

APPLICATIONS OF A FACTORIZATION THEOREM
FOR NINTH-ORDER ABERRATION OPTICS

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A B S T R A C T We describe the results and objectives of a research line to develop Lie-algebraic and group-theoretic methods for the study of optics with aberration. We have applied REDUCE programming to obtain the phase-space transformation due to the refracting interface between two media, using a recent result on factorization into simpler 'root' transformations. The latter is given by a pair of implicit equations.

1. Description of the field. The Hamiltonian formulation of optics describes passive optical devices in terms of transformations of optical phase space. For the general reader we refer to the work of Dragt (1982a) Optical phase space has two configuration coordinates $(q_1, q_2)^T = \underline{q}$, the object point, measured at a plane perpendicular to the optical axis z (see Figure 1), and two momentum coordinates $(p_1, p_2)^T = \underline{p}$, which determine the direction of the beam through $\underline{p} = \underline{n} \sin \theta$; here \underline{n} is a vector in the plane which is the projection of the beam, of magnitude $n = n(\underline{q}, z)$, the index of refraction at (\underline{q}, z) , and θ is the angle between the beam and the optical axis.

In contradistinction to classical mechanics of point particles, here $|\underline{p}| \leq n$ and the Hamiltonian is $h = -\sqrt{n^2 - p^2} \approx -n + p^2/2n + p^4/8n^3 + p^6/16n^5 + 5p^8/128n^7 + 7p^{10}/256n^9 + \dots$. Classical mechanics contains Hamiltonians to second order in \underline{p} only; in optics, this is the Gaussian (paraxial) approximation, which involves only linear transformations of a phase-space plane. Referring to Figure 1, $\underline{p}'(\underline{p}, \underline{q}) = a\underline{p} + b\underline{q}$, $\underline{q}'(\underline{p}, \underline{q}) = c\underline{p} + d\underline{q}$, ($ad - bc = 1$). Higher-order terms in the system's Hamiltonian will aberrate the mapping of the object on to the image. This means that now \underline{p}' and \underline{q}' will be given by higher-order polynomials in \underline{p} and \underline{q} which will of N^{th} degree for N^{th} aberration order.

2. Mathematical methods used. The treatment of optical systems with aberration may be done using Lie techniques which involve the Poisson bracket between functions of \underline{p} and \underline{q} . For axis-symmetric systems, the only quadratic functions which appear are p^2 , $\underline{p} \cdot \underline{q}$, and q^2 .

All higher-order polynomials are functions of these five objects. Even-order ones, only of the latter three. For N^{th} order aberration calculations, Poisson brackets must be iterated up to $(N-1)/2$ times following the exponential series to produce N^{th} order phase space terms. For these systems, there is a distinct advantage in not considering all monomials $q_1^{a_1} q_2^{a_2} p_1^{b_1} p_2^{b_2}$, but only $(p^2)^{c_1} (\underline{p} \cdot \underline{q})^{c_2} (q^2)^{c_3}$, or these expression times a single \underline{p} or \underline{q} . For asymmetric magnetic optics, the former monomials are necessary: see Dragt (1982b) and Forest (1984).

The description of a refracting interface between two media, n and n' , produced by a surface $z = \mathcal{S}(\underline{q}) = \alpha q^2 + \beta q^4 + \gamma q^6 + \delta q^8 + \epsilon q^{10}$, may be found using a recent theorem motivated by Lie methods. This theorem appears in Navarro-Saad and Wolf (1984), and states that the transformation $S(n, n', \mathcal{S})$ factorizes into a product $R(n, \mathcal{S}) \cdot R(n', \mathcal{S})^{-1}$. The root transformation $R(n, \mathcal{S})$ is locally canonical and is given by the following implicit equations

$$\begin{aligned} R(n, \mathcal{S}): \underline{p} &= \underline{p}' = \underline{p} + \sqrt{n^2 - p^2} (\nabla \mathcal{S})(\underline{q}), \\ R(n, \mathcal{S}): \underline{q} &= \underline{q}' = \underline{q} + \underline{p} \mathcal{S}'(\underline{q}) / \sqrt{n^2 - p^2}. \end{aligned} \quad (*)$$

Self-replacement of the left-hand side of the second equation into the right hand side and subsequent replacement in the first, yields a recursive computation method for the explicit form of $\underline{p}'(\underline{p}, \underline{q})$ and $\underline{q}'(\underline{p}, \underline{q})$ to the predetermined aberration order. It is used thereafter for further group-theoretic manipulations. We report on this computation to aberration order nine, published in: Navarro-Saad (1985).

