Lie algebras for systems with mixed spectra. I. The scattering Pöschl–Teller potential

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(Received 23 March 1984; accepted for publication 4 October 1984)

Starting from an N-body quantum space, we consider the Lie-algebraic framework where the Pöschl–Teller Hamiltonian, $-\frac{1}{2}\partial_\chi^2 + c\,\mathrm{sech}^2\chi + s\,\mathrm{csch}^2\chi$, is the single sp(2,$R$) Casimir operator. The spectrum of this system is *mixed*: it contains a finite number of negative-energy bound states and a positive-energy continuum of free states; it is identified with the Clebsch–Gordan series of the $\mathscr{D}^+ \times \mathscr{D}^-$ representation coupling. The wave functions are the sp(2,R) Clebsch–Gordan coefficients of that coupling in the parabolic basis. Using only Lie-algebraic techniques, we find the asymptotic behavior of these wave functions; for the special pure-trough potential ($s = 0$) we derive thus the transmission and reflection amplitudes of the scattering matrix.

I. INTRODUCTION

Symmetry methods involving dynamical algebras have been long used to study the eigenstates and spectra of Schrödinger equations for certain one-dimensional potentials.[1–4] Notable among them are the hydrogen atom[1] (bound states and scattering states), the harmonic oscillator,[3] and the radial oscillator[5] (bound states only); among the systems with continuous spectra we know the free-fall (or linear) potential, the free particle, and the repulsive oscillator,[4] and the latter two in their welled versions. Here the symmetry method builds the dynamical algebra, and —in all but the first case, where it is the *pseudo*-Coulomb system which enjoys the algebra—the Hamiltonian is an element of this Lie algebra, which is[4,6] sp(2,R) = so(2,1) = sl(2,R) = su(1,1), and which we refer to as the two-dimensional real symplectic algebra sp(2,R). (We note that Lie *algebras* are involved, rather than Lie *groups*, as it is often stated.) Symmetries have a longer history, of course, since the angular properties of any central potential Hamiltonian and the rigid rotator acquainted physicists with group theory in the first place.[7] We are here concerned with *dynamical* algebras, i.e., those whose representations correspond with the whole energy spectrum of the system.

It is important to state that the only spectra that have been obtained from a dynamical algebra of which the Hamiltonian is an element are equally spaced spectra[8,9] if discrete (with a lower bound if realistic), a lower-bound continuum, or a double non-lower-bound continuum. These cases correspond to the Hamiltonian being on the elliptic, parabolic, or hyperbolic subalgebras of sp(2,R).[6]

Next, the Hamiltonian may be a simple function of one or more of the generators, the spectrum now being that function of the integers or subset thereof. This construction may be made for the bound hydrogen atom states, for its scattering states separately,[1] and has been used recently for the Morse[10] and Pöschl–Teller[11] potentials, among others, by Alhassid, Gürsey, and Iachello.

The Morse potential[12] is very well-known for its role in molecular physics while the Pöschl–Teller potential[13] emerges in connection with diverse physical systems, such as completely integrable many-body systems in one dimension,[14] the solitary wave solutions to the Korteweg–de Vries equation,[15] and in the Hartree mean field equation of many-body systems interacting through a δ force[16] among others. The Pöschl–Teller Schrödinger equation also stems from the Klein–Gordon equation on a space of constant curvature, with an appropriate set of separating variables, the D'Alembertian being the Laplace–Beltrami operator on a sphere or hyperboloid.[17]

The Pöschl–Teller potential has two free parameters:

$$V^{cs}(\chi) = c\,\mathrm{sech}^2\chi + s\,\mathrm{csch}^2\chi\,. \tag{1.1}$$

See Fig. 1. There is a $\sim s/\chi^2$ core at the origin plus a trough $\sim \mathrm{sech}^2\chi$. When $0 < -s/c < 1$, the two may combine to a potential with a core ($s > 0$) and a trough ($c > 0$). This trough may capture one or more quantum bound states when $\sqrt{2s+\frac{1}{4}} < \sqrt{-2c+\frac{1}{4}}$, which will be part of the spectrum of the Pöschl–Teller quantum Hamiltonian $\mathbb{H}^{\mathrm{PT}}$. The number of bound states is the integer part of the difference between $\sqrt{2s+\frac{1}{4}}$ and $\sqrt{-2c+\frac{1}{4}}$.

Alhassid,[18] Gürsey,[19] and Iachello[11] used the algebra

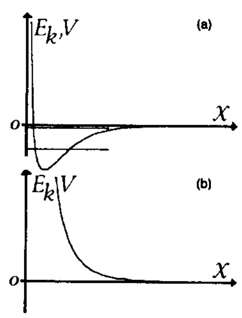

FIG. 1. (a) The Pöschl–Teller potential with a core and trough, exhibiting two bound states and the continuum. (b) A Pöschl–Teller potential where the trough parameter is smaller than the core parameter; it has only a continuum of positive-energy states.

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 0022-2488/85/050973-11$02.50

so(3) with one subalgebra generator for full representation l and row m classification. The Pöschl–Teller equation is then found to be the square of that generator and thus the bound-state spectrum is accounted for, being $\sim -m^2$ over the multiplet. This potential also has a continuum of positive-energy scattering states, and the Weyl analytic continuation is used to turn the algebra into so(2,1), where the positive continuous energy eigenvalue is the square of the eigenvalue of a noncompact generator of the algebra. They also investigate a more general version of the Pöschl–Teller potential, which is obtained from a representation of the direct sum algebra su(1,1) ⊕ su(1,1) realized by the symmetric top system in which one of the Euler angles is made hyperbolic.[20] They are thus able to show that the Pöschl–Teller Hamiltonian emerges as essentially the Casimir operator of the algebra and that it has mixed spectrum, including the bound and scattering states of the potential,[20] a result also found by Basu and Wolf.[21]

In this article we shall reexamine the Pöschl–Teller potential, showing that the Clebsch–Gordan *series*[22] of sp(2,R) yields the spectrum of the system, while the eigenstates turn out to be the sp(2,R) Clebsch–Gordan *coefficients* in the parabolic chain of Basu and Wolf[21] for a lower- and an upper-bound sp(2,R) discrete series representation, coupling into a finite sum of discrete series plus an integral over continuous-series representation. The energy values are determined by the coupled-sp(2,R) representations, while the potential parameters in (1.1) are determined by the two-factor sp(2,R) representations. The action of the raising and lowering operators in the conjugate so(2,2) algebra allow us to relate potentials (1.1) with *different values* of *the potential parameters* s and c for eigenstates of the *same energy*. In particular they can be made to relate a given potential with the free-particle potential $V^{00}(\chi)=0$, the eigenstates of the two systems then being related through an algebra with shift operators, thus allowing a derivation of the reflection and transmission coefficients of the S matrix by purely algebraic means. These will be functions of the potential parameters and the energy of the state.

The mixed-spectrum character of the Pöschl–Teller potential makes it attractive for nuclear physics models of scattering. It is shown in Sec. II and III that this potential arises in an N-particle space out of the quadratic operators in position and momentum, forming an oscillator sp(2N,R) algebra, which contains sp(2,R) through the maximal subalgebra sp(2,R) ⊕ so(n,m). In this reduction, the representations of the two summands are conjugate. We further decompose so(n,m) ⊃ so(n) ⊕ so(m), each direct summand algebra having a conjugate sp(2,R), which provide the Pöschl–Teller potential parameters s and c with restrictions to discrete values. In Sec. IV we use the so(2,2) algebra generators to raise and lower[2,23] these values: the dimensions n and m are not crucially important for the structure of the system, and so(2,2) has most of the general features, plus some particularly useful ones. In this way we find the reflection and transmission coefficients[24] and the scattering matrix for this potential. The closing section offers some conclusions as to the place of the system treated here within the general systems whose spectrum is given by the Clebsch–Gordan series for sp(2,R), which may include the Coulomb system in the proper representation coupling class.

II. THE OSCILLATOR REALIZATION OF sp(2n,R) ⊃ sp(2,R) ⊕ so(n)

We consider the Schrödinger realization of the quantum operators of position and momentum in an n-dimensional Euclidean space R^n,

$$\mathbb{Q}_a f(\mathbf{x}) := x_a f(\mathbf{x}), \quad \mathbb{P}_a f(\mathbf{x}) := -i\frac{\partial f(\mathbf{x})}{\partial x_a}, \quad a=1,2,\dots,n. \tag{2.1a}$$

They are self-adjoint in a common invariant domain dense in $\mathscr{L}^2(R^n)$, and satisfy the well-known Heisenberg commutation relations

$$[\mathbb{Q}_a, \mathbb{P}_b] = i\delta_{a,b}\mathbf{1}, \tag{2.1b}$$

where $\mathbf{1}$ is the unit operator.[25] Next, we build all bilinear self-adjoint operators in $\mathbb{Q}_a$ and $\mathbb{P}_b$, denoting them as

$$\mathbb{J}^1_{ab} := \tfrac{1}{4}(\mathbb{P}_a\mathbb{P}_b - \mathbb{Q}_a\mathbb{Q}_b), \tag{2.2a}$$

$$\mathbb{J}^2_{ab} := \tfrac{1}{4}\mathbb{N}_{ab} := \tfrac{1}{4}(\mathbb{Q}_a\mathbb{P}_b + \mathbb{Q}_b\mathbb{P}_a - i\delta_{a,b}\mathbf{1}), \tag{2.2b}$$

$$\mathbb{J}^0_{ab} := \tfrac{1}{4}(\mathbb{P}_a\mathbb{P}_b + \mathbb{Q}_a\mathbb{Q}_b), \tag{2.2c}$$

$$\mathbb{M}_{ab} := \mathbb{Q}_a\,\mathbb{P}_b - \mathbb{Q}_b\,\mathbb{P}_a, \quad a,b=1,2,\dots,n. \tag{2.2d}$$

This set of operators closes under commutation, with the commutation relations defining the $2n$-dimensional real symplectic algebra[2] sp(2n,R). Since $\mathbb{J}^k_{ab}=\mathbb{J}^k_{ba}$ and $\mathbb{M}_{ab} = -\mathbb{M}_{ba}$, there are $2n^2+n$ operators in the set and they are self-adjoint in $\mathscr{L}^2(R^n)$. On this space, they yield the *oscillator*[3] (or *metaplectic*[26]) representation of sp(2n,R). On this space this representation is not irreducible, since the inversion commutes with the set (2.2), and decomposes into two irreducible representations, one in the subspace of even functions and one in the subspace of odd functions.

