Lie Algebras for Potential Scattering

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We consider a new Lie-algebraic framework for the Pöschl-Teller potential, in which a single [sp(2,R) Casimir] operator has both bound and scattering states. This approach allows the determination of the S matrix by purely algebraic means.

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Symmetry methods have been successfully applied in a wide variety of systems in physics. Symmetry and dynamical algebras (and their groups) have proved useful tools in the analysis of bound-state problems, ranging from exact solutions, as in the Coulomb[1] and harmonic oscillator[2] potentials, to algebraic models of physical systems, such as the interacting-boson-model description of collective states in nuclei.[3] In contrast, little has been done in the application of algebraic methods to *scattering,* with the exception of the Coulomb system, where the bound and scattering states have been analyzed separately in terms of the groups SO(4) and SO(3,1), respectively. In that case, one may also imbed the two into a SO(4,2) dynamical group.[4]

In a recent series of papers,[5–7] Alhassid, Gürsey, and Iachello have shown that bound and scattering states of certain one-dimensional potentials can be related to unitary representations of a certain group and its analytic continuation to a noncompact group, respectively.[5–7] These ideas are illustrated on the Morse and Pöschl-Teller potentials, where the reflection and transmission coefficients (or the S matrix) are evaluated by obtaining the explicit wave functions and studying their asymptotic behavior. In these cases the relevant group chains are $U(2)\supset O(2)$ for bound states and $U(1,1)\supset O(1,1)$ for scattering states. This procedure seems to provide a fruitful pathway for the study of continuous spectra as the analytic continuation of bounded spectra.

We deem it desirable, however, to study a different formulation for systems with mixed spectra. The objective of this Letter is to present one such alternative applied to the Pöschl-Teller potential,[8] which emerges in connection with diverse physical systems, such as completely integrable many-body systems in one dimension,[9] the soliton solutions to the Korteweg–de Vries equation,[10] and in the Hartree mean-field equation of many-body systems interacting through a δ force[11] among others. This approach has two attractive features: (i) *Both* the bound and the scattering states belong to representations of the *same* group and (ii) the S matrix can be determined by purely *Lie-algebraic* manipulations, with no need for the explicit form of the wave functions.

We start our discussion by considering an sp(2,R) [= so(2,1)][4] algebra generated by operators J_x, J_y, and J_z, satisfying the well-known commutation relations[4]

$$[J_x,J_y]=-iJ_z,\quad [J_z,J_x]=iJ_y,\quad [J_y,J_z]=iJ_x, \tag{1}$$

with Casimir invariant $J^2=J_z^2-J_x^2-J_y^2$. This algebra has the realization

$$J_x=-i(x\,\partial_y+y\,\partial_x),\quad J_y=i(x\,\partial_z+z\,\partial_x),$$
$$J_z=-i(x\,\partial_y-y\,\partial_x). \tag{2}$$

With the introduction of polar hyperbolic coordinates $x=r\cosh\rho\cos\theta$, $y=r\cosh\rho\sin\theta$, $z=r\sinh\rho$, and the similarity transformation by $\Omega^{1/2}$, where $\Omega=\cosh\rho$ is the weight function in the hyperboloid measure on $(r=\text{const},\rho,\theta)$, the Casimir invariant and J_z take form

$$J^2=-\partial_\rho^2-\operatorname{sech}^2\rho(-\partial_\theta^2-\tfrac{1}{4})+\tfrac{1}{4},$$
$$J_z=-i\,\partial_\theta. \tag{3}$$

At this point we use the twofold cover of the hyperboloid, allowing θ to range over $[0,4\pi)$ and thus J_z to have half-integer as well as integer eigenvalues.

We now build the simultaneous, normalized eigenfunctions $|km\rangle$ of J^2 and J_z,[4] classified by their eigenvalues:

$$J^2|km\rangle = k(1-k)|km\rangle, \qquad J_z|km\rangle = m|km\rangle. \tag{4}$$

