Quantization, symmetry, and natural polarization

V. Aldaya, J. A. de Azcárraga, and K. B. Wolf[a)]
Departamento de Física Teórica, Facultad de C. Físicas, Burjasot (Valencia), Spain

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We discuss the notion of polarization, as defined in a geometric quantization scheme recently introduced, in terms of the role played by the evolution operator of the quantum system. The analysis uses an integral transform representation of the group WSp(2,ℝ). This clarifies the group theoretic origin of the natural polarizations and the meaning of the polarization changing transformations.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

A method of geometric quantization has been introduced very recently[1], based on the definition of a group $\widetilde{G}$, the quantum group, from which all the essential ingredients of the theory are derived.[2] In the terminology of the usual quantization scheme[4] this means that given $\widetilde{G}$ there exists a procedure to construct the quantum manifold and the basic quantum operators (which prequantizes the system) and that a full quantization may be achieved by defining the appropriate polarization. This method uses the definition of the canonical left 1-form on $\widetilde{G}$, and in particular of its vertical component Θ {the "verticality" being defined by the fact that $\widetilde{G}$ [U(1), $\widetilde{G}$/U(1)$\equiv G$] is a principal bundle of structure group U(1)} as well as of the right and left invariant vector fields on $\widetilde{G}$ in order to define quantum operators and polarizations, respectively. In fact, the use of $\widetilde{G}$ as the starting point allows us to work directly on the evolution space, making it unnecessary to base the theory on the usual (contact) quantum manifold which, nevertheless may be obtained from $\widetilde{G}$ if one so wishes,[1] the contact 1-form being derived from Θ.

The so-called quantum group $\widetilde{G}$ is defined as a central extension by U(1) of a group $G_{(k)}$ (dynamical group[5] of the system under study) which contracts to the usual Galilei group for the case of the free system. If the problem under consideration is that of an interacting particle, the constant k in $G_{(k)}$ is a constant which switches off the potential in the limit $k \to 0$. Once $\widetilde{G}$ has been determined, the quantization may be performed by means of the following steps[1]:

(a) derivation of the left and the right invariant vector fields (LIVF and RIVF);

(b) construction of the canonical left 1-form;

(c) definition of the basic quantum operators (by means of RIVF);

(d) definition of the polarization (by means of LIVF).

In the usual approach to geometric quantization, the last step—that of defining a suitable polarization—is the least precise one. In the approach based on the quantum group $\widetilde{G}$, a polarization may be defined as a subspace of $\mathfrak{X}^L(\widetilde{G})$—the space of L-vector fields on $\widetilde{G}$—which contains $\mathscr{C}_\Theta$—the characteristic module of Θ[6]—which is projected onto a subalgebra of the Lie algebra of $\mathfrak{X}^L(G)$. It is the purpose of this paper to explore further the proposed definition to exhibit how it leads to the definition of a natural polarization associated with the system (and accordingly, with the corresponding quantum group) under consideration.

This will be done by considering the following four *one-dimensional* systems: the free particle, the free fall, the harmonic oscillator, and the repulsive "oscillator." The potentials which correspond to these situations are the representative of four orbits of the wsp(2,ℝ) algebra under the adjoint action of the corresponding group (Sec. 2). These are *all* orbits which include sp(2,ℝ) elements (containing the kinetic energy, as we shall see in Sec. 2). Thus, the group WSp(2,ℝ) includes the quantum dynamical groups of the above one-dimensional systems and will be used as the starting point for their study. To this aim, Sec. 2 will be devoted to describe the WSp(2,ℝ) group, its algebra, and the group of integral transform associated with it. Section 3 will present the simplest case of the free particle and the derivation of its Schrödinger equation. It will be found that the natural polarization leads to the momentum space Schrödinger equation, and that configuration space is obtained by means of an automorphism in Sp(2,ℝ) (the Fourier-integral transform). The essentially similar case of the free fall will be considered in Sec. 4. Sections 5 and 6 will be devoted to the harmonic oscillator and the repulsive "oscillator" (which is obtained through a change of sign in the harmonic oscillator potential). In all cases the vector fields of $\mathfrak{X}^L(\widetilde{G})$ defining the polarizations will be found to define maximal invariant subgroups in the corresponding "classical groups" $G = \widetilde{G}/\mathrm{U}(1)$ in accordance with the given definition. Moreover, the condition that the polarization vector fields must include $\mathscr{C}_\Theta$ will turn out to be very helpful in finding the polarization. Indeed, $\mathscr{C}_\Theta$ characterizes the time evolution of the system, and so the determination of the polarization will be associated with the problem of diagonalizing the time evolution (Hamiltonian) part in $\widetilde{G}$. This will immediately yield the suitable polarization for each case. In particular, the polarization which corresponds to the harmonic oscillator case will be found to be the one which leads to the Bargmann–Fock–Segal picture, and the "rotation" which leads to the usual harmonic oscillator eigenfunctions will be seen to be the Bargmann transform.