Now we construct the linear combinations

$$\mathbb{J}^k := \sum_{a=1}^{n} \mathbb{J}^k_{aa}, \quad k=1,2,0. \tag{2.3}$$

These three operators generate an algebra sp(2,R) which commutes with the operators $\mathbb{M}_{ab}$ in (2.2d). The latter commute among themselves and generate the n-dimensional orthogonal algebra so(n). We thus consider the algebra chain

$$sp(2n,R) \supset \mathrm{sp}(2,R) \oplus \mathrm{so}(n), \tag{2.4}$$

where the subalgebra is maximal in the parent algebra. The two direct summands in the subalgebra are, moreover, *conjugate*, i.e., within the oscillator representation of sp(2n,R), the representation of one direct summand determines the representation of the other. Indeed, the Casimir operator of sp(2,R) is[6]

$$\mathbb{C}^{\mathrm{sp}} := (\mathbb{J}^1)^2 + (\mathbb{J}^2)^2 - (\mathbb{J}^0)^2, \tag{2.5a}$$

while the second-order Casimir operator of so(n) is [2]

$$\mathbb{C}^{\mathrm{so}} := \frac{1}{2}\sum_{a,b=1}^{n} (\mathbb{M}_{ab})^2, \tag{2.5b}$$

and all higher-order Casimir operators of the latter are zero since the algebra is realized on the n-sphere[27] S^{n-1}. The second-order Casimir operator (2.5b) is the Laplace–Beltrami operator on that $(n-1)$-dimensional space S^{n-1}, with constant curvature related to the radius of the sphere.[28] One may show directly replacing (2.2) that the two operators (2.5) are related by

$$\mathbb{C}^{\mathrm{sp}} = -\tfrac{1}{4}\mathbb{C}^{\mathrm{so}} + \tfrac{1}{16}n(4-n). \tag{2.6}$$

The eigenvalues of $\mathbb{C}^{\mathrm{so}}$ on S^{n-1} (i.e., the spectrum of the D'Alembertian) are given by

$$c^{\mathrm{so}} = l(l+n-2), \quad l = 0,1,2,..., \tag{2.7}$$

and thus through (2.6), the eigenvalues of $\mathbb{C}^{\mathrm{sp}}$ are

$$c^{\mathrm{sp}} = k(1-k), \tag{2.8a}$$

$$k = \tfrac{1}{2}(l+\tfrac{1}{2}n) = \tfrac{1}{4}n, \tfrac{1}{4}n+\tfrac{1}{2}, \tfrac{1}{4}n+1,..., \tag{2.8b}$$

where k is referrred to as the *Bargmann* sp(2,R) representation index.[6] The representations of sp(2,R) present in the decomposition (2.4) are thus the lower-bound discrete-series representations $\mathscr{D}_k^+$. The parity of the so(n) representation l on S^{n-1} is well known to be $(-1)^l$. It follows that on the irreducible subspace of even functions, the oscillator realization (2.2) decomposes into the direct sum of sp(2,R) ⊕ so(n) representations $(k,l) = (\tfrac{1}{4}n,0) + (\tfrac{1}{4}n+1,2) + (\tfrac{1}{4}n+2,4) + \cdots$, while in the subspace of odd functions it is $(k,l) = (\tfrac{1}{4}n+\tfrac{1}{2},1) + (\tfrac{1}{4}n+\tfrac{3}{2},3) + (\tfrac{1}{4}n+\tfrac{5}{2},5) + \cdots$.

In the case $n=1$, the generatorless algebra "so(1)" is replaced by the inversion operator with eigenvalues $+1$ and -1 on the two-point space S^0. The former goes with $k=\tfrac{1}{4}$ and the latter with $k=\tfrac{3}{4}$. This "so(1)" also effects the "algebra reduction" of so(2) to eigenvalues $m = \pm l$ of the latter's single generator, the sign being the "so(1)" eigenlabel. In the general-n case, we need not concern ourselves with the representation row labeling.

Regarding the subalgebra reduction of sp(2,R), the better-known[6] chain involves the compact subalgebra with generator $\mathbb{J}^0$. This operator is the n-dimensional harmonic oscillator Hamiltonian with angular momentum l, whose spectrum is lower bounded by k and is linearly spaced by integers.[2] In this work we shall use the *parabolic* subalgebra generator[29]

$$\mathbb{J}^- := \mathbb{J}^0 - \mathbb{J}^1 = \frac{1}{2}\sum_{a=1}^{n} \mathbb{Q}_a\mathbb{Q}_a. \tag{2.9}$$

This has been implicitly used whenever sp(2,R) is realized in terms of up-to-second-order differential operators, but has not often appeared as an abstract subalgebra chain in the physics literature. The operator $\mathbb{J}^-$ is noncompact and its spectrum therefore continuous; in the discrete-series representations $\mathscr{D}_k^+$, it is positive, as seen here, and simple. In the continuous-series representations of the next section, it is still simple but both positive *and* negative.[30]

The algebra sp(2,R) also has *negative* discrete-series representations, denoted by $\mathscr{D}_k^-$. These may be obtained from the positive-series operators (2.2) and (2.3) on $\mathscr{L}^2(R^n)$ through the mapping[6]

$$A:\{\mathbb{J}^1,\mathbb{J}^2,\mathbb{J}^0\} \mapsto \{-\mathbb{J}^1,\mathbb{J}^2,-\mathbb{J}^0\}. \tag{2.10}$$

This is an automorphism of sp(2,R), so the Casimir operator eigenvalues (2.8) are unaffected. It is involutive, but not within the group generated by it, *i.e.*, it is outer. It inverts the harmonic-oscillator spectrum of $\mathbb{J}^0$ to negative values, so the eigenvalues of the latter are upper bound by $-k$. The spectrum of $A\mathbb{J}^-$ is now the negative half-axis.

III. COUPLING AND REDUCTION IN sp(2N,R) ⊃ sp(2,R) ⊕ so(n,m)

We now consider the following Euclidean spaces: R^n, R^m, and R^N, $N = n+m$, where the two first spaces are disjoint subspaces of the latter, arranged so that $x_a \in R^n$ for $1 < a < n$ and $x_a \in R^m$ for $n+1 < a < n+m = N$. On the $\mathscr{L}^2(R^N)$ space of functions $f(\mathbf{x})$ we may build the oscillator representation of the symplectic algebra sp(2N,R), which has been presented in the last section and given in (2.2), letting all index ranges grow to N. We reproduce the structure for R^n and R_m placing their oscillator algebras sp(2n,R) and sp(2m,R) as subalgebras of sp(2N,R). Each of the former two will be decomposed as sp(2n,R) ⊃ $\mathrm{sp}_{(n)}(2,R)$ ⊕ so(n) and sp(2m,R) ⊃ $\mathrm{sp}_{(m)}(2,R)$ ⊕ so(m), where the generators of the first factors will be labeled as $\mathbb{J}^k_{(n)}$ and $\mathbb{J}^k_{(m)}$ for $k = 1,2,0$, built as in (2.3) with the appropriate summation index range. Now, if we follow the same procedure with sp(2N,R), we are *coupling*[21,31] the representations of $\mathrm{sp}_{(n)}(2,R)$ and $\mathrm{sp}_{(m)}(2,R)$ to a representation of $\mathrm{sp}_{(N)}(2,R)$. If the two factor representations belong to the $\mathscr{D}^+$ series, their product[21,22,31,32] will be reducible in terms of irreducible representations of the latter also belonging to the $\mathscr{D}^+$ series. If the former are given by their Bargmann indices $k_{(n)}$ and $k_{(m)}$, the Clebsch–Gordan series will contain the $\mathrm{sp}_{(N)}(2,R)$ representations $k_{(N)} = k_{(n)} + k_{(m)}, k_{(n)} + k_{(m)} + 1,...$ and its Casimir operator would have eigenvalues $k_{(N)}(1-k_{(N)})$ with $k_{(N)}$ on the series. These facts may be easily seen in the compact subalgebra reductions, where $\mathbb{J}^0_{(N)}$ is the sum of the harmonic oscillators $\mathbb{J}^0_{(n)}$ and $\mathbb{J}^0_{(m)}$ with the consequent sum of their discrete, lower-bound spectra to a "radial" discrete, lower-bound spectrum which, were we to follow this coupling, would lead to the constraining Pöschl–Teller potential (of the *first* type) $V(\chi) = c\sec^2\chi + s\csc^2\chi$, $0 < \chi < \pi/2$.