The unitary nonexceptional representations[12] of sp(2,R) may be divided into a continuous series C and a discrete series D. The Bargmann index[12] k and the so(2) content m follow:

$$C:\quad k=\tfrac{1}{2}+i\kappa,\quad \begin{cases} \text{in } \mathscr{C}^{0}_{1/4-\kappa^2}: & \kappa \geqslant 0, \quad m=0,\pm1,\pm2,\dots, \\ \text{in } \mathscr{C}^{1/2}_{1/4-\kappa^2}: & \kappa > 0, \quad m=\pm\tfrac{1}{2},\pm\tfrac{3}{2},\dots, \end{cases} \tag{5a}$$

$$D:\quad k=\tfrac{1}{2},1,\tfrac{3}{2},2,\dots,\quad \begin{cases} \text{in } D_k^+: & m=k,k+1,k+2,\dots, \\ \text{in } D_k^-: & m=-k,-k-1,-k-2,\dots. \end{cases} \tag{5b}$$

The Casimir operator has thus a *mixed* spectrum which, as we show below, corresponds to the full spectrum of the Pöschl-Teller (PT) potential. The wave functions in (4) have the form

$$|km\rangle = u_m^k(\rho)e^{im\theta}, \tag{6}$$

where $u_m^k(\rho)$ satisfies the equation

$$[-\partial_\rho^2 - \operatorname{sech}^2\rho\,(m^2-\tfrac{1}{4})]u_m^k(\rho) = -(k-\tfrac{1}{2})^2 u_m^k(\rho), \tag{7}$$

which is a PT Schrödinger equation where the depth of the well is given by the so(2) eigenvalue m. The *group* SO(2,1) [twice covered by Sp(2,R)] generated by the above algebra was called "the potential group" in Ref. 6, but is properties were not examined further.

We see that (5b) together with (7) correctly describe the bound energy spectrum of the PT potential,[6] since for a fixed m (i.e., for fixed potential) they imply

$$E_k = -(k-\tfrac{1}{2})^2, \quad k=\tfrac{1}{2},1,\tfrac{3}{2},\dots,|m|. \tag{8}$$

Our main interest in this Letter, however, is to consider Eq. (7) for the C representation series. Since now $k=\frac{1}{2}+i\kappa$, with m being either integer (in $\mathscr{C}^0$) or half-integer (in $\mathscr{C}^{1/2}$), Eq. (7) now reads

$$[-\partial_\rho - \operatorname{sech}^2\rho\,(m^2-\tfrac{1}{4})]u_m^k(\rho) = \kappa^2 u_m^k(\rho). \tag{9}$$

This is the scattering PT equation. To obtain the transmission and reflection coefficients or, equivalently, the S matrix, we proceed as follows. Using (2) in the covered hyperbolic coordinates, we define the raising and lowering operators

$$J_\pm = (iJ_x \mp J_y) = ie^{\pm i\theta}[\mp\partial_\rho + \tanh\rho\,(\pm\tfrac{1}{2} - i\,\partial_\theta)]; \tag{10}$$

$$J_\pm|km\rangle = [(\tfrac{1}{2}\pm m - i\kappa)(-\tfrac{1}{2}\mp m - i\kappa)]^{1/2}|k\ m\pm1\rangle. \tag{11}$$

Noting that $\lim_{\rho\to\pm\infty}\tanh\rho = \pm1$, we define the asymptotic operator

$$J_+^{(\pm\infty)} = \lim_{\rho\to\pm\infty} J_\pm = ie^{i\theta}[-\partial_\rho \pm \tfrac{1}{2} \mp i\,\partial_\theta]. \tag{12}$$

(An operator $J_-^{(\pm\infty)}$ may be defined similarly.) We write the sp(2,R) asymptotic basis functions as

$$|\psi^-\rangle = \lim_{\rho\to-\infty}|km\rangle = a_m e^{im\theta}e^{i\kappa\rho} + c_m e^{im\theta}e^{-i\kappa\rho}, \tag{13a}$$

$$|\psi^+\rangle = \lim_{\rho\to+\infty}|km\rangle = b_m e^{im\theta}e^{i\kappa\rho}. \tag{13b}$$

These are related to the PT asymptotic wave functions through

$$u^\pm = \lim_{\rho\to\pm\infty} u_m^k(\rho) = e^{-im\theta}|\psi^\pm\rangle. \tag{14}$$