2. THE WSp(2,ℝ) GROUP

This group is the semidirect product of the two-dimensional real symplectic group Sp(2,ℝ) [$\approx$SU(1,1)$\approx$SL(2,ℝ)]

[a)]Permanent address: IIMAS, Universidad Nacional Autónoma de México, Apdo. Postal 20-726, 01000, México D.F., Mexico.

by the Weyl group W as a normal factor. We shall present this group in the context of a particularly convenient basis. Consider the three abstract operators $\mathbb{Q}$, $\mathbb{P}$, $\mathbb{I}$, as a basis for a Lie algebra with brackets

$$[\mathbb{Q},\mathbb{P}] = i\mathbb{I}, \quad [\mathbb{Q},\mathbb{I}] = 0, \quad [\mathbb{P},\mathbb{I}] = 0, \tag{2.1}$$

which is the well-known Heisenberg–Weyl algebra w.[7] Consider now the universal enveloping algebra $\bar{w}$ as a Lie algebra induced by (2.1) through the Leibnitz rule and in particular the finite-dimensional subalgebra whose basis is the set of the three independent second-order generators

$$\mathbb{P}^2, \quad \{\mathbb{Q},\mathbb{P}\}_+ \equiv \mathbb{Q}\mathbb{P} + \mathbb{P}\mathbb{Q}, \quad \mathbb{Q}^2 \tag{2.2}$$

in the representation where $\mathbb{I}$ is the unit operator. The operators (2.2) constitute a basis for the metaplectic representation[8] of the symplectic algebra sp(2,$\mathbb{R}$). When we sum (2.1) and (2.2) as vector spaces, we naturally obtain the six-dimensional algebra wsp(2,$\mathbb{R}$), where w is a three-dimensional ideal. The exponential mapping of this Lie algebra yields the WSp(2,$\mathbb{R}$) group together with a local parametrization of its manifold[9]

$$\exp(i[\alpha\mathbb{P}^2 + \beta\{\mathbb{Q},\mathbb{P}\}_+ + \gamma\mathbb{Q}^2 + \delta\mathbb{Q} + \epsilon\mathbb{P} + \theta\mathbb{I}])$$
$$\equiv g\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \ (p,q), \ z\right\}, \tag{2.3a}$$

where $ad - bc = 1$ and

$$\begin{aligned}
&a = \cos 2s - \beta s^{-1}\sin 2s, \quad s = \pm\sqrt{(\alpha\gamma - \beta^2)},\\
&b = -\alpha s^{-1}\sin 2s,\\
&c = \gamma s^{-1}\sin 2s,\\
&d = \cos 2s + \beta s^{-1}\sin 2s,\\
&p = \tfrac{1}{2}(\gamma\epsilon - \beta\delta)s^{-2}(1-\cos 2s) + \tfrac{1}{2}\delta s^{-1}\sin 2s,\\
&q = \tfrac{1}{2}(\beta\epsilon - \alpha\delta)s^{-2}(1-\cos 2s) + \tfrac{1}{2}\epsilon s^{-1}\sin 2s,\\
&z = \theta - \tfrac{1}{4}(\alpha\delta^2 + \gamma\epsilon^2 - 2\beta\delta\epsilon)s^{-2}(1 - \tfrac{1}{2}s^{-1}\sin 2s).
\end{aligned} \tag{2.3b}$$

The group composition law, $*$, may be established to be

$$g'\{\mathbf{M}',\mathbf{u}',z'\} * g\{\mathbf{M},\mathbf{u},z\}$$
$$= g''\{\mathbf{M}'\mathbf{M}, \mathbf{u}'\mathbf{M} + \mathbf{u}, \ z' + z + \tfrac{1}{2}\mathbf{u}'\mathbf{M}\Omega\mathbf{u}^T\}, \tag{2.4a}$$

where

$$\mathbf{M} \equiv \begin{pmatrix} a & b \\ c & d\end{pmatrix}, \quad \mathbf{u} \equiv (p,q), \quad \mathbf{u}^T \equiv \begin{pmatrix} p \\ q\end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}. \tag{2.4b}$$