Our interest in this paper lies in the *scattering* Pöschl–Teller potential (i.e., of the *second* kind):

$$V^{cs}(\chi) = c\,\mathrm{sech}^2\chi + s\,\mathrm{csch}^2\chi, \tag{3.1}$$

which, according to the values of the two parameters, c and s, will have a lower-bound continuum of scattering states, with the possibility of a finite number of bound states.

To achieve this, we couple the two sp(2,R) subalgebras in sp(2n,R) and sp(2m,R) to the $\mathrm{sp}_{(N)}(2,R)$ algebra in sp(2N,R) through essentially the *difference* of the generators, following the linear combinations[2,31,32]

$$\mathbb{J}^1_{(N)} = \mathbb{J}^1_{(n)} - \mathbb{J}^1_{(m)}, \tag{3.2a}$$

$$\mathbf{J}^2_{(N)} = \mathbf{J}^2_{(n)} + \mathbf{J}^2_{(m)}, \tag{3.2b}$$

$$\mathbf{J}^0_{(N)} = \mathbf{J}^0_{(n)} - \mathbf{J}^0_{(m)}. \tag{3.2c}$$

This corresponds to coupling one $\mathscr{D}^+_{k(n)}$ with one $\mathscr{D}^-_{k(m)}$ irreducible representation. The so-algebra commuting with this $\mathrm{sp}_{(N)}(2,R)$ is the pseudo-orthogonal algebra so(n,m) whose set of generators is the union of the so(n) generators $\mathbf{M}_{ab}$, $a,b = 1,\ldots,n$ (communting nontrivially with $\mathbf{J}^k_{(m)}$), the so(m) generators $\mathbf{M}_{ab}$, $a,b = n+1,\ldots,n+m$ (commuting nontrivially with $\mathbf{J}^k_{(n)}$), and the "cross" noncompact boost generators $\mathbf{N}_{ab}$, $a = 1,\ldots,n$, $b = n+1,\ldots,n+m$ in (2.2b). We thus work with the subalgebra chain

$$\mathrm{sp}(2N,R) \supset \mathrm{sp}(2,R) \oplus \mathrm{so}(n,m). \tag{3.3}$$

The second-order Casimir operator of this so(n,m) may be expressed in the following form, in terms of the three constituent sets of generators:

$$\mathbb{C}^{\mathrm{so}(n,m)} = \mathbb{C}^{\mathrm{so}(n)} + \mathbb{C}^{\mathrm{so}(m)} - \sum_{a=1}^{n} \sum_{b=n+1}^{n+m} (\mathbf{N}_{ab})^2, \tag{3.4}$$

while that of $\mathrm{sp}_{(N)}(2,R)$ is given by (2.5a) in terms of (3.2). The two direct summand algebras in (3.3), sp(2,R) and so(n,m) are again conjugate in sp(2N,R), and their Casimir operators are related as in (2.6), with N replacing n; the eigenvalues relate accordingly.

The spectrum of the so(n,m) Casimir on the (n,m) hyperboloid H^{N-1} may be written as in (2.7), but with a different range of values of l. Through (2.8) we conclude that the conjugate $\mathrm{sp}_{(N)}(2,R)$ representation is labeled by $k = (1 + i\lambda)/2$ and thus belongs to the *continuous* (nonexceptional[6]) representation series $\mathscr{C}^{\epsilon}_{c}$ with Casimir eigenvalue $c = (1+\lambda^2)/4 \geqslant \frac{1}{4}$ and multivaluation index $\epsilon = 0, \frac{1}{2}$ resolved in the $\mathscr{L}^2(R^N)$ subspaces of even and odd functions.

When $n = 1 = m$, both "so(1)" Casimir operators are zero and $\mathbb{C}^{\mathrm{so}(1,1)}$ is the square of a single boost generator $\mathbf{N}_{12}$, with a negative sign, i.e., $c^{\mathrm{so}(1,1)} = -\lambda^2$ corresponding to $l = i\lambda$, $\lambda \geqslant 0$.

For general $n,m > 1$, the coupling of $\mathscr{D}^+_{k_{(n)}}$ and $\mathscr{D}^-_{k_{(m)}}$ representations of sp(2,R) has the following Clebsch–Gordan series[21,22,31,32]:

$$\mathscr{D}^+_{k_{(n)}} \dot{\times} \mathscr{D}^-_{k_{(m)}} = \sum_{|k_{(n)} - k_{(m)}| \geqslant k_{(N)} > \frac{1}{2}} \mathscr{D}^{\mathrm{sgn}[k_{(n)} - k_{(m)}]}_{k_{(N)}} + \int_{\kappa=0}^{\infty} \mathscr{C}^{k_{(n)} - k_{(m)} \bmod 1}_{\frac{1}{4} + \kappa \frac{2}{4}}, \tag{3.5}$$

i.e., for $k_{(n)} > k_{(m)}$, a direct finite sum of lower-bound discrete-series representations $\mathscr{D}^+_{k(N)}$ from $k_{(N)} = k_{(n)} - k_{(m)}$ in integer steps down to (but not including) $\frac{1}{2}$, *plus* a direct integral over all nonexceptional continuous representation series with the appropriate multivaluation index $\epsilon = 0$ or $\epsilon = \frac{1}{2}$, according to the total space inversion parity. The discrete part of the spectrum is absent if $k_{(n)} - k_{(m)} \leqslant \frac{1}{2}$. The Casimir operator of $\mathrm{sp}_{(N)}(2,R)$ has thus the *mixed* spectrum

$$c^{\mathrm{sp}_{(N)}(2,R)} = \begin{cases} k(1-k) < \frac{1}{4}, & k = k_{\min}, k_{\min} - 1, \ldots, > \frac{1}{2}, \quad k_{\min} = k_{(n)} - k_{(m)}, \\ \frac{1}{4}(1+\kappa^2) \geqslant \frac{1}{4}, & \kappa \geqslant 0. \end{cases} \tag{3.6}$$

This is the form of the spectra fitting into our coupling scheme: mixed spectra with a continuum of positive energy and a finite number of bound states with a characteristic quadratically downward-increasing separation for negative energy.

The previous statements are basis independent. In order to see how (3.6) becomes the spectrum of the scattering Pöschl–Teller potential Schrödinger Hamiltonian, we introduce the appropriate coordinates in R^N. These are (n,m)-bipolar-hyperbolic coordinates x_j $(\sigma, \rho, \chi, \{v_k\}, \{\omega_k\})$:

$$x_j = : v\nu_j, \quad j = 1,2,\ldots,n; \quad \gamma \geqslant 0, \quad \sum_j (\nu_j)^2 = 1, \tag{3.7a}$$

$$x_j = : s\omega_j, \quad j = n+1,\ldots,n+m; \quad s \geqslant 0, \quad \sum_j (\omega_j)^2 = 1; \tag{3.7b}$$

$$\sigma = +1 \quad (r > s) \quad \begin{cases} r = : \rho \cosh \chi, & \rho,\chi \geqslant 0, \\ s = : \rho \sinh \chi, \end{cases} \tag{3.8a}$$

$$\sigma = -1 \quad (r < s) \quad \begin{cases} r = : \rho \sinh \chi, \\ s = : \rho \cosh \chi. \end{cases} \tag{3.8b}$$

The $\{\nu_j\}^n_{j=1}$ and $\{\omega_k\}^{n+m}_{k=n+1}$ are coordinates on the spheres S^{n-1} and S^{m-1}, and the Casimir operators of so(n) and so(m) are second-order differential operators in them, while r and s do not enter their expression. The so(n,m) Casimir operator will be a differential operator in all variables but ρ. One should be careful to note that these coordinates are *not global*, i.e., two *charts*, labeled by $\sigma = \pm 1$, are needed to cover the $r,s \geqslant 0$ quadrant by ρ, $\chi \geqslant 0$, (2.8a) *and* (3.8b). The so(n,m) Casimir operator will have two forms, one in each chart.[21,31]

We now detail the forms of the six operators and their eigenvalues, whose eigenfunctions are—once appropriately normalized—the $\mathscr{D}^+_{k(n)} \dot{\times} \mathscr{D}^-_{k(m)}$ Clebsch–Gordan coefficients.[21]

$k_{(n)}$: $\mathbb{C}^{\mathrm{sp}(n)(2,R)}$ has eigenvalue $k_{(n)}(1 - k_{(n)})$. This fixes the $\boldsymbol{\nu}$ dependence of the eigenfunction to be an so(n) harmonic with angular momentum $l_{(n)} = 2k_{(n)} - \frac{1}{2}n$.

$k_{(m)}$: $\mathbb{C}^{\mathrm{sp}(m)(2,R)}$ has eigenvalue $k_{(m)}(1 - k_{(m)})$. The $\boldsymbol{\omega}$ dependence is that of an so(m) harmonic $l_{(m)} = 2k_{(m)} - \frac{1}{2}m$.

