\mathbf{p}$ in (5.1a), and $(p^2)^{m-1}\mathbf{p}\cdot\mathbf{q}\mathbf{p}$ and $(p^2)^m\mathbf{q}$ in (5.1b).

The first term on the right-hand sides of (5.1) is the *linear* part of the mapping. This falls within Gaussian (paraxial, linear) optics: $\mathbf{p}' = e^{-\alpha}\mathbf{p}$ is a contraction of ray momentum that necessitates (for canonicity) the expansion $\mathbf{q}' = e^{\alpha}\mathbf{q}$ of ray positions.

The rest of the series (5.1) is nonlinear and contains the *aberration* due to boost. It should be noted carefully that the only smallness parameter is p^2. Indeed, in α, the magnification part $e^{-\alpha}\mathbf{p} \simeq (1-\alpha)\mathbf{p}$ and the aberration part $\simeq \alpha(p^2/2n^2 + (p^2)^2/8n^4 + \cdots)\mathbf{p}$ in relativistic coma are of the same order; similarly for $\mathbf{q}'$.

In the expansion (5.1b) [*and* in the exact form (4.7)], it is useful to note that the particular function form $C(\mathbf{p},\mathbf{q}) = A\mathbf{p}\cdot\mathbf{q}\mathbf{p} + Bp^2\mathbf{q}$ maps a cone of rays around the optical axis ($\mathbf{q}$ and $p = |\mathbf{p}|$ fixed, *twice* (for $\pm\mathbf{p}$) onto a *circle* in the spot diagram, with center at $(A/2 + B)p^2\mathbf{q}$, of radius $Ap^2q/2$, and extending between $(A+B)p^2\mathbf{q}$ (the image of the two *meridional* rays, i.e., in a plane with the optical axis, $\mathbf{p}\cdot\mathbf{q} = \pm pq$) and $Bp^2\mathbf{q}$ (the image of the two *saggital* rays across, $\mathbf{p}\cdot\mathbf{q} = 0$).

In Lie optics[2] the generator of *circular coma* aberration of order $2m+1$ is $f^c = (p^2)^m\mathbf{p}\cdot\mathbf{q}$. (This is M_{m10} in the monomial basis[10] and ${}^{m+1}\chi_m^{m+1}$ in the symplectic basis.[11]) The action of exp $\kappa\hat{f}^c$ on phase space, to the aberration order, is

$$\mathbf{p} \to \mathbf{p} + \kappa(p^2)^m\mathbf{p}$$

and

$$\mathbf{q} \to \mathbf{q} - \kappa[2m(p^2)^{m-1}\mathbf{p}\cdot\mathbf{q}\mathbf{p} - (p^2)^m\mathbf{q}].$$

On this basis we recognize the relativistic aberration as *circular coma*. The third-order ($m=1$) comatic parameter is thus

$$\kappa^{(3)} = \tfrac{1}{2}n^{-2}e^{-\alpha}\sinh\alpha. \qquad (5.2a)$$

In the factorization order[2]

$$e^{\alpha\hat{b}_z} = \cdots e^{\kappa^{(7)}\hat{f}_7^c}e^{\kappa^{(5)}\hat{f}_5^c}e^{\kappa^{(3)}\hat{f}_3^c}e^{-\alpha\widehat{\mathbf{p}\cdot\mathbf{q}}},$$

the fifth- and seventh-order coma parameters are found to be[9]

$$\kappa^{(5)} = \tfrac{1}{16}n^{-4}e^{-2\alpha}\sinh 2\alpha, \qquad (5.2b)$$

$$\kappa^{(7)} = \tfrac{1}{32}n^{-6}e^{-3\alpha}\sinh 3\alpha. \qquad (5.2c)$$

In Figs. 2 and 3 we show the spot(s) diagram of relativistic coma at values $\alpha = 0.3$ and $\alpha = -0.3$, respectively, for a 4×4 array of sources. The tips of the "comets" (wherefrom the name for *coma* aberration) exhibit the familiar 60° opening angle characteristic of third-order Seidel coma.[4] The angles τ which the circles subtend from the tip are not constant, however, but

$$\sin\tfrac{1}{2}\tau(p^2) = \tfrac{1}{2} + p^2(2e^{-2\alpha} - 1)/16n^2 + \cdots.$$