The unit element is given by $e = g\{\mathbb{I},\mathbf{0},0\}$ and the inverse is $[g\{\mathbf{M},\mathbf{u},z\}]^{-1} = g\{\mathbf{M}^{-1}, -\mathbf{u}\mathbf{M}^{-1}, -z\}$. The adjoint action of the group on the algebra may be found, and it can be ascertained to consist of six distinct orbits.[10] Representative of these orbits are $\mathbb{P}^2$, $\mathbb{P}^2 + \mathbb{Q}$, $\mathbb{P}^2 + \mathbb{Q}^2$, $\mathbb{P}^2 - \mathbb{Q}^2$, $\mathbb{P}$, and $\mathbb{I}$, the first four of which are germane to the present work since they include a $\mathbb{P}^2$ term. The algebra wsp(2,$\mathbb{R}$) is thus the common dynamical algebra for the free particle, the linear potential (or free fall, $\mathbb{P}^2 + \mathbb{Q}$), the harmonic ($\mathbb{P}^2 + \mathbb{Q}^2$) and the repulsive ($\mathbb{P}^2 - \mathbb{Q}^2$) oscillators. Moreover, this six-dimensional algebra is the largest finite algebra with a semisimple factor within $\bar{w}$.[11]

Let us now turn to the generalized representation basis where $\mathbb{Q}$ is diagonal, i.e., $\mathbb{Q}f(x) = xf(x)$. In this basis, the Weyl group acts as a Lie transformation group through

$$g{:}f(x) \mapsto [g\{\mathbb{I},(p,q),z\}f](x)$$
$$= \exp[i(z + \tfrac{1}{2}pq + px)]f(x+q). \tag{2.5a}$$

The Sp(2,$\mathbb{R}$) group is generated by *up-to-second*-order operators in these generators, and its action is that of a group of integral transforms[12]:

$$g{:}f(x) \mapsto \left[g\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix},0,0\right\}f\right](x) = \int_{\mathbb{R}} dx'\, C_{\mathbf{M}}(x,x')f(x') \tag{2.5b}$$

with kernel[13]

$$C_{\mathbf{M}}(x,x') = (2\pi b)^{-1/2}e^{-i\pi/4}\exp[i(ax'^2 - 2xx' + dx^2)/2b] \tag{2.5c}$$

which is unitary in $\mathscr{L}^2(\mathbb{R})$. In the two-parameter subgroup of Sp(2,$\mathbb{R}$) of lower triangular matrices [those with $b = 0$, generated by all algebra elements with no $\mathbb{P}^2$ summand in (2.3a), $\alpha = 0$] the integral transform (2.5b) collapses to a Lie transformation group action,

$$g{:}f(x) \mapsto \left[g\left\{\begin{pmatrix} a & 0 \\ b & a^{-1}\end{pmatrix},0,0\right\}f\right](x)$$
$$= a^{-1/2}\exp[(icx^2/2a)]f(x/a). \tag{2.6}$$

The full WSp(2,$\mathbb{R}$) group action may be obtained through the composition of (2.5a) and (2.5b). The composition of integral transforms (2.5) follows the group composition property (2.4) modulo a sign[12] [it is a faithful representation of the *two*-fold covering of Sp(2,$\mathbb{R}$), called the metaplectic group Mp(2,$\mathbb{R}$)]. Thus, there exists a local isomorphism between WSp(2,$\mathbb{R}$), the hiperdifferential operators (2.3a), and the integral transforms (2.5). In the limit $\mathbf{M}\to\mathbb{I}$ (or $b\to 0$, with $\arg b \in [-\pi,0]$), the integral kernel (2.5c) has as weak limit a Dirac δ, as can be inferred from (2.6). Finally, the inverse of the integral transform (2.5) associated with a matrix $\mathbf{M}$ is that corresponding to $\mathbf{M}^{-1}$; it has a kernel which is the complex conjugate of the original one.