$k_{(N)}$: $\mathbb{C}^{\mathrm{sp}_{(N)}(2,R)}$ has eigenvalue $c^{\mathrm{sp}_{(N)}(2,R)}$ given by the Clebsch–Gordan series (3.6), expressible through (2.6), (3.4), and (3.7) as the differential operator on $(\sigma,\chi,\boldsymbol{\nu},\boldsymbol{\omega})$,

$$
\begin{aligned}
\mathbb{C}^{\mathrm{sp}_{(N)}(2,R)} &= \tfrac{1}{16}N(4-N)-\tfrac{1}{4}\left[\mathbb{C}^{\mathrm{so}(n)}+\mathbb{C}^{\mathrm{so}(m)}+\sum_{a=1}^{n}\sum_{b=n+1}^{n+m}\left(x_a\frac{\partial}{\partial x_b}+x_b\frac{\partial}{\partial x_a}\right)\right]\\
&=\frac{-1}{4}\left[N\left(N-\frac{1}{4}\right)+\left(1-\frac{s^2}{r^2}\right)\mathbb{C}^{\mathrm{so}(n)}+\left(1-\frac{r^2}{s^2}\right)\mathbb{C}^{\mathrm{so}(m)}\right.\\
&\quad\left.+\left(r\frac{\partial}{\partial s}+s\frac{\partial}{\partial r}\right)^2+\left((n-1)\frac{s}{r}+(m-1)\frac{r}{s}\right)\left(r\frac{\partial}{\partial s}+s\frac{\partial}{\partial r}\right)\right] \qquad (3.9)\\
&=\frac{-1}{4}\left[N\left(N-\frac{1}{4}\right)+\frac{\partial^2}{\partial\chi^2}+\left((n-1)\begin{Bmatrix}\tanh\chi\\ \coth\chi\end{Bmatrix}+(m-1)\begin{Bmatrix}\coth\chi\\ \tanh\chi\end{Bmatrix}\right)\frac{\partial}{\partial\chi}\right.\\
&\quad\left.+\begin{Bmatrix}\operatorname{sech}^2\chi\\ -\operatorname{csch}^2\chi\end{Bmatrix}\mathbb{C}^{\mathrm{so}(n)}+\begin{Bmatrix}-\operatorname{csch}^2\chi\\ \operatorname{sech}^2\chi\end{Bmatrix}\mathbb{C}^{\mathrm{so}(m)}\right],\quad\text{for}\quad\begin{Bmatrix}\sigma=+1\\ \sigma=-1\end{Bmatrix}.
\end{aligned}
$$

Note that this operator has one form on each chart. Now we come to the parabolic subalgebra "row" labels.

r: $\mathbb{J}^-_{(n)}=\frac{1}{2}r^2$ with nonnegative eigenvalue.

s: $\mathbb{J}^-_{(m)}=\frac{1}{2}s^2$ with nonnegative eigenvalue.

(σ,ρ): $\mathbb{J}^-_{(N)}=-\mathbb{J}^-_{(n)}-\mathbb{J}^-_{(m)}=\frac{1}{2}(r^2-s^2)=\frac{1}{2}\sigma\rho^2$.

This eigenvalue is fixed by r and s, which determine the chart σ on which $\mathbb{C}^{\mathrm{sp}_{(n)}(2,R)}$ lies, and $\rho\geqslant 0$.

The normalized eigenfunctions of the first three operators, valuated at the eigenvalues of the last three, are the numerical Clebsch–Gordan coefficients. Only two of the latter three are independent. We may fix (σ,ρ) and, say, r, to determine s. We decide to fix σ and ρ, and let the coefficient be a function of the single free coordinate χ. Then, the Clebsch–Gordan coefficients are obtained as functions of χ satisfying the differential eigenfunction equation (3.9) with (3.6) for its spectrum.[21] It is particularly important to fix σ since this places us on a single chart σ, which we choose hereafter to be $\sigma=+1$. (Choosing $\sigma=-1$ only exchanges n and m.)

Clebsch–Gordan coefficients, even for noncompact algebras, are best known when reduced with respect to a compact subalgebra,[7,21,32] so that the row indices are integers m_1, m_2, and $m=m_1+m_2$, for example. These satisfy three-term recursion relations—a second-order difference equation—which stem from the coupled Casimir operator. Their proper summation for normalization is a rather difficult problem. In the noncompact parabolic subalgebra basis, the row labels are continuous and the eigenfunctions of (3.9) satisfy an ordinary second-order differential equation when the so(n) and so(m) eigenfunction subspaces are taken. We anticipate that the solutions of (3.9) are $_2F_1$ Gauss hypergeometric functions,[21] while the elliptic or hyperbolic subalgebras lead to $_3F_2$ functions of unit argument.[31,32]

The original $\mathscr{L}^2(R^N)$ eigenfunctions of the Casimir operator are orthogonal under a maximal set of commuting operators under the measure

$$d^N\mathbf{x}=\rho^{N-1}\,d\rho\,\Omega_{nm}(\chi)d\chi\,d^{n-1}\mathbf{v}\,d^{m-1}\boldsymbol{\omega}, \qquad (3.10a)$$

$$\Omega_{nm}(\chi)=\sigma\sinh^{n-1}\chi\cosh^{m-1}\chi. \qquad (3.10b)$$

The integration on $\mathbf{v}$ and $\boldsymbol{\omega}$ leads to orthogonality in the so(n) and so(m) representation labels $k_{(n)}$ and $k_{(m)}$, and row labels which are absent from the sp(2,R) coefficient. Definite $\mathbb{J}^-_{(N)}$ eigenfunctions restrict to a definite (σ,ρ) value, on the χ half-line, the operator (3.9) is symmetric with respect to the measure $\Omega_{nm}(\chi)d\chi$. By similarity we may transform (3.9) to an operator symmetric with respect to $d\chi$, containing thus no first-order derivative terms:

$$
\begin{aligned}
\widetilde{\mathbb{C}}^{\mathrm{sp}} &:= \Omega^{1/2}\mathbb{C}^{\mathrm{sp}_{(N)}(2,R)}\Omega^{-1/2}\big|_{k_{(n)},k_{(m)}}\\
&=\tfrac{1}{16}N(4-N)-\tfrac{1}{4}\left[\partial^2_\chi-\{(2k_{(n)}-1)^2-\tfrac{1}{4}\}\operatorname{sech}^2\chi\right.\\
&\quad\left.+\{(2k_{(m)}-1)^2-\tfrac{1}{4}\}\operatorname{csch}^2\chi-\tfrac{1}{4}(N-2)^2\right]. \qquad (3.10c)
\end{aligned}
$$

To obtain the usual $-\frac{1}{2}\partial^2_\chi+V(\chi)$ form of Schrödinger equations, we define

$$\mathbb{H}^{\mathrm{PT}}=2\widetilde{\mathbb{C}}^{\mathrm{sp}}-\tfrac{1}{2}=-\tfrac{1}{2}\partial^2_\chi+V^{cs}(\chi), \qquad (3.11a)$$

where $V^{cs}(\chi)$ is the scattering Pöschl–Teller potential (3.1) with parameters

$$c=-\tfrac{1}{2}[(2k_{(n)}-1)^2-\tfrac{1}{4}]<\tfrac{1}{8}, \qquad (3.11b)$$

$$s=\tfrac{1}{2}[(2k_{(m)}-1)^2-\tfrac{1}{4}]\geqslant-\tfrac{1}{8}, \qquad (3.11c)$$

and spectrum

$$E_k=2k_{(N)}(1-k_{(N)})-\tfrac{1}{2}=-\tfrac{1}{2}(2k_{(N)}-1)^2, \qquad (3.11d)$$

where the range of k in the Clebsch–Gordan series (3.5) and (3.6) yields negative-energy bound states for couplings to the discrete series: $k=k_{\min},k_{\min}-1,\ldots>\frac{1}{2}$, $k_{\min}=k_{(n)}-k_{(m)}$. The continuum of positive-energy scattering states appears for couplings to the continuous series for $k=\frac{1}{2}(1+i\kappa)$, $\kappa>0$. The multivaluation index is determined by $k_{(n)}-k_{(m)}$ mod 1; $\epsilon=0$ allows the $\kappa=0$ value, while $\epsilon=\frac{1}{2}$ excludes it since the representation belongs to the exceptional type and is not square-integrable.[29]

Some remarks about the allowed values of the Pöschl–Teller parameters c and s, and the proper spectrum of $\mathbb{H}^{\mathrm{PT}}$ follow.

The coefficient s in (3.11b) multiplies the $\operatorname{csch}^2\chi$ term of the potential, which is singular as $\sim\chi^{-2}$ at the origin. The coefficient s represents thus a *core* parameter. There may be three cases.

(a) The core may be a singular, negative *well* $(0>s\geqslant-\frac{1}{8})$ for $\frac{1}{4}<k_{(m)}<\frac{3}{4}$. Among the representations of $\mathrm{sp}_{(m)}(2,R)$ contained in the oscillator representation of sp(2N,R), only $k_{(m)}=\frac{1}{2}$ leads to a potential with an attractive well $s=-\frac{1}{8}$ at the origin. This is what we would call a *weak*[8] well for the purposes of investigating the conditions under which the Pöschl–Teller Hamiltonian has a unique spectrum.

(b) When $k_{(m)}=\frac{1}{4}$ or $\frac{3}{4}$, the core parameter s is zero.[24] These values of $k_{(m)}$ are allowed within sp(2N,R) [see (2.8b)] only when $l=0$, $m=1$, i.e., when the reduction of so$(n,1)$ is the canonical one to so(n), and (3.7) and (3.8) are spherical coordinates. For $n=2$, this is the chain $\mathrm{sp}(6,R)\supset \mathrm{sp}(2,R)\oplus \mathrm{so}(2,1)$, and the algebra so(2,1) is doubly at play. Using only so(2,1) with its Casimir operator and generators, thus, we cannot get a nonzero core parameter. We do get, however, the two parity values which allow the system to be extended to the full line $\chi\in R$.