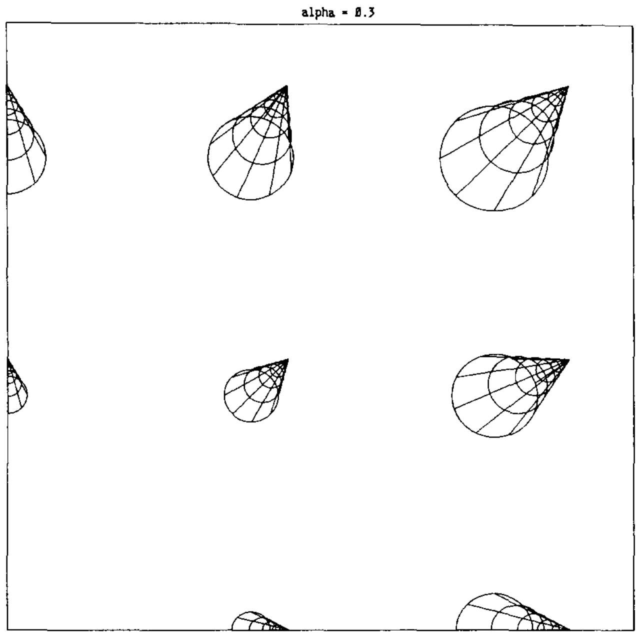

FIG. 2. Spots diagram of the relativistic coma transformation with positive $\alpha = 0.3$. A 4×4 array of object sources (the last row and column of which fall entirely outside the figure) is shown for ray angles of up to 45° (the values of momentum p are spaced by 0.101, up to 0.7071, corresponding to seven circles. The optical center is at the lower left corner.

For $\alpha = 0.3$, the p^2 coefficient is positive and so the comet opens; for values beyond $\alpha = \tfrac{1}{2}\ln 2 \simeq 0.3466$, from 60°, τ closes somewhat before opening again for p^2 in the far-metaxial region, to be discussed below.

The figures were drawn for p^2 up to $n^2/2$, i.e., for rays with angles θ of up to 45° from the direction of motion. This is more than what most instruments are designed for, but it allows us to discuss relativistic coma as a *global* aberration

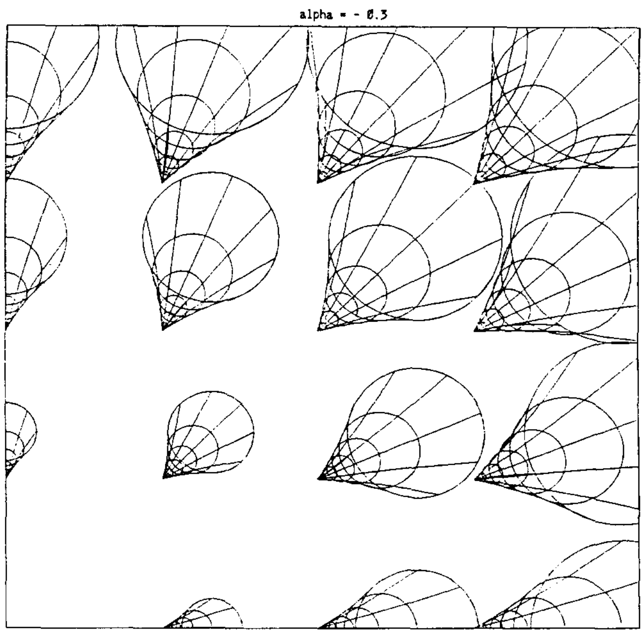

FIG. 3. Spots diagram of the relativistic coma transformation with negative $\alpha = -0.3$. The array of sources and angles are the same as in Fig. 2.

phenomenon. The figures were plotted using the *exact* expression (4.7) rather than any truncated aberration expansion (5.1b).

Consider what happens for negative α: as the screen moves in the $-z$ direction, (1.1) shows that some "critical" rays with angle θ_c to the optical axis will map onto rays with angle $\theta'_c = \pi/2$. This happens for $\tan \frac{1}{2}\theta_c = e^{\alpha}$ or

$$p_c = n \sin \theta_c = n \operatorname{sech} \alpha, \quad h_c = n \cos \theta_c = -n \tanh \alpha.$$

At this value, the denominator in (4.7) vanishes, and that cone of rays will map to infinity at the moving screen. The Poisson bracket $\{\mathbf{q}',\mathbf{p}'\}$ remains constant: the blowup of $\mathbf{q}'(\mathbf{q},\mathbf{p}_c)$ at θ_c is compensated by $p'(p_c)$ reaching its maximum at $\theta' = \pi/2$ and having zero derivative there. There is, of course, no physical singularity, as there is none for rotations in (3.3c) vis-à-vis Fig. 1, when $\theta + \gamma \to \pi/2$. Up to θ_c, the circles subtend angles up to 180°, while the distance from the circle to the comet tip slowly increases up to and beyond θ_c.

The *global* picture of the relativistic coma aberration is the mapping of the whole direction *sphere*. We note that forward rays ($h>0$) under backward motion ($\alpha<0$) are the same as backward rays ($h<0$) under forward motion ($\alpha>0$); indeed, Eq. (4.7) is invariant under the exchange $(h,\alpha)\leftrightarrow(-h,-\alpha)$. Thus while Fig. 2 is the image of rays around the *forward* pole of the direction sphere, $\alpha>0$, Fig. 3 is the image of rays around the *backward* pole, also for $|\alpha|$. To see the spot diagram of the whole direction sphere we may superpose both figures: the spot will extend from $\mathbf{q}'_f = e^{\alpha}\mathbf{q}$ (the image of the forward ray, along the optical axis) to $\mathbf{q}'_B = e^{-\alpha}\mathbf{q}$ (the image of the backward ray, counter to that axis). The full comatic *caustic* acquires a diamond shape, with two 60° angles at the two finite tips, $\mathbf{q}_F$ and $\mathbf{q}_B$, and two "vertices" at infinity in the perpendicular direction. The location of the asymptotic caustic line may be found in (4.7) as the position of the saggital rays ($\mathbf{p}\cdot\mathbf{q}=0$) at the critical angle θ_c: it is $\mathbf{q}_c = \operatorname{sech}\alpha\ \mathbf{q}$. The three points: $\mathbf{q}_B$, $\mathbf{q}_c$, and $\mathbf{q}_F$ lie in a line in that order. The region of the direction sphere accessible to optical focusing is in practice very limited, of course.