One can subject the WSp(2,$\mathbb{R}$) integral transform action to analytic continuation in the complex parameter plane, with certain restrictions. In order that the domain remain $\mathscr{L}^2(\mathbb{R})$, one needs that $\mathrm{Im}\, b^*a \geqslant 0$ and, when $a = 0$, then $\mathrm{Im}\, b = 0$. [This defines a subsemigroup of Sp(2,$\mathbb{C}$), called HSp(2,$\mathbb{C}$).] The elements of the complex-extended integral transform semigroup can be made to have the *unitarity* property when we consider them as mappings between $\mathscr{L}^2(\mathbb{R})$ and Bargmann–Fock–Segal–type Hilbert spaces of analytic functions[14] $\mathscr{B}_{\mathbf{M}}$ whose defining inner product is on the complex plane:

$$(f,g)_{\mathbf{M}} = \int_C d\mu_{\mathbf{M}}(\xi,\xi^*)f^*(\xi)g(\xi), \tag{2.7a}$$

$$d\mu_{\mathbf{M}}(\xi,\xi^*) = 2(2\pi\nu)^{-1/2}\exp[(v\xi^2 - 2\xi\xi^* + v^*\xi^{*2})/2\nu]\,d\,\mathrm{Re}\,\xi\,d\,\mathrm{Im}\,\xi, \tag{2.7b}$$

$$v = a^*d - b^*c, \quad \nu = 2\,\mathrm{Im}\, b^*a. \tag{2.7c}$$

The transform kernel inverse to that corresponding to $\mathbf{M}$ [(2.7b)] is obtained by putting $\mathbf{M}^{-1}$ in (2.7a), and thus maps $\mathscr{B}_{\mathbf{M}}$ back to $\mathscr{L}^2(\mathbb{R})$.

In essence, we use HSp(2,$\mathbb{C}$) in order to act through similarity on the relevant ("Galilei"-type) quantum group $\widetilde{G}$, and correspondingly on its generators (adjoint action of the group on a subalgebra). This is required to perform the task of diagonalizing the Hamiltonian-generated time evolution group [see (2.3a)]. If this matrix cannot be diagonalized—the first two cases to be considered—then it should at least reduce to a triangular matrix having an invariant subspace.

In terms of generators, diagonalizing the Hamiltonian under consideration means rotating it onto the change-of-scale operator $\frac{1}{2}\{\mathbb{P},\mathbb{Q}\}_+ = -i(x\partial/\partial x + \frac{1}{2})$. This may be done using Sp(2,$\mathbb{R}$) transformations in the repulsive oscillator case, where $\frac{1}{2}(\mathbb{P}^2 - \mathbb{Q}^2)$ and $\frac{1}{2}\{\mathbb{P},\mathbb{Q}\}_+$ belong to the same orbit under Sp(2,$\mathbb{R}$). In the harmonic oscillator case, $\frac{1}{2}(\mathbb{P}^2 + \mathbb{Q}^2)$ and $\frac{1}{2}\{\mathbb{P},\mathbb{Q}\}_+$ are in different orbits under Sp(2,$\mathbb{R}$), but on the same orbit under HSp(2,$\mathbb{C}$). They are mapped onto each other by the Bargmann transform; this will change the structure of (i.e., the inner product defining) the Hilbert space.

Having defined this operational machinery, we now turn to the study of polarizations for the approach described in the previous section and for the four one-dimensional systems mentioned above.

3. THE CASE OF THE FREE PARTICLE

This is the simplest case; since there is no interaction, $k = 0$ (Sec. 1), and $\widetilde{G}$ is directly given by the central extension of the ordianry Galilei group G by U(1), i.e., $\widetilde{G} \equiv \widetilde{G}_{(m)}$.[15] Since for the free particle only the kinetic term is relevant, the elements of $\widetilde{G}_{(m)}$ are given by the subgroup of WSp(2,$\mathbb{R}$), which is generated by $H_{\text{free}} = \mathbb{P}^2/2m$ ($\alpha = -t/2m$) and the Weyl subgroup in (2.3a), i.e., by

$$\widetilde{G}_{(m)} = g\left\{\begin{pmatrix}1 & t/m\\ 0 & 1\end{pmatrix},\ (p,q),\ \theta\right\} = \exp\left(-i\frac{\mathbb{P}^2}{2m}t\right)\exp[i(p\mathbb{Q} + qP + \theta\mathbb{I})]. \tag{3.1}$$

The composition law (2.4) applied to (3.1) reproduces thus the usual one:

$$\begin{aligned} t'' &= t' + t,\\ p'' &= p' + p,\\ q'' &= q' + q + (p'/m)t,\\ \theta'' &= \theta' + \theta + \tfrac{1}{2}(pq' - p'q) + (1/2m)pp't \end{aligned} \tag{3.2}$$

in terms of the evolution space variables (t, p, q) and the Bargmann cocyle[15] for the U(1) part of $\widetilde{G}_{(m)}$.