(c) Finally, when $s>0$ we get true core potentials. We must note the interval $(0,\frac{1}{4})\cup(\frac{3}{4},1)$ for which $0<s<\frac{3}{8}$. It is in fact a *weak* core,[8] and the problem is having *two* values of $k_{(m)}$ leading to a single potential with fixed s; the interval, luckily, does not include values allowed by sp$(2N,R)$. For $k_{(m)}\geqslant 1$ the core is *strong*[33] ($\frac{3}{8}\leqslant s=\frac{3}{8},1,\frac{15}{8},3,\ldots$, for $k=1,\frac{5}{4},\frac{3}{2},\frac{7}{4},\ldots$) and the square-integrable solutions of the Schrödinger equation must be zero at the origin.

The second parameter c is factor to a $\mathrm{sech}^2\chi$ term in the Pöschl–Teller potential. It is a trough parameter for $c<0$, and a smooth bump for $0<c<\frac{1}{8}$. There are also three cases: (a) a $c=\frac{1}{8}$ bump for $k_{(n)}=\frac{1}{2}$, (b) a $c=0$ zero potential for $k_{(n)}=\frac{1}{4}$ and $\frac{3}{4}$ allowing the system to be extended to $\chi\in R$, and (c) troughs of coefficient $c=-\frac{3}{8},-1,-\frac{15}{8},-3,\ldots$ for $k_{(n)}=1,\frac{5}{4},\frac{3}{2},\frac{7}{4},\ldots$.

The core and trough parameters combine to a Pöschl–Teller potential with a trough when $0<-s/c<1$ (i.e., $k_{(n)}>k_{(m)}$) at the position $\chi_{\min}=\operatorname{arctanh}(-s/c)^{1/4}$ and of depth $V(\chi_{\min})=-(s-\sqrt{-c})^2$. This potential is able to hold bound states for $\frac{1}{2}<k_{\min}=k_{(n)}-k_{(m)}$ or $\sqrt{-2c+\frac{1}{4}}-\sqrt{2s+\frac{1}{4}}>1$, and the number of these states is the integer part of $k_{\min}$. The bound state energies are given by (3.11) for $k_{(N)}=k_{\min}, k_{\min}-1,\ldots>\frac{1}{2}$, each corresponding to a discrete-series term in the Clebsch–Gordan series. For all sp$(2N,R)$-allowed Pöschl–Teller potential parameters, there is a continuum extending over positive energies, and corresponding to the continuous-series representations in the integral in the Clebsch–Gordan series for $k=\frac{1}{2}(1+i\kappa),\kappa\in R^+$. It should finally be remarked again that for n even, $k_{(n)}$ may be fixed to be integer or half-integer, while for n odd, $k_{(n)}$ may only be a quarter-integer.

The Pöschl–Teller bound and free wave functions

$$\mathbb{H}^{\mathrm{PT}}\,\psi_k=E_k\,\psi_k \tag{3.12}$$

may be obtained directly, in normalized form, from the work of Basu and Wolf on sp$(2,R)$. They relate through[34]

$$C_>\begin{pmatrix} k_1 & k_2 & ; & k\\ +1,r & -1,r & ; & +1,\rho\end{pmatrix} =\delta(\tfrac{1}{2}r^2-\tfrac{1}{2}s^2-\tfrac{1}{2}\rho^2)(\rho^2\cosh\chi\sinh\chi)^{-1/2}\,\psi_k^{(k_1,k_2)}(\chi), \tag{3.13}$$

where $k_1=k_{(n)}$ and $k_2=k_{(m)}$ determine the c and s potential parameters, $-\frac{1}{2}(2k-1)^2$ is the energy [see (3.11), setting $k=k_{(N)}$], and $\tanh\chi=s/r[s/r<1$ on the $\sigma=+1$ chart] is the position in the Pöschl–Teller Schrödinger equation.

The bound states have the following wave functions normalized on $[0,\infty)$:

$$\psi_k^{(k_1,k_2)}(\chi)=c_k^{k_1k_2}(\cosh\chi)^{-2k_1+3/2}(\sinh\chi)^{2k_2-1/2}\,F\left[\begin{matrix}-k_1+k_2+k,-k_1+k_2-k+1\\ 2k_2\end{matrix};-\sinh^2\chi\right], \tag{3.14a}$$

$$c_k^{k_1k_2}=\frac{1}{\Gamma(2k_2)}\left[\frac{(2k-1)\Gamma(k_1+k_2-k)\Gamma(k_1+k_2+k-1)}{\Gamma(k_1-k_2+k)\Gamma(k_1-k_2-k+1)}\right]^{1/2}, \tag{3.14b}$$

where $F\left[{a,b \atop c};z\right]$ is the ${}_2F_1$ Gauss hypergeometric function. These also may be put in terms of Jacobi polynomials of degree k_1-k_2-k and argument $2\,\mathrm{sech}^2\chi-1$. The scattering states are described by the wave functions

$$\psi_k^{(k_1,k_2)}(\chi)=c_k^{k_1k_2}(\cosh\chi)^{2k_1-1/2}(\sinh\chi)^{2k_2-1/2}\times F\left[\begin{matrix}k_1+k_2-k,k_1+k_2+k-1\\ 2k_2\end{matrix};-\sinh^2\chi\right], \tag{3.15a}$$

$$c_k^{k_1k_2}=[1/\pi\Gamma(2k_2)]\,[\tfrac{1}{2}\kappa\sinh\pi\kappa\times\Gamma(k_1+k_2-k)\Gamma(-k_1+k_2+k)\times\Gamma(k_1+k_2+k-1)\Gamma(-k_1+k_2-k+1)]^{1/2}. \tag{3.15b}$$

We note the symmetry relations

$$\psi_k^{(1-k_1,k_2)}(\chi)=\psi_k^{(k_1,k_2)}(\chi),\quad \text{for } k_1\in R, \tag{3.15c}$$

$$\psi_k^{(k_1,1-k_2)}(\chi)=\psi_k^{(k_1,k_2)}(\chi),\quad \text{for } 2k_2 \text{ integer}, \tag{3.15d}$$

for the scattering states. This is an invariance transformation for the potential parameter c in (3.11b), obviously. Not so for s, however, as we shall see in the next section. In particular, from the coupling of two oscillator represention to the continuous-series representations[35] we obtain

$$\psi_k^{(3/4,3/4)}(\chi)=\psi_k^{(1/4,3/4)}(\chi)=\sqrt{2/\pi}\,\sin\kappa\chi, \tag{3.16a}$$

$$\psi_k^{(3/4,1/4)}(\chi)=\psi_k^{(1/4,1/4)}(\chi)=\sqrt{2/\pi}\,\cos\kappa\chi, \tag{3.16b}$$

normalized on $[0,\infty)$.

The asymptotic behavior of the scattering states under $\chi\to\infty$ corresponds to the oscillatory behavior of the Clebsch–Gordan coefficient (3.13) in the neighborhood of the cone $r=s$, where ρ becomes vanishingly small. Out of the tabulated asymptotic properties of the hypergeometric function, one may find, for $k=\frac{1}{2}(1+i\kappa)$,

$$\psi_k^{(k_1,k_2)}(\chi)\underset{\chi\to\infty}{\sim}\alpha_k^{k_1k_2}e^{i\kappa\chi}+\beta_k^{k_1k_2}e^{-i\kappa\chi}, \tag{3.17a}$$

$$\alpha_k^{k_1k_2}=\frac{2^{-i\kappa}}{\sqrt{2\pi}}\left[\frac{\Gamma(i\kappa)}{\Gamma(-i\kappa)}\times\frac{\Gamma(k_1+k_2-k)\Gamma(-k_1+k_2-k+1)}{\Gamma(k_1+k_2+k-1)\Gamma(-k_1+k_2+k)}\right]^{1/2}, \tag{3.17b}$$

$$\beta_k^{k_1k_2}=\alpha_{1-k}^{k_1k_2}=(\alpha_k^{k_1k_2})^*=\beta_k^{1-k_1,k_2}. \tag{3.17c}$$

In the next section we shall rederive these asymptotic coefficients out of pure Lie-algebraic considerations. In particular, we shall use them to find the reflection and transmission amplitudes of the pure trough potential $V^{c0}(\chi)$.

IV. so(2,2) SHIFT OPERATORS FOR THE SCATTERING STATES

In Sec. II we started with the parent algebra sp($2N,R$) in its oscillator representation (2.2) in order to provide an underlying N-particle phase space. There, the sp($2,R$) subalgebra in the chain (3.3) has for Casimir operator (in bipolar-hyperbolic coordinates) the Pöschl–Teller equation. The price we pay is to be able to account only for certain values for the potential coefficients c and s. In particular, we do *not* obtain (for $n,m > 1$) the null potential $V^{00}(\chi)$.

We may do away with this restriction in one important case:

$$\mathrm{sp}(8,R) \supset \mathrm{sp}(2,R) \oplus \mathrm{so}(2,2). \tag{4.1a}$$

There, the dimensional accident occurs that[2]

$$\mathrm{so}(2,2) = \mathrm{sp}_a(2,R) \oplus \mathrm{sp}_b(2,R). \tag{4.1b}$$

In fact, it also allows us to present the Pöschl–Teller equation as the *Klein–Gordon* equation[17] $(\Delta - \mu^2)\psi = 0$ with Δ being the Laplace–Beltrami operator on the three-dimensional surface of the (2,2) hyperboloid H^3 [(3.7) and (3.8] $\rho = \text{const}$, $\chi \in [0,\infty)$, θ, $\phi \in S^1$,

$$x_1 = \rho \cosh\chi \cos\theta, \quad x_3 = \rho \sinh\chi \cos\phi,$$

$$x_2 = \rho \cosh\chi \sin\theta, \quad x_4 = \rho \sinh\chi \sin\phi. \tag{4.2}$$

The eigenvalues μ may be interpreted as the masses allowed in such a model.

