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Diffraction-Free Beams Remain Diffraction Free under All Paraxial Optical Transformations

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It is pointed out that the nondiffracting J_0 beams of Durnin, Miceli, and Eberly are subgroup-reduced basis functions for the group of paraxial (symplectic, linear) transformations of optical phase space. This allows the action of the group on such and similar beams to be computed through 2×2 matrix algebra.

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The diffraction-free beams predicted by Durnin[1] and observed by him together with Miceli and Eberly[2] have attracted attention for their properties of being extremely narrow and (almost paradoxially) nondiffracting. They have been called *J_0 beams* and have the form[2]:

$$\Phi_\alpha(q_x,q_y,z;\kappa)=e^{i\beta z}J_0(\alpha q),\quad 0<\alpha<\kappa,$$
$$\alpha^2+\beta^2=\kappa^2,\quad q_x^2+q_y^2=q^2, \tag{1}$$

oscillating in time with a factor $\exp(-i\kappa ct)$ (i.e., a phase κct). The half width of the central intensity peak is nearly $1/\alpha$, and is *independent* of z. They are separated solutions of the wave equation in cylindrical coordinates. As for plane waves (also nondiffracting), (1) are not square integrable. Indeed, the necessarily finite extent of the experimental wave front brings in diffraction in the form of a shadow zone[1]; this was one of the determining properties measured in Ref. 2.

In this Letter I want to point out that the J_0 beams (1) are a class of functions with the property of self-reproducing under *paraxial optical transformations*: small angles, thin lenses, parabolic-profile graded media, and similar mirrors. This is a consequence of the fact that (1) are subgroup-classified representation functions of $\mathrm{Sp}(2,R)=\mathrm{SL}(2,R)$, the group of paraxial transformations by axis-symmetric optical systems.

Bacry and Cadilhac[3] derived the (double cover of) $\mathrm{Sp}(4,R)$ as symmetry group of the wave equation in the paraxial limit. The recognition that the optical transfer functions are *canonical transform* kernels[4] is due to Nazarathy and co-workers[5]; a monographic account of canonical transforms in paraxial wave optics may be found in Castaño, López-Moreno, and Wolf.[6] It is to be expected (*post factum*—with apology) that the J_0 beams should be identified with a clear group-theoretic structure.

In 1980 Basu and Wolf[7] studied all representations of the $\mathrm{SL}(2,R)$ group in all subgroup reductions. I consider the oscillator representation of the full $\mathrm{Sp}(4,R)$ group of linear (asymmetric) transformations of optical phase space, reduced[6] by the symmetry $\mathrm{O}(2)$ of rotations and inversions around the optical z axis, as $\mathrm{Sp}(4,R)\supset\mathrm{Sp}(2,R)\otimes\mathrm{O}(2)$. The three generators of the "radial" $\mathrm{Sp}(2,R)$ are given by

$$K_0^\gamma=\tfrac{1}{4}(-d^2/dq^2+\gamma/q^2+q^2), \tag{2a}$$

$$K_2^\gamma=-(\tfrac{1}{2}i)(q\,d/dq+\tfrac{1}{2}), \tag{2b}$$

$$K_+^\gamma=K_0^\gamma+K_1^\gamma+\tfrac{1}{2}(-d^2/dq^2+\gamma/q^2), \tag{2c}$$

$$K_-^\gamma=K_0^\gamma-K_1^\gamma=\tfrac{1}{2}q^2. \tag{2d}$$

The Casimir invariant is $-\gamma/4+\frac{3}{16}=k(1-k)$, for $\gamma=(2k-1)^2-\frac{1}{4}$, where k is Bargmann's label[8] for the (positive) discrete representations $D_k^+, k=\frac{1}{2},1,\frac{3}{2},\ldots$; this range is related by integer $|m|=2k-1$ to the representations of the conjugate $\mathrm{O}(2)$ subgroup.