The left invariant vector fields are easily derived from (3.2) with the result

$$\begin{aligned} X_t^L &= \frac{\partial}{\partial t} + \frac{p}{m}\frac{\partial}{\partial q}, \quad [X_v^L, X_t^L] = X_q^L,\\ X_q^L &= \frac{\partial}{\partial q} - \frac{1}{2}p\Xi, \quad [X_q^L, X_v^L] = mX_\zeta^L,\\ X_v^L &= m\frac{\partial}{\partial p} + \frac{m}{2}q\Xi, \quad [\text{all others}] = 0,\\ X_\zeta^L &= i\zeta\frac{\partial}{\partial \zeta} \equiv \Xi, \end{aligned} \tag{3.3}$$

where $\zeta = \exp i\theta$ and the subindices t, q, and v indicate that they parametrize translations in time, position, and velocity ("boosts"). The "vertical" component of the canonical form satisfies $\Theta(X_t; X_q; X_v) = 0$, $\Theta(X_\zeta) = 1$ and is given by $\Theta = \frac{1}{2}(p\,dq - q\,dp) - (p^2/2m)dt + d\zeta/i\zeta$; it is easily verified that $\mathscr{C}_\Theta$ generated by X_t.

It is now clear that the definition of polarization given in Sec. 1 requires us to take as wave functions $\mathbb{C}$-valued functions on $\widetilde{G}_{(m)}$ satisfying the conditions

$$X_t^L \cdot \psi(q,p,\zeta,t) = 0, \tag{3.4a}$$

$$X_q^L \cdot \psi(q,p,\zeta,t) = 0, \tag{3.4b}$$

and[16]

$$\Xi \cdot \psi = i\psi. \tag{3.5}$$

Thus, X_q and X_t generate the polarizations (X_q and X_t determine a maximal—and abelian—invariant subgroup in G.[17]) In fact, X_t gives in (3.4a) the Schrödinger equation in momentum space $i(\partial/\partial t)\psi = (p^2/2m)\psi$ once (3.4b) and (3.5) have been taken into account. ($\hbar = h/2\pi$ has been put equal to unity throughout.)

The polarization defined by X_q we may now call the *natural* polarization by observing that the part of $\widetilde{G}_{(m)}$, which corresponds to the evolution operator

$$g\left\{\begin{pmatrix}1 & t/m\\ 0 & 1\end{pmatrix},(0,0),0\right\} \tag{3.6}$$

leaves the subspace $(0, q)$ invariant; this is a way of rephrasing the definition of Sec. 1. The fact that the triangular matrix (3.6) is not diagonizable indicates that this is the only polarization which may be defined for the free particle in a natural way. Of course, we may now give the Schrödinger equation in configuration space. To do this, it is necessary to apply a transformation [an automorphism external to $\widetilde{G}_{(m)}$ but internal to Sp(2,$\mathbb{R}$)] capable of interchanging the roles of p and q. Such a transformation is (up to a phase $e^{-i\pi/4}$) the Fourier transform given by

$$g_{\mathscr{F}}\left\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},0,0\right\}, \tag{3.7}$$

which determines the well-known integral kernel $1/\sqrt{2\pi}\,\exp(-ipq)$ in (2.5c).

4. THE FREE FALL

We now turn to the quantization of the particle in free fall. The dynamical group for this case is the subgroup of WSp(2,$\mathbb{R}$) given by

$$\exp(-itH_{ff})\exp[i(p\mathbb{Q} + q\mathbb{P} + \theta\mathbb{I})], \tag{4.1}$$

where

$$H_{ff} = \mathbb{P}^2/2m + F\mathbb{Q}. \tag{4.2}$$

Using (2.3) for each factor and (2.4) for the product of the two exponentials, we obtain

$$\tilde{g}\left\{\begin{pmatrix}1 & t/m\\ 0 & 1\end{pmatrix},\ \left(p - Ft, q - \frac{F}{2m}t^2\right),\ \theta + \frac{1}{2}qFt - p\frac{Ft^2}{4m} + \frac{F^2t^3}{12m}\right\}. \tag{4.3}$$