The so(2,2) algebra in the decomposition (4.1b) may be written explicitly in terms of the generators (2.2) as

$$\mathbb{K}_a^0 = \tfrac{1}{2}(\mathbb{M}_{12} + \mathbb{M}_{34}) = -(i/2)(\partial_\theta + \partial_\phi),$$

$$\mathbb{K}_b^0 = \tfrac{1}{2}(-\mathbb{M}_{12} + \mathbb{M}_{34}) = (i/2)(\partial_\theta - \partial_\phi), \tag{4.3a}$$

$$\mathbb{K}_a^1 = \tfrac{1}{2}(\mathbb{N}_{23} + \mathbb{N}_{14}), \quad \mathbb{K}_b^1 = \tfrac{1}{2}(-\mathbb{N}_{23} + \mathbb{N}_{14}), \tag{4.3b}$$

$$\mathbb{K}_a^2 = \tfrac{1}{2}(-\mathbb{N}_{13} + \mathbb{N}_{24}), \quad \mathbb{K}_b^2 = \tfrac{1}{2}(-\mathbb{N}_{13} - \mathbb{N}_{24}). \tag{4.3c}$$

We note that on H^3 the two Casimir operators of the two sp($2,R$)'s in (4.1b) are equal to each other and related to the sp($2,R$) Casimir in (4.1a) through

$$\mathbb{C}^{\mathrm{sp}_c} = (\mathbb{K}_c^1)^2 + (\mathbb{K}_c^2)^2 - (\mathbb{K}_c^0)^2$$

$$= -\tfrac{1}{4}\mathbb{C}^{\mathrm{so}(2,2)} = \mathbb{C}^{\mathrm{sp}(2,R)}, \quad c = a,b. \tag{4.4}$$

This means we have a "square" (k,k) representation of so(2,2) corresponding to the degenerate representation with Casimir eigenvalue $l(l+2)$, $k = \tfrac{1}{2}l + 1$, as before. The so(2,2) representation basis elements, classified through their eigenvalues M_a and M_b under $\mathbb{K}_a^0$ and $\mathbb{K}_b^0$, and k under $\mathbb{C}^{\mathrm{sp}_a} = \mathbb{C}^{\mathrm{sp}_b}$, may be written as functions over H^3 through

$$\varphi_k^{m_a m_b}(\theta,\phi,\chi) = e^{i(m_1\theta + m_2\phi)}\psi_k^{(k_1,k_2)}(\chi), \tag{4.5}$$

where m_1 and m_2 are the eigenvalues under $\mathbb{M}_{12}$ and $\mathbb{M}_{34}$,

$$m_a - m_b = m_1 = 2k_1 - 1, \quad m_a + m_b = m_2 = 2k_2 - 1 \tag{4.6}$$

(on H^3, m_1 and m_2 can be all and only integers), and $\psi_k^{(k_1,k_2)}(\chi)$ is the normalized Pöschl–Teller wave function (3.14) and (3.15) of energy $-\tfrac{1}{2}(2k-1)^2$.

The four so(2,2) shift operators are

$$\mathbb{K}_a^\updownarrow = \mathbb{K}_a^1 \pm i\mathbb{K}_a^2$$

$$= \tfrac{1}{2}e^{\pm i(\theta+\phi)}[\mp\partial_\chi + \tanh\chi(-i\partial_\theta \pm \tfrac{1}{2})$$

$$+ \coth\chi(-i\partial_\phi \pm \tfrac{1}{2})], \tag{4.7a}$$

$$\mathbb{K}_b^\updownarrow = \mathbb{K}_b^1 \pm i\mathbb{K}_b^2$$

$$= \tfrac{1}{2}e^{\pm i(-\theta+\phi)}[\mp\partial_\chi + \tanh\chi(i\partial_\theta \pm \tfrac{1}{2})$$

$$+ \coth\chi(-i\partial_\phi \pm \tfrac{1}{2})]. \tag{4.7b}$$

Shift operators, independently of their realization, have the well-known constants and action[6] on any normalized eigenbasis of $\mathbb{C}^{\mathrm{sp}_c}$ and $\mathbb{K}^0$ given by

$$\mathbb{K}^\updownarrow \varphi_k^m = [(k \pm m)(1 - k \pm m)]^{1/2}\varphi_k^{m\pm 1}. \tag{4.8}$$

The shifts produced by (4.7) on a fixed Pöschl-Teller potential generates the elements of its *multiplet*:

$$\mathbb{K}_a^\updownarrow : (m_1,m_2) \mapsto (m_1 \pm 1, m_2 \pm 1),$$

$$\mathbb{K}_b^\updownarrow : (m_1,m_2) \mapsto (m_1 \mp 1, m_2 \pm 1), \tag{4.9a}$$

$$\mathbb{K}_a^\updownarrow : (k_1,k_2) \mapsto (k_1 \pm \tfrac{1}{2}, k_2 \pm \tfrac{1}{2}),$$

$$\mathbb{K}_b^\updownarrow : (k_1,k_2) \mapsto (k_1 \mp \tfrac{1}{2}, k_2 \pm \tfrac{1}{2}). \tag{4.9b}$$

We noted in (3.15) that reflection in k_1 through $\tfrac{1}{2}$, $\psi_k^{(k_1,k_2)}(\chi) = \psi_k^{(1-k_1,k_2)}(\chi)$ holds for all k_1 due to hypergeometric function identities. This is as expected, since m_1 and $-m_1$ provide the same $\mathrm{sech}^2\chi$-potential coefficient c in (3.11b). When $2k_2$ is an integer (i.e., when m_2 is an integer, but *only* then) it holds that k_2 may be reflected through $\tfrac{1}{2}$, corresponding to the same $\mathrm{csch}^2\chi$-potential core coefficient s in (3.11c).

We thus arrive at the point of view of so(2,2) as the "dynamical potential group" (named in Ref. 11, Sec. 7) of the Pöschl–Teller system, where the irreducible representation basis elements are the wave functions, with the same energy, of *different potentials*. We may thus speak of the multiplet $\{m_1,m_2\}$ of potentials for a given energy level E_k.

For the bound states of a given energy $E_k = -\tfrac{1}{2}(2k-1) < 0$ allowed for $k = k_1 - k_2, k_1 - k_2 - 1, \ldots > \tfrac{1}{2}$, m_a and m_b range over $k, k+1, k+2, \ldots$. See Fig. 2(a), where the axes are drawn for m_1 and m_2. The set of dots constitute the so(2,2) multiplet; each dot in the first quadrant is associated to a given potential, the second quadrant being a reflection through the m_2 axis of the first. The $m_1 = 0$, $m_2 = 2k$ potential has a minimal (negative) core $s = -\tfrac{1}{8}$; shifting along the lattice boundary with $\mathbb{K}_a^\uparrow$ we increase both the repulsive core over positive values of s, and the trough parameter c to ever more negative values. The lowest allowed state in the first potential remains the lowest allowed one in all potentials of the "boundary" of the lattice in Fig. 2(a). Moving into the lattice we deepen the trough and thicken—to a lower degree—the core. Our energy E_k eigenstate will have more bound states below it, one for every nested layer we cross.

For the scattering states of a given energy $E_k = -\tfrac{1}{2}(2k-1)^2 > 0$, $k = \tfrac{1}{2}(1 + i\kappa)$, $\kappa \geqslant 0$, only the integer representations $\mathscr{C}^0_{k(1-k)}$ of the conjugate sp($2,R$) are allowed. The so(2,2) multiplet is shown in Fig. 2(b). Points in the first quadrant (including the axes) represent Pöschl–Teller potentials. The multiplet may be traversed on diagonals by means

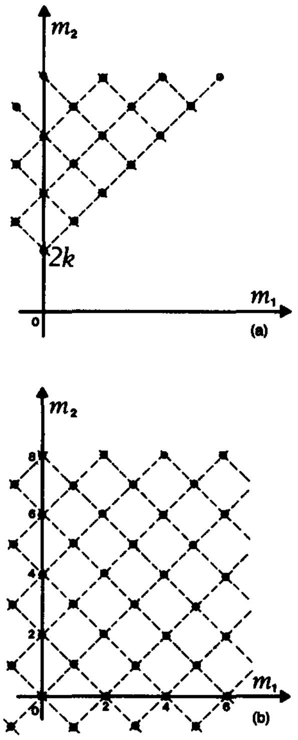

FIG. 2. (a) The so(2,2) multiplet of Pöschl–Teller potentials corresponding to a (bound) energy level $E_k < 0$. (b) The so(2,2) multiplet corresponding to a (free) energy level $E_k > 0$.

of the shift operators (4.7). Potentials along the m_2 axis ($m_1 = 0$) have a $c = \frac{1}{8}$ bump and all others a $c < 0$ trough, while those along the m_1 axis ($m_2 = 0$) have a $s = -\frac{1}{8}$ weak attractive core, and all others a true $s > 0$ core. Potentials lying above the diagonal in the first quadrant have a trough deep enough to hold bound states.

This is the situation on the (2,2) hyperboloid H^3. We would like to be able to include other Pöschl–Teller potentials with different values of the c and s parameters, in particular the null potential $V^{00}(\chi)$.