3. Systems used. The symbolic computations have been made at the Departamento de Cómputo installations in IIMAS. The machine is a Foonly F2 (1.7 MB) containing REDUCE-2 (Hearn (1973)) which occupies up to 329 KB. The resulting expressions were printed converting the REDUCE output file to a TEX (Knuth (1979)) input file, with Replace String commands (in EMACS) and complementary formatting, without touching coefficients and signs. As an aid, we used a Foovision system for TEX output prepared by Max Díaz at IIMAS. For printing, the file was transferred to a PDP-11/34 at the same installation, and from there to a VERSATEC V-80 electrographic printer. See Figure 2.

4. Experiences and present development. The concrete results achieved have been the iteration of equations (*) to ninth order in p and q , the inversion of this transformation through (*) for p' and q' , and the concatenation of these two transformations through the factorization theorem. Aberration order nine seems to be the upper capacity of the system (it took 2 1/2 hours real time). In Figure 2 we reproduce the results of Navarro-Saad (1985) for the first few lines of the TEX output. The full two formulas run for seven pages of text. Through the REDUCE output flag ON FORT, a FORTRAN file for further numerical computation and graphical representation of the results by spot diagrams in optics may be achieved. An example of this diagram is given in Figure 3 (from Navarro-Saad (1985)) which shows third order aberration. This is the output of a FORTRAN-based interactive program to create and handle third-order systems, which we plan to expand to higher orders, and for inhomogeneous optical systems.

The objective of our research in Lie methods is to incorporate the unwidely result output into a representing group element for surfaces, labelled by a truly minimal

set of parameters and subject to an economical product rule which does not make explicit use of the Baker-Campbell-Hausdorff series (Dragt (1982a)).

To find this minimal set of parameters, the results have to be compared with the analogue result which derives from the calculation of $\exp(:f_{10}:) \times \exp(:f_8:) \cdots \exp(:f_2:)$ on \underline{p} and \underline{q} , where $:f:$ denotes the Poisson bracket operator used by Dragt (1982a) for polynomial expressions in p^2 , $\underline{p} \cdot \underline{q}$, and q^2 . For the calculation of the Poisson brackets in $(\underline{p}, \underline{q})$ vector notation, the REDUCE Procedures were written with extensive use of LET commands.

The replacement of the phase-space variables by the Schrödinger generators of the Heisenberg-Weyl algebra should provide the transver integral kernel of wave optics. We expect the observables of order higher than two in phase space to be quantized according to the Weyl rule (see García-Bullé, Lassner, and Wolf (1985)). For these wave optics calculations (the quantization procedure and replacement of Poisson brackets by commutators) we use a method of symbol representation for the noncommuting algebra of Schrödinger operators \underline{p} and \underline{q} , (Lassner (1980)). We expect to obtain in this way, confirming preliminary results, plots for the wavefronts and the patterns of diffraction in aberration.

5. References

- Dragt, A.J. (1982a). Lie algebraic theory of geometrical optics and optical aberrations, J. Opt. Soc. Am. 72, 372-379.

- Dragt, A.J. (1982b), Lectures on Nonlinear Orbit Dynamics, American Institute of Physics Conference Proceedings, Vol. 87 (New York, NY)
- Forest, E. (1984), Lie Algebraic Methods for Charged Particle Beams and Light Optics, Ph. D. Thesis, University of Maryland, College Park, MD.
- García-Bullé, M., Lassner, W., and Wolf, K.B., (1985) The Metaplectic Group within the Heisenberg-Weyl Ring, Universidad Autónoma Metropolitana (Iztapalapa) preprint II, No. 20.
- Hearn, A.C. (1973), REDUCE-2 User's Manual, University of Utah.
- Knuth, D.E. (1979), TAU EPSILON CHI, a System for Technical Text (fourth printing), Stanford Artificial Intelligence Laboratory, Stanford CA.
- Lassner, W. (1980), Noncommutative Algebras Prepared for Computer Calculations. In: Proceedings of the International Conference on Systems and Techniques of Analytical Computing and Their Applications in Theoretical Physics. Joint Institute of Nuclear Research (Dubna) report D11-80-13.
- Navarro-Saad, M., Wolf, K.B. (1984). Factorization of the Phase-Space Transformation Produced by an Arbitrarily Refracting Surface. Accepted for publication in Journal of the Optical Society of America.
- Navarro-Saad, M. (1985). Cálculo de aberraciones en sistemas ópticos con teoría de grupos. Tesis, Facultad de Ciencias, Universidad Nacional Autónoma de México, México D.F.