This group gives for the evolution variables the same composition law as for the free particle, except for the U(1) part for whose exponent θ we now obtain

$$\theta'' = \theta' + \theta + \frac{1}{2}\left[(pq' - p'q) + \frac{pp'}{m}t\right] - F\left(q't + \frac{1}{2m}p't^2\right). \quad (4.4)$$

The left invariant vector fields are given by

$$\begin{aligned} X^L_t &= \frac{\partial}{\partial t} + \frac{p}{m}\frac{\partial}{\partial q} - (Fq)\Xi, & [X^L_v, X^L_t] &= X^L_q, \\ X^L_q &= \frac{\partial}{\partial q} - \frac{1}{2}p\Xi, & [X^L_q, X^L_v] &= mX^L_\zeta, \\ X^L_v &= m\frac{\partial}{\partial p} + \frac{m}{2}q\Xi, & [X^L_t, X^L_q] &= FX^L_\zeta, \\ X^L_\zeta &= i\zeta\frac{\partial}{\partial\zeta} \equiv \Xi, & [\text{all others}] &= 0. \end{aligned} \quad (4.5)$$

It is clear that, when $F \to 0$, (4.5) reproduces (3.3). The fact that two parameters (m and F) label the extension ($\widetilde{G}_{(m,F)}$) of the Galilei group G is due to its one-dimensional character; this group has a *two*-dimensional space of extensions (the space of the extensions of the ordinary 10-dimensional G is labeled by the mass only). The fact that G is the common starting point for the free particle and the free fall case merely reflects that the invariance group of the *one-dimensional* equation $F = m\, d^2q/dt^2$ is the same for $F = 0$ and for $F = \text{const} \neq 0$.

To define polarization, we now observe that $[X^L_t, X^L_q] \neq 0$ contrarily to what happened in the free case. We now take the invariant subalgebra generated by X^L_q and

$$X^L_C = X^L_t + \frac{F}{m}X^L_v = \frac{\partial}{\partial t} + \frac{p}{m}\frac{\partial}{\partial q} + F\frac{\partial}{\partial p} - \frac{F}{2}q\Xi; \quad (4.6)$$

note that the basic commutator $[X_q, X_v] = m\Xi$ is not altered by the above redefinition, but that now

$$[X^L_C, X^L_q] = 0 \quad (4.7)$$

as in the free particle case.

To obtain the Schrödinger equation, we now require that the functions on $\widetilde{G}$ satisfy the polarization conditions

$$X^L_C \cdot \psi(q,p,t,\zeta) = 0, \quad X^L_q \cdot \psi = 0 \quad (4.8)$$

and the equivariance condition $\Xi \cdot \psi = i\psi$. This condition and $X^L_C \cdot \psi = 0$ give for ψ the form

$$\psi = \zeta\varphi(p,t)e^{iqp/2} \quad (4.9)$$

and then $X_C \cdot \psi = 0$ yields

$$\left(i\frac{\partial}{\partial t} - \frac{p^2}{2m} + Fi\frac{\partial}{\partial p}\right)\varphi(p,t) = 0, \quad (4.10)$$

i.e., the Schrödinger equation for the linear potential in the momentum representation. Again configuration space expressions are gained through the Fourier transform (3.7).

To fully characterize the process leading to the determination of the natural polarization, it is now important to ascertain the meaning of X^L_C, the polarization which determines the temporal evolution of the system and, accordingly, the quantum wave equation once all the other conditions have been taken into account. To do this, it is necessary to evaluate again the canonical 1-form on $\widetilde{G}$ and, more specifically, its "vertical" component Θ^L. The result is

$$\Theta^L = \tfrac{1}{2}(p\,dq - q\,dp) - (p^2/2m - Fq)dt + d\zeta/i\zeta. \quad (4.11)$$

Let us now derive the equations of the characteristic module generated by the vector fields such that

$$i_X\Theta = 0, \quad i_X\, d\Theta = 0, \quad (4.12a)$$

where X is the vector field of generic components

$$X = X^t\frac{\partial}{\partial t} + X^q\frac{\partial}{\partial q} + X^v\frac{\partial}{\partial p} + X^\zeta i\zeta\frac{\partial}{\partial\zeta}. \quad (4.12b)$$