Thus far we referred to the boost aberration as coma, because of its striking appearance when the screen movement is in the z direction. The effect of boosts *in* the screen plane, specifically, $\hat{b}_x$, will be described now more succinctly.

The boost function b_x in (4.2a) for $\sigma=0$ will generate the Lie transformation on the ray position plane,

$$\begin{aligned} e^{\alpha \hat{b}_x} q_x &= q'_x \\ &= (\cosh\alpha + p_x n^{-1}\sinh\alpha) \\ &\quad \times (q_x\cosh\alpha + \mathbf{p}\cdot\mathbf{q}n^{-1}\sinh\alpha), \end{aligned} \tag{5.3a}$$

$$e^{\alpha \hat{b}_x} q_y = q'_y = (\cosh\alpha + p_x n^{-1}\sinh\alpha) q_y, \tag{5.3b}$$

while on the ray direction sphere it yields the familiar stellar aberration in the x direction.

To first order in α, the aberration of phase space produces spot diagrams with elliptical spots. If $q_y/q_x = \tan\tau$, τ is the angle between the object point and the direction of motion (the x axis here), the ellipses are tilted by $\kappa=\tau/2$, have "major" axis $a(\tau) = \alpha pq(1+\cos\tau)/n$, and "minor" axis $b(\tau) = \alpha pq(1-\cos\tau)/n$. We note that $a(\tau) = b(\pi-\tau)$, $b(\tau) = a(\pi-\tau)$, so that the longer axes of the ellipses are closer to the x axis. For object points *on* the x axis the ellipse degenerates into a line segment of length $2a(0) = 4pq\alpha/n$, mimicking third-order Seidel astigmatism[2] on that axis. Object points on the y axis have their spots circular, as in third-order Seidel curvature of field.

This first-order description holds up to $p=n$, i.e., for the whole forward ($0\leqslant\theta\leqslant\pi/2$) hemisphere of rays. The *same* spots are obtained from the backward ($\pi/2\leqslant\theta\leqslant\pi$) hemisphere: note that (5.3) do not depend on the sign of h. The global mapping of the direction sphere on the image plane generated by b_x is thus *also* a 2:1 mapping. The global coma of boosts has a variety of faces according to the orientation of the observer screen with respect to the boost direction, resembling Seidel coma in the z direction, and an asymmetric kind of Seidel astigmatism/curvature of field aberration for directions of boost in the x-y plane. Intermediate orientations should interpolate between these faces.

Regarding the observability of relativistic coma, stellar aberration is the ray direction aspect of the phenomenon. To observe it in ray *position* space (regardless of the imaging apparatus one may contrive) we may present the following estimate of aberration *size*: at satellite speeds of $V\sim 10$ km/sec, $\alpha\sim 3\times10^{-5}$, a $\theta = 0.1r\sim 5.7°$ cone of ray directions around the optical axis will yield a factor of $\alpha p/n \sim 3\times10^{-6}$. Under z boosts this will spread into a circle of radius $1.5\times10^{-7}q$ in a coma whose *caustic* has a relative size of the order of $10^{-7}q$. For boosts in the screen plane, we may have from circles of radius $3\times10^{-6}q$ to caustic segments of length $6\times10^{-6}q$. The linear factor q gives the relative scale of the aberration to object size.

VI. CONCLUDING REMARKS

The spirit of our prediction of a relativistic comatic aberration due to screen motion has been Lie theoretical. In that vein we should add the following glossary and comments beyond geometric optics.

The three-dimensional Euclidean group is the dynamical group of optics in a homogeneous medium.[12] The deformation[5,6] of this group on the (ray direction) sphere leads to the Lorentz group of special relativity acting on the same sphere. When the projection of this sphere is called momentum space for a system, the canonically conjugate position space will undergo spherical aberration and circular coma when the screen is boosted (in *first* aberration order these are free flight and pure magnification) perpendicular to itself.

The Casimir invariant of the Lorentz group is related to the freedom in translating position space without affecting ray direction. This role of the invariant seems to be new and needs to be further exploited.

Finally, the relativistic transformation is *global* over the phase space of geometric optics, singularities notwithstanding.

The group theoretical objects mentioned above were seen here in the geometric optics realization. They possess other realizations, however, that are better known,[13] and that will be explored to clarify further the "wavization"[14,15] process. It seems this should be parallel to quantization,[7] but based on the Euclidean, rather than Heisenberg–Weyl, algebra.

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