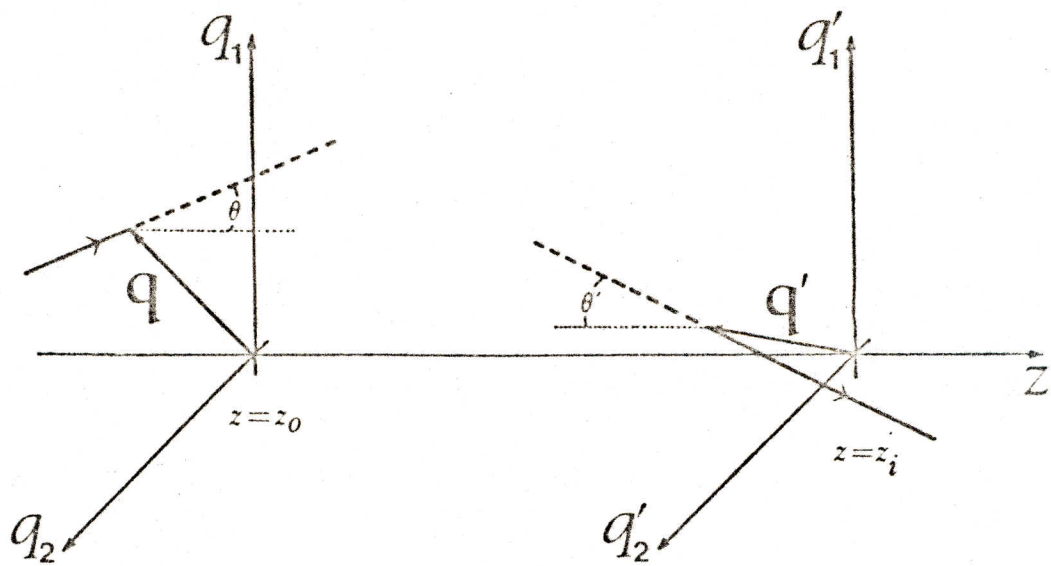


Figure 1. A light beam at the object plane $z = z_0$ is characterized by position \underline{q} and direction, yielding \underline{p} . After an intervening optical system, the image plane is at $z = z_i$.

$$\begin{aligned}
& \mathbf{p}' = \mathbf{p} \\
& +q(2n\alpha - 2n'\alpha) \quad (1^{\text{st}} \text{ order}) \\
& +p^2q\left(-\frac{1}{n}\alpha + \frac{1}{n'}\alpha\right) \\
& +q^2p\left(-2\frac{1}{n}n'\alpha^2 + 2\alpha^2\right) \\
& +q^2q\left(4n^2\frac{1}{n'}\alpha^3 - 8n\alpha^3 + 4n\beta + 4n'\alpha^3 - 4n'\beta\right) \\
& +(p \cdot q)q\left(4n\frac{1}{n'}\alpha^2 - 4\alpha^2\right) \quad (3^{\text{rd}} \text{ order}) \\
& +p^4q\left(-\frac{1}{4}\frac{1}{n^2}\alpha + \frac{1}{4}\frac{1}{n'^2}\alpha\right) \\
& +p^2q^2p\left(\frac{1}{n}\frac{1}{n'}\alpha^2 - \frac{1}{n^2}n'\alpha^2\right) \\
& +p^2q^2q\left(2n^2\frac{1}{n'^2}\alpha^3 - 2\frac{1}{n}\beta - 2\frac{1}{n'}\alpha^3 + 2\frac{1}{n'}\beta\right) \\
& +p^2(p \cdot q)q\left(2n\frac{1}{n'^2}\alpha^2 - 2\frac{1}{n}\frac{1}{n'}\alpha^2\right) \\
& +q^4p\left(4n\frac{1}{n'}\alpha^4 + 4\frac{1}{n}n'\alpha^4 - 8\frac{1}{n}n'\alpha\beta - 8\alpha^4 + 8\alpha\beta\right) \\
& +q^4q\left(4n^4\frac{1}{n'^2}\alpha^5 - 24n^2\frac{1}{n'}\alpha^5 + 24n^2\frac{1}{n'}\alpha^2\beta + 32n\alpha^5\right. \\
& \quad \left.- 48n\alpha^2\beta + 6n\gamma - 12n'\alpha^5 + 24n'\alpha^2\beta - 6n'\gamma\right) \\
& +q^2(p \cdot q)p\left(-4\frac{1}{n^2}n'\alpha^3 + 4\frac{1}{n'}\alpha^3\right) \\
& +q^2(p \cdot q)q\left(8n^3\frac{1}{n'^2}\alpha^4 - 16n\frac{1}{n'}\alpha^4 + 16n\frac{1}{n'}\alpha\beta + 8\frac{1}{n}n'\alpha^4 - 8\frac{1}{n}n'\alpha\beta - 8\alpha\beta\right) \\
& +(p \cdot q)^2q\left(4n^2\frac{1}{n'^2}\alpha^3 - 4\frac{1}{n'}\alpha^3\right) \quad (5^{\text{th}} \text{ order}) \\
& +p^6q\left(-\frac{1}{8}\frac{1}{n^5}\alpha + \frac{1}{8}\frac{1}{n'^5}\alpha\right) \\
& +p^4q^2p\left(\frac{1}{4}\frac{1}{n}\frac{1}{n'^2}\alpha^2 + \frac{1}{2}\frac{1}{n^2}\frac{1}{n'}\alpha^2 - \frac{3}{4}\frac{1}{n^5}n'\alpha^2\right) \\
& +p^4q^2q\left(\frac{3}{2}n^2\frac{1}{n'^2}\alpha^3 - \frac{1}{n^2}\alpha^3 - \frac{1}{2}\frac{1}{n^2}\beta - \frac{1}{2}\frac{1}{n'^2}\alpha^3 + \frac{1}{2}\frac{1}{n'^2}\beta\right) \\
& +p^4(p \cdot q)q\left(\frac{3}{2}n\frac{1}{n'^2}\alpha^2 - \frac{1}{n}\frac{1}{n'^2}\alpha^2 - \frac{1}{2}\frac{1}{n^2}\frac{1}{n'}\alpha^2\right) \\
& + \dots
\end{aligned}$$