On $\mathcal{L}^2(R^+)$, with inner product

$$(f,g)=\int_0^\infty dq\, f(q)^* g(q),$$

eigenfunctions of K_0^γ are *harmonic* oscillator $+\gamma/q^2$ potential wave functions; those of K_1^γ are similar *repulsive* oscillators; those of K_2^γ are the Mellin transform basis functions; and those of K_-^γ are $\delta_a(q)=\delta(q-a)$, the generalized basis of q-localized states. Our interest here is in the $\mathcal{L}^2(R^+)$ generalized eigenbasis of K_+^γ, since they are Bessel functions:

$$\Phi_a^k(q)=e^{i\pi k}(aq)^{1/2}J_{2k-1}(aq), \tag{3}$$

with eigenvalue $a^2/2 \in R^+$; they are Dirac orthonormal and complete.

The group Sp(2,R) generated by the second-order differential operators (2) is a group of unitary integral transforms[5] on $\mathcal{L}^2(R^+)$ of the Hankel type.[7] The matrix representation and optical elements of the various subgroups are

$$\exp(i\theta K_0^\gamma)\leftrightarrow\begin{pmatrix}\cos\theta/2 & -\sin\theta/2\\ \sin\theta/2 & \cos\theta/2\end{pmatrix}\leftrightarrow\begin{cases}\text{SO(2)}\\ \text{quadratic fibers.}\end{cases} \tag{4a}$$

$$\exp(i\beta K_2^\gamma)\leftrightarrow\begin{pmatrix}\exp(-\beta/2) & 0\\ 0 & \exp(\beta/2)\end{pmatrix}\leftrightarrow\begin{cases}\text{SO(1,1) } of \text{ pure}\\ \text{magnifiers.}\end{cases} \tag{4b}$$

$$\exp(ibK_+^\gamma)\leftrightarrow\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix}\leftrightarrow\begin{cases}\text{E(2) parabolic group}\\ \text{of free flight } z=Kb.\end{cases} \tag{4c}$$

$$\exp(icK_-^\gamma)\leftrightarrow\begin{pmatrix}1 & 0\\ C & 1\end{pmatrix}\leftrightarrow\begin{cases}\text{E(2) of thin lenses}\\ \text{of power } C.\end{cases} \tag{4d}$$

The paraxial group composition law is that of matrix multiplication (here no metaplectic sign). (See Ref. 7 with the notation correspondence of its footnote 1.) The group action is the integral transform $\mathbb{C}^k(\mathbf{M})$, $\det\mathbf{M}=1$,

$$\left[\mathbb{C}^k\begin{pmatrix}a & b\\ c & d\end{pmatrix}f\right](q)=\int_0^\infty dq'\, C^k\begin{pmatrix}a & b\\ c & d\end{pmatrix}(q,q')f(q'), \tag{5a}$$

with a kernel that is the optical transfer function

$$C^k\begin{pmatrix}a & b\\ c & d\end{pmatrix}(q,q')=e^{-i\pi k}b^{-1}(qq')^{1/2}\exp\left(\frac{i(dq^2+aq'^2)}{2b}\right)J_{2k-1}(qq'/b). \tag{5b}$$

In the singular case $b=0$, the action is geometrical:

$$\left[\mathbb{C}^k\begin{pmatrix}a & 0\\ c & a^{-1}\end{pmatrix}f\right](q)=a^{-1/2}\exp\left(\frac{icq^2}{2a}\right)f(q/a),\quad a>0. \tag{5c}$$

Using the matrix representation (4) and products thereof, one can see that a paraxial transformation (5) applied to the K_+^γ (free flight) eigenbasis Φ_a^k *reproduces* the *form* of the wave function:

$$\left[\mathbb{C}^k\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Phi_a^k\right](q)=\left[\mathbb{C}^k\begin{pmatrix}a & 0\\ c & a^{-1}\end{pmatrix}\exp(-iba^{-1}K_+^\gamma)\Phi_a^k\right](q)=a^{-1/2}\exp\left(\frac{-iba^2}{2a}\right)\exp\left(\frac{icq^2}{2a}\right)\Phi_a^k(q/a). \tag{6}$$