Now, the limits of (12) and (13) yield

$$\lim_{\rho\to\pm\infty}(J_+|km\rangle)=J_+^{(\pm\infty)}|\psi^{\pm}\rangle, \tag{15}$$

which lead us directly to the following recurrence relations obtained from (11):

$$a_{m+1}=i\left[\frac{-\frac{1}{2}-m-i\kappa}{\frac{1}{2}+m-i\kappa}\right]^{1/2}a_m, \tag{16a}$$

$$b_{m+1}=i\left[\frac{\frac{1}{2}+m-i\kappa}{-\frac{1}{2}-m-i\kappa}\right]^{1/2}b_m, \tag{16b}$$

$$c_{m+1}=-i\left[\frac{\frac{1}{2}+m-i\kappa}{-\frac{1}{2}-m-i\kappa}\right]^{1/2}c_m. \tag{16c}$$

Finally, since the PT potential with $m=\frac{1}{2}$ in (7) corresponds to the *free* wave system, we see that $a_{1/2}=b_{1/2}$, $c_{1/2}=0$. Equations (16) can be solved to give the transmission and reflection coefficients,

$$T_m=\frac{b_m}{a_m}=\frac{\Gamma(\frac{1}{2}+m-i\kappa)\Gamma(\frac{1}{2}-m-i\kappa)}{\Gamma(1-i\kappa)\Gamma(-i\kappa)}, \tag{17a}$$

$$R_m=\frac{c_m}{a_m}=0, \tag{17b}$$

for m half-integer. These expressions have been obtained with use of only the Lie-algebraic properties of sp$(2,R)$. Once T_m is known for all half-integer values of m we can analytically continue (17a) to real m. This can be readily justified.[13] It then follows from the unitarity and symmetry of the S matrix,[14] which in this case is

$$S_m=\begin{pmatrix}R_m & T_m\\ T_m & R_m\end{pmatrix},$$

that

$$R_m=\frac{\Gamma(i\kappa)\Gamma(\frac{1}{2}-m-i\kappa)\Gamma(\frac{1}{2}+m-i\kappa)}{\Gamma(-i\kappa)\Gamma(\frac{1}{2}-m)\Gamma(\frac{1}{2}+m)}, \tag{17c}$$

for real m. For half-integer m we recover of course (17b).

It should be emphasized that the new viewpoint presented here is the association of the sp$(2,R)$ *Casimir* operator of an algebra with the Schrödinger PT Hamiltonian. We quote here the work of Basu and Wolf,[15] which identifies the spectrum of this system with the Clebsch-Gordan *series* of a certain class of sp$(2,R)$ couplings and the wave functions with Clebsch-Gordan *coefficients.* We have dealt with sp$(2,R)$ since an N-particle sp$(2N,R)$ formulation of the most general PT system is yet to be completed.[13] This should be relevant for N-body systems describing scattering processes in atomic and nuclear collisions.

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[1]See, e.g., M. J. Engelfield, *Group Theory and the Coulomb Problem* (Wiley, New York, 1972).

[2]See, e.g., M. Moshinsky, *Group Theory and the Many-Body Problem* (Gordon and Breach, New York, 1968).

[3]A. Arima and F. Iachello, Ann. Phys. (N.Y.) **99**, 253 (1976), and **111**, 201 (1977), and **123**, 469 (1979).

[4]B. G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).

[5]Y. Alhassid, F. Gürsey, and F. Iachello, Phys. Rev. Lett. **50**, 873 (1983).

[6]Y. Alhassid, F. Gürsey, and F. Iachello, Ann. Phys. (N.Y.) **148**, 346 (1983).

[7]Y. Alhassid, in Proceedings of the Seventh Oaxtepec Symposium on Nuclear Physics, 1984 (Instituto de Física, Universidad Nacional Autónoma de México, to be published).

[8]G. Pöschl and E. Teller, Z. Phys. **83**, 143 (1933).

[9]F. Calogero, Lett. Nuovo Cimento **13**, 411 (1975); M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **94**, 313 (1983).

[10]P. Lax, Commun. Pure Appl. Math. **27**, 97 (1968).

[11]F. Calogero and A. Degasperis, Phys. Rev. A **11**, 265 (1975); B. Yoon and J. W. Negele, Phys. Rev. A **16**, 1451 (1977).

[12]V. Bargmann, Ann. Math. **48**, 568 (1947).

[13]A. Frank and K. B. Wolf, to be published.

[14]See, e.g., E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970).

[15]D. Basu and K. B. Wolf, Czech. J. Phys. **B23**, 584 (1982), and Physica (Utrecht) **144**A, 472 (1982), and J. Math. Phys. (N.Y.) **24**, 478 (1983).