Figure 2. TEX output file of the first few lines of the phase-space transformation produced by a refracting surface $\alpha q^2 + \beta q^4 + \gamma q^6 + \delta q^8 + \xi q^{10} + \dots$ between two media n, n' .

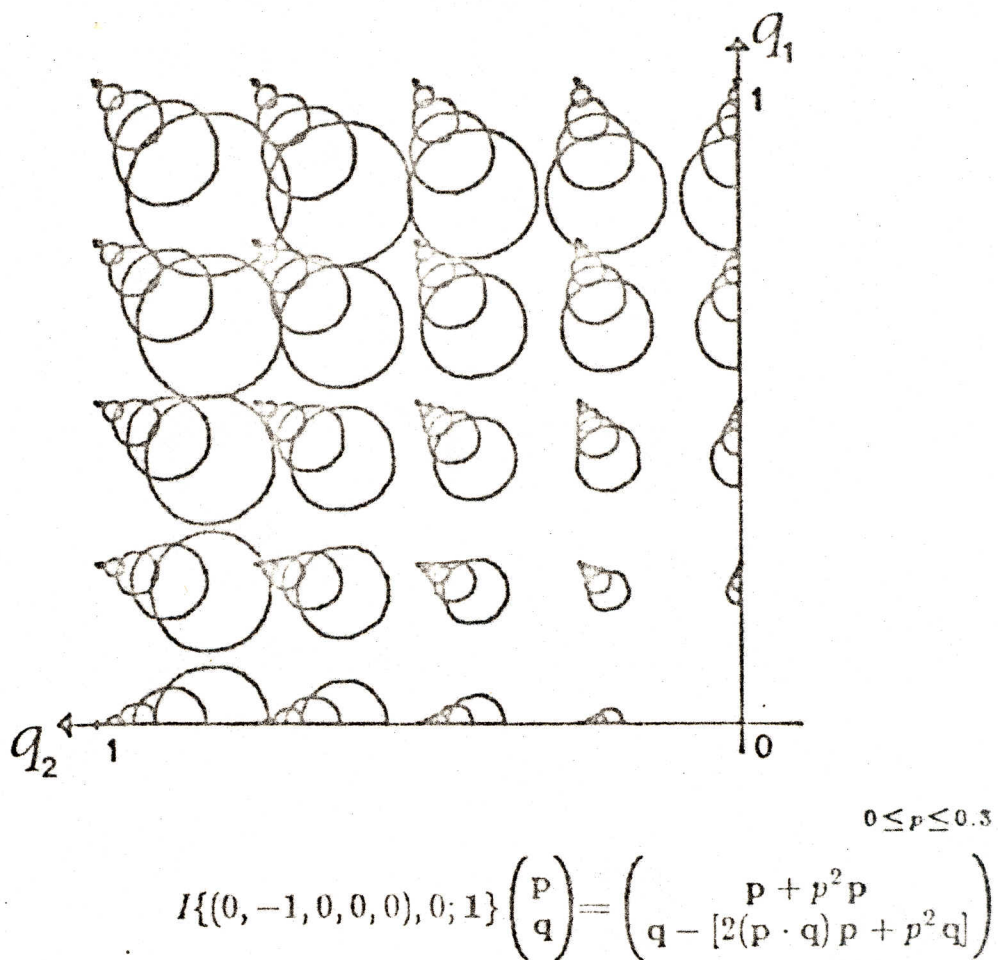


Figure 3. Sample spot diagram for a system with third-order pure coma aberration of unit coefficient (Navarro-Saad and Wolf (1984)).