The coefficient c of the $\operatorname{sech}^2 \chi$ well, we saw, vanishes for $m_1 = \pm \frac{1}{2}$ ($k_1 = \frac{1}{4},\frac{3}{4}$) and the coefficient s of the $\operatorname{csch}^2 \chi$ core for $m_2 = \pm \frac{1}{2}$ ($k_2 = \frac{1}{4},\frac{3}{4}$). But note, only integer m's are allowed on H^3. This situation may be remedied for the well and managed for the barrier in the following way.

The three-dimensional hyperboloid space $H^3(\theta, \phi, \chi)$ projected on $\phi = \phi_0, \phi_0 + \pi$ is a two-dimensional space: a one-sheeted hyperboloid; the coordinate which circles it is θ. The $\theta = \theta_0, \theta_0 + \pi$ subspace, on the other hand, is two-sheeted hyperboloid circled by ϕ. We may *cover* the original H^3 hyperboloid n times in θ to a space of constant curvature (θ, ϕ, χ), where $\theta \in [0, 2\pi n)$, $\phi \in [0, 2\pi]$, $\chi \in [0, \infty)$, with the proper identifications, including $\theta \equiv \theta \bmod 2\pi n$. In the *double covered* three-hyperboloid $\overline{H^3}$, $\mathbf{M}_{12} = -i\,\partial_\theta$ may have integer as well as *half*-integer eigenvalues m_1. The zero-trough Pöschl–Teller system $m_1 = \pm\frac{1}{2}$ may be thus placed in the same multiplet with other potentials with half-integer m_1's and integer m_2's. Note that we may *not* do the same covering using ϕ, so the barrierless potential $V^{00}(\chi)$ *cannot* be realized on $\overline{H^3}$, nor partake in a multiplet belonging to a self-adjoint representation of so(2,2).

In Ref. 24 we took up the barrierless case through working with so(2,1) instead of so(2,2). In that case, only the one-sheeted two-hyperboloid space $H^2(\theta, \chi)$ is used, the algebra so(2) generated by $\mathbf{M}_{34}$ leaves an inversion "so(1)" group which in turn allows the unfolding of χ to the full line, where only coreless Pöschl–Teller potentials are allowed. The shift operators (4.7) become a single pair which lead to the scattering matrix, as detailed there.

In the full so(2,2) case followed here we note that up to this point we are consistent in having $+m_1$ and $-m_1$ half-integer (i.e., k_1 and $1 - k_1$ quarter-integer) describing the same potential constant $c = -\frac{1}{2}[m_1^2 - \frac{1}{4}]$ and the same eigenfunctions (3.15c). The question of allowing the core parameter s to vanish is more delicate. On a pedestrian level it would seem that one could "analytically continue"[11] the $\mathbf{M}_{34}$ eigenvalue m_2 to half-integer values, in spite of the fact that the hyperboloid coordinate ϕ allows *no* covering. In fact, we may do just that provided we realize that the resulting representation of the algebra so(2,2) will no longer be self-adjoint,[36] since, as we shall see below, the action of the shift operators does *not* leave the space of square-integrable wave functions invariant. Such representations of the algebra are *not* integrable to representations of the group. The results of Basu and Wolf[21] on sp(2,R) Clebsch–Gordan coefficients which provide the explicit eigenfunctions (3.14) and (3.15) continue to be valid for any real $k_2 > 0$ ($m_2 > -1$) since they were built out of the algebra, not the group.

The continuous-spectrum wave functions for the null Pöschl–Teller potential ($c = 0$, $s = 0$) may be found from (3.16a) and (3.16b) and are, as expected,

$$\varphi^{1/2,0}(\theta, \phi, \chi) = \sqrt{2/\pi}\, e^{i(\theta + \phi)/2} \sin \kappa\chi , \tag{4.10a}$$

$$\varphi_k^{-1/2,0}(\theta, \phi, \chi) = \sqrt{2/\pi}\, e^{i(-\theta + \varphi)/2} \cos \kappa\chi , \tag{4.10b}$$

and similar ones for $m_1 = -\frac{1}{2}$. Both (4.10a) and (4.10b) are solutions to the same, null potential $V^{00}(\chi)$ but they are obviously not the same. If χ were extended to the full real line, they would be the odd and even solutions of the free Schödinger equation for energy $\frac{1}{2}\kappa^2$. If we repeatedly apply $\mathbf{K}_a^\uparrow$ to $\varphi_k^{1/2,0}$ we find through (4.8) $\varphi_k^{3/2,0}$, $\varphi_k^{5/2,0}$, etc., which coincide with the functions on the hyperboloid built with $\psi_k^{(5/4,5/4)}$, $\psi_k^{(7/4,7/4)}$, etc. See Fig. 3. Applying $\mathbf{K}_a^\downarrow$ to $\varphi_k^{1/2,0}$ we obtain ($\kappa/2$ times) $\varphi_k^{-1/2,0}$. If we apply $\mathbf{K}_a^\downarrow$ to $\varphi_k^{-1/2,0}$, generating $\varphi_k^{-3/2,0}$, $\varphi_k^{-5/2,0}$,..., we find as expected that these functions are obtained out of $\psi_k^{(-1/4,-1/4)}$, $\psi_k^{(-3/4,-3/4)}$,.... The surprising

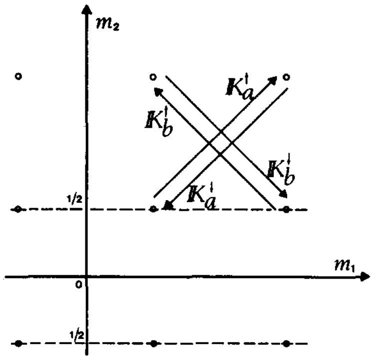

FIG. 3. The so(2,2) shift operators acting on a multiplet of Pöschl–Teller potentials (in a level of positive energy) which includes the null potential $V^{00}(\chi) = 0$ at $m_1 = \pm\frac{1}{2}$, $m_2 = \pm\frac{1}{2}$.

element in this construction is that while $\psi_k^{(k_1,k_1)}$ and $\psi_k^{(1-k_1,1-k_1)}$ are solutions to the *same* potential, they are the *two independent* solutions. Recall that (3.15b) holds only for integer $2k_2$. For $m_2 > \frac{1}{2}$, $\psi_k^{(k_1,k_2)}(\chi) \underset{\chi\to\infty}{\to} 0$ is the "good" solution, while its companion $m_2' = -m_2 < -\frac{1}{2}$, $\psi_k^{(k_1,1-k_2)}(\chi) \underset{\chi\to\infty}{\to} \infty$ is the "bad" solution to the same potential. For $m_2' < -1$ ($k_2<0$) it is not even square-integrable.

Bases for algebra representations built out of "good" and "bad" (i.e., non-square-integrable) functions are known in other contexts[36,37] in which the wave functions to raise and lower are, for instance, the Bessel functions $J_m(x)$.

We make use now of the so(2,2) shift operators in order to obtain algebraic relations between the asymptotic expansion coefficients $\alpha_k^{k_1k_2}$ and $\beta_k^{k_1k_2}$ in (3.17) for different values of k_1 and k_2, in particular, to relate them to those of (4.10). This will lead to the reflection and transmission coefficient of the coreless Pöschl–Teller potentials.

To this end we examine the asymptotic form of the shift operators (4.7) and recall that, as $\chi\to+\infty$, $\tanh\chi\to 1$ and $\coth\chi\to 1$. We define

$$\mathbb{K}_a^{\updownarrow(\infty)} := \lim_{\chi\to\infty} \mathbb{K}_a^{\updownarrow} = \tfrac{1}{2}e^{i(\theta+\phi)}[\mp\partial_\chi - i(\partial_\theta+\partial_\phi)\pm 1], \tag{4.11a}$$

$$\mathbb{K}_b^{\updownarrow(\infty)} := \lim_{\chi\to\infty} \mathbb{K}_b^{\updownarrow} = \tfrac{1}{2}e^{i(-\theta+\phi)}[\mp\partial_\chi - i(-\partial_\theta+\partial_\phi)\pm 1]. \tag{4.11b}$$

Next, we propose the asymptotic form of the so(2,2) basis wave functions (4.5) to be

$$\lim_{\chi\to\infty} \varphi_k^{m_am_b}(\theta,\phi,\chi) = e^{i(m_1\theta+m_2\phi)}(A_k^{m_am_b}e^{i\kappa\chi} + B_k^{m_am_b}e^{-i\kappa\chi}). \tag{4.12}$$

Now, consider (4.8) with the appropriate labels m_a or m_b, and its limit as $\chi\to\infty$. The left-hand side entails applying (4.11) to (4.12), while the right-hand side retains the square root factor and (4.12), with the replacement $m_a\mapsto m_a+1$ or $m_b\mapsto m_b+1$. This yields the following recursion relations between neighboring coefficients through the phase factor:

$$\begin{aligned} F(k,m) &= [F(1-k,m)]^{-1} = [(1-k+m)/(k+m)]^{1/2} \\ &= \exp[-\tfrac{1}{2}\arg\{k+m\}], \end{aligned} \tag{4.13}$$

viz.

$$\begin{aligned} A_k^{m_a+1,m_b} &= F(k,m_a)A_k^{m_am_b}, \\ B_k^{m_a+1,m_b} &= F(1-k,m_a)B_k^{m_am_b}, \end{aligned} \tag{4.14a}$$

$$\begin{aligned} A_k^{m_a,m_b+1} &= F(1-k,m_b)A_k^{m_am_b}, \\ B_k^{m_a,m_b+1} &= F(k,m_b)B_k^{m_am_b}, \end{aligned} \tag{4.14b}$$

Once the appropriate replacements are made, namely $m_a = k_1+k_2-1$, $m_b = -k_1+k_2$, it may be seen that the coefficients in (3.17b) and (3.17c) obey (4.14). Finally, the asymptotic coefficients in (3.17) are found when we start the recurrence (4.14) from the $V^{00}(\chi)$ eigenfunctions (4.10), where

$$\begin{aligned} A_k^{1/2,0} &= -i\sqrt{2\pi} = -B_k^{1/2,0}, \\ A_k^{0,-1/2} &= 1/\sqrt{2\pi} = B_k^{0,-1/2}. \end{aligned} \tag{4.15}$$

We may thus identify the α's and A's, and the β's and B's. In this way we derive the asymptotic behavior, the "good" as well as the "bad" solutions to the Pöschl–Teller potential using only algebraic techniques.