In words: A thin lens (4d) will multiply the diffraction-free states Φ_a^k by a phase $\exp(icq^2/2)$ (with no effect on the intensity); pure magnifiers (4b) will yield $a^{-1/2}\Phi_a^k(q/a)=a^{1/2}\Phi_{a/a}^k(q)$, a diffraction-free state of a different width; a length of quadratic-profile fiber (4a) will produce a line of integral transforms of the (Fourier) Hankel type. Finally, free propagation (4c) by $b=z/\kappa$ multiplies Φ_a^k by the phase factor $\exp(-iza^2/2\kappa)$. This phase is the paraxial approximation to the phase anomaly factor of Durnin's function (1) for $\alpha\ll\kappa$, $\beta\simeq\kappa-\alpha^2/2\kappa-\alpha^4/8\kappa^3-\cdots$. The paraxial approximation with integral transforms (5) does violence to the relation $\beta=(\kappa^2-\alpha^2)^{1/2}$ that keeps a minimum beam width: It frees the conjugate optical Hamiltonian momentum from the constraint of taking values only inside a circle of radius n, the refractive index of the medium.[6] This is the price of the application of Schrödinger quantization to model paraxial optics. The cutoff to $\alpha<\kappa$ comes from the restriction of having a finite smallest wavelength in the spectrum of optical wave functions.

Modulo these caveats, we identify Durnin's diffraction-free J_0 states (1) with the K_+^γ-classified eigenbasis (3) of the group of paraxial transformations, for $k=\frac{1}{2}$ ($\gamma=-\frac{1}{4}$): thus

$$\Phi_a(q_x,q_y,0;\kappa)\leftrightarrow\Phi_a^{1/2}((q_x^2+q_y^2)^{1/2}). \tag{7}$$

The $(aq)^{1/2}$ factors in (3) and (5b) yield the proper measure $\int_0^\infty q'dq'\ldots$ to integrate over the circles' radii. We should not overlook that also $k\neq\frac{1}{2}$ transforms exist and they originate from wave syntheses [Eq. (2) in Ref. 1] with azimuthal dependence $A(\phi)\sim e^{im\phi}$.

The title of the Letter refers to the self-reproducing form of Eq. (6) with respect to Φ_α^k. Paraxial optical systems normally end with free flight ζ, i.e., the image on the screen is the canonical transform $\mathbb{C}^k(\mathbf{M}_\zeta)$ of the object J_k state corresponding to the matrix

$$\mathbf{M}_\zeta=\begin{pmatrix}1 & \zeta\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a+\zeta c & b+\zeta d\\ c & d\end{pmatrix}. \tag{8}$$

Since the 1-1 element of $\mathbf{M}_\zeta$ depends on ζ, the half-width α of the outcoming beam will depend on ζ. This converging or diverging "diffraction-free" beam differs only by scale and quadratic phase from the object J_k state; the transversal illumination pattern, however, will be the same, up to scale.

Self-reproducing wave functions in this sense[9] are all the subgroup-reduced eigenfunctions including radial harmonic and repulsive oscillator functions, and, complexifying, Gaussians (with a modified Bessel function in αq in place of the usual exponential factor), and Barut-Girardello coherent states.[10] The last two are, respectively,

$$G_\omega(q-\alpha)=\omega^{-1}(\alpha q)^{1/2}e^{-\alpha^2/2\omega}I_{2k-1}(\alpha q/\omega)e^{-q^2/2\omega}=\left[\mathbb{C}^k\begin{pmatrix}i\omega & 1\\ -1 & 0\end{pmatrix}\Phi_\alpha^k\right](q), \tag{9a}$$

$$\Upsilon_\alpha(q)=(2\pi)^{1/2}\left[\mathbb{C}^k\begin{pmatrix}1/\sqrt{2} & -i/\sqrt{2}\\ -i/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\delta_\alpha\right](q)=(2n)^{1/2}\left[\mathbb{C}^k\begin{pmatrix}1/\sqrt{2} & -i/\sqrt{2}\\ -i/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\mathbb{C}^k\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Phi_\alpha^k\right](q). \tag{9b}$$

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