Scattering *through* a $\operatorname{csch}^2\chi$ core or well does not make sense since the $\chi>0$ and $\chi<0$ regions of the Schrödinger equation are uncoupled. We may remain with $\chi>0$, but we cannot in general impose an arbitrary asymptotic behavior on the wave functions so as to make them "incoming" or "outgoing" without receiving a linear combination of "good" and "bad" solutions. We can speak about scattering for the coreless class of potentials, however. These, we saw, lie at $m_2=\pm\frac{1}{2}$, and for them it is the full real line which makes sense in the scattering process. From (3.16) or (4.5)–(4.10) it is obvious that the basis functions $(m_1,m_2)=(\pm\frac{1}{2},\frac{1}{2})$ are odd in χ and $(\pm\frac{1}{2},-\frac{1}{2})$ are even. Use of the χ parity changing shift operators (4.7) shows that all $m_2=\frac{1}{2}$ ($k_2=\frac{3}{4}$) basis functions are odd, and all $m_2=-\frac{1}{2}$ ($k_2=\frac{1}{4}$) ones are even. Since we may cover the H^3 hyperboloid any number of times, this is also true for real m_1 (or k_1).

For a fixed, real trough parameter we may build the general wave function for the system with energy E_k as a linear combination:

$$\begin{aligned} &\psi_k^{(k_1)}(\chi) := \sigma_1\psi_k^{(k_1,1/4)}(\chi) + \sigma_3\psi_k^{(k_1,3/4)}(\chi), \\ &|\sigma_1|^2 + |\sigma_3|^2 = 1. \end{aligned} \tag{4.16}$$

The asymptotic behavior of this at $\chi\to\pm\infty$ is obtained from (3.17) and the parity of the two summands. Denoting $\alpha^1 := \alpha_k^{k_1,1/4}$, $\beta^1 := \beta_k^{k_1,1/4}$, $\alpha^3 := \alpha_k^{k_1,3/4}$, and $\beta^3 := \beta_k^{k_1,3/4}$, we have

$$\psi_k^{(k_1)}(\chi) \underset{\chi\to+\infty}{\sim} (\sigma_1\alpha^1+\sigma_3\alpha^3)e^{i\kappa\chi} + (\sigma_1\beta^1+\sigma_3\beta^3)e^{-i\kappa\chi}, \tag{4.17a}$$

$$\psi_k^{(k_1)}(\chi) \underset{\chi\to-\infty}{\sim} (\sigma_1\beta^1-\sigma_3\beta^3)e^{i\kappa\chi} + (\sigma_1\alpha^1-\sigma_3\alpha^3)e^{-i\kappa\chi}. \tag{4.17b}$$

Now, a *scattering* state for that potential and energy is a wavefunction which represents a flux of particles incoming far from the right ($ae^{-i\kappa\chi}$), part of the wave reflecting back ($be^{i\kappa\chi}$), and part of it transmitting towards the left ($ce^{i\kappa\chi}$), i.e.,

$$\psi_k^s(\chi) \underset{\chi\to+\infty}{\sim} be^{i\kappa\chi} + ae^{-i\kappa\chi}, \tag{4.18a}$$

$$\psi_k^s(\chi) \underset{\chi\to-\infty}{\sim} ce^{i\kappa\chi}. \tag{4.18b}$$

When this behavior at $\chi\to-\infty$ is imposed on the general solution (4.16), it implies $\sigma_1\beta^1=\sigma_3\beta^3$ and this in turn fixes the coefficients a, b, and c. The *transmission* T and *reflection* R amplitudes[38] are then found to be

$$T := \frac{c}{a} = \frac{1}{2}\left(\frac{\alpha^1}{(\alpha^1)^*} - \frac{\alpha^3}{(\alpha^3)^*}\right) = iC(k_1,\kappa)\sinh \pi\kappa, \quad (4.19a)$$

$$R := \frac{b}{a} = \frac{1}{2}\left(\frac{\alpha^1}{(\alpha^1)^*} + \frac{\alpha^3}{(\alpha^3)^*}\right) = -C(k_1,\kappa)\cos 2\pi k_1, \quad (4.19b)$$

where

$$C(k_1,\kappa) = \frac{1}{\pi}\frac{\Gamma(i\kappa)}{\Gamma(-i\kappa)}\Gamma\left(2k_1 - \frac{1}{2} - i\kappa\right) \times \Gamma\left(-2k_1 + \frac{3}{2} - i\kappa\right). \quad (4.19c)$$

These results agree—as expected—with those of Ref. 39, once we replace $k \mapsto \kappa$ and $j \mapsto m_1 - \frac{1}{2} = 2k_1 - \frac{3}{2}$. They are here obtained as a consequence of the form of the asymptotic coefficients (3.17).

V. CONCLUSION

We have worked with the $2N$-dimensional real symplectic algebra sp($2N,R$) so as to have an N-particle configuration space in its oscillator representation. This may be reduced with respect to its sp($2,R$) $\oplus$ so(n,m) subalgebras. Their conjugate Casimir operator is then the system's Hamiltonian with a Pöschl–Teller potential. Seen as a sp($2,R$) Casimir operator, the spectrum of this Hamiltonian becomes the Clebsch–Gordan series of sp($2,R$) which has a mixed spectrum. Seen as an so(n,m) Casimir operator, the Pöschl–Teller Schrödinger Hamiltonian becomes the Laplace–Beltrami operator in the so(n) $\oplus$ so(m) reduction, i.e., a Klein–Gordon equation on a space with constant curvature. In either case we worked specifically on so(2,2) which has already the essence of the properties on any more general so(n,m).

We devoted little space to mention the reduction sp($2N,R$) $\supset$ sp($2,R$) $\oplus$ so(N), where instead of a hyperboloid we have a sphere. This is the trigonometric Pöschl–Teller potential of the first type, which contains only bound states and no continuum. The constraining of the sphere—a point rotor—yields the familiar quadratically increasing eigenvalues (2.7) associated with angular momentum. In so(4), in particular, we have the rigid rotator system which belongs to the canonical reduction so(4) $\supset$ so(3) $\supset$ so(2), while the trigonometric Pöschl–Teller potential belongs to so(4) = so(3) $\oplus$ so(3) $\supset$ so(2) $\oplus$ so(2).

Beyond so(4), we have so(3,1), which is real, semisimple, and has not been treated explicitly here, but which can be shown to correspond to Hamiltonians built as Casimir operators with a spectrum given by the Clebsch–Gordan series $\mathscr{D} \dot{\times} \mathscr{C}$, which decompose[21,22,31,32] into an *infinite* set of quadratically decreasing values (for the full discrete series), plus a continuum of positive-energy "scattering" states. We did not pursue this line further since, as shown in the work of Basu and Wolf, the Hamiltonians—which are indeed of the Pöschl–Teller type but with a strong attractive core well—have the additional feature of being multichart operators [i.e., the index σ in (3.8) can no longer be fixed by a parabolic subalgebra representation] and both charts must be coupled properly.[21] Multichart operators with these kind of wells and non-lower-bound spectra do not make for attractive physical models. The *inverse* (reciprocal) of the spectrum, however, shifted by $\frac{1}{4}$, is the *full* spectrum of the hydrogen atom system, so it may well be that a good description of this system will lead to the $\mathscr{D} \times \mathscr{C}$ Clebsch–Gordan series and coefficients in the parabolic basis.

We worked here with the general case so(n,m) which leads to the $\mathscr{D}^+ \dot{\times} \mathscr{D}^-$ coupling to a mixed spectrum with a finite number of bound states. It must be mentioned that one may also effect the reduction so(2,2) $\supset$ so(1,1) $\oplus$ so(1,1), as a particular case for so(n,m) $\supset$ so(p,q) $\oplus$ so(r,s) ($p + r = n$, $q + s = m$) leading to the last sp($2,R$) representation coupling,[21,22,31,32] $\mathscr{C} \times \mathscr{C}$ which has the same reduction as $\mathscr{D} \times \mathscr{C}$, doubled by parity, and possibly containing *one* exceptional representation. Unfortunately, the resulting Pöschl–Teller Hamiltonian is a three-chart operator, and no attractive physical interpretation can be attached to it. Beyond these reductions, one has the nonsubgroup reductions of Winternitz and collaborators[40] which probably lead to the periodic potentials studied in Ref. 11. We intend to pursue their inclusion in this scheme to provide a unified sp($2,R$)-based description of mixed and other spectra. In any case, we hope to have made the point that Pöschl–Teller systems are quite general with a clear-cut geometric interpretation.

ACKNOWLEDGMENTS

We would like to thank Y. Alhassid, O. Castaños, J. Lomnitz, M. Moshinsky, J. Plebański, and L. F. Urrutia for their interest and helpful comments.

This work was supported in part by CONACyT project PCCBCEU-020061.

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