Equation (4.12a) gives

$$X^t = 1, \quad X^q = p/m, \quad X^v = F, \quad X^\zeta = -\tfrac{1}{2}Fq, \quad (4.13)$$

i.e., the characteristic vector field is X^L_C, the part of the polarization (Sec. 1) which generates the quantum equations of motion. This was to be expected; the integral curves of X^L_C give also the *classical* equations of motion,

$$p = Ft + p_0, \quad q = \tfrac{1}{2}(F/m)t^2 + (p_0/m)t + q_0, \quad (4.14a)$$

plus

$$\zeta = z\exp\{-\tfrac{1}{2}iF[\tfrac{1}{6}(F/m)t^3 + (p_0/2m)t^2 + q_0t]\}. \quad (4.14b)$$

(4.14a) shows that one could have started from the quantum group $\widetilde{G}_{(m,F)}$ without physically identifying its parameters since the equations of motion provide directly the adequate correspondence with the evolution space variables.

It is interesting to remark at this stage that one can establish the connection with the usual quantization formalism by defining a contact 1-form on the manifold of solutions of the classical problem as parametrized by the constants of the motion (initial position q_0 and momentum p_0). Indeed one may check that on such manifold ($\widetilde{G}/\mathscr{C}_\Theta$, where $\mathscr{C}_\Theta$ is the characteristic manifold), Θ is written

$$\Theta = \tfrac{1}{2}(p_0\,dq_0 - q_0\,dp_0) + dz/iz. \quad (4.15)$$

We have not pursued this latter path so as to emphasize how the procedure outlined in Sec. 1[1] allows us to perform the quantization directly on $\widetilde{G}$.

To conclude this section, we mention that the basic quantum operators may be obtained from the right invariant vector fields: from

$$X^R_q = \frac{\partial}{\partial q} + \left(\frac{p}{2} - Ft\right)\Xi \quad (4.16a)$$

and

$$X^R_v = t\frac{\partial}{\partial q} + m\frac{\partial}{\partial p} + \left(-\frac{m}{2}q + \frac{p}{m}t + \frac{F}{2}t^2\right)\Xi \quad (4.16b)$$

it may be easily derived that, on $\varphi(p,t)$, $\hat{p}$ and $\hat{q}$ are represented by p and $i\partial/\partial p$ through imposing that the eigenvalues of $\hat{P} \equiv -iX^R_q$ and $\hat{K} \equiv (i/m)X^R_v$ be the corresponding constants of motion.

5. THE HARMONIC OSCILLATOR

For the case of the harmonic oscillator, the quantum dynamical group is the subgroup of WSp(2,$\mathbb{R}$) obtained from $\exp(-iH_{h_0}t)\cdot\exp[i(p\mathbb{Q} + q\mathbb{P} + \theta\mathbb{I})]$. Since the quantum

Hamiltonian is given by

$$H_{h_0} = \mathbb{P}^2/2m + \tfrac{1}{2}m\omega^2\mathbb{Q}^2, \quad (5.1)$$

the result of evaluating $\exp(-iH_{h_0}t)\exp[i(p\mathbb{Q} + q\mathbb{P} + \theta\mathbb{I})]$ is

$$\widetilde{G}_{(m,\omega)} = \tilde{g}\left\{\begin{pmatrix} \cos\omega t & \frac{1}{m\omega}\sin\omega t \\ -m\omega\sin\omega t & \cos\omega t \end{pmatrix}, \ (p,q), \ \theta\right\} \quad (5.2)$$

[see Eqs. (2.3) with $\alpha = -t/2m, \gamma = -m\omega^2 t/2, s = \pm\omega t/2$], which characterizes the group we take as our starting point.

The composition law (2.4) induces the following one for the evolution space variables and θ:

$$\begin{aligned} t'' &= t' + t, \\ p'' &= p + p'\cos\omega t - m\omega q'\sin\omega t, \\ q'' &= q + (1/m\omega)p'\sin\omega t + q'\cos\omega t, \\ \theta'' &= \theta' + \theta' + \tfrac{1}{2}[(pq' - qp')\cos\omega t \\ &\quad + (pp'/m\omega - m\omega qq')\sin\omega t], \end{aligned} \quad (5.3)$$

which in the free limit $\omega \to 0$ gives (3.2) as it should. Clearly, we could continue and define left invariant vector fields on $\widetilde{G}_{(m,\omega)}$. Nevertheless, it is already evident that the basis (5.3) of the evolution space is not the adequate one; the matrix which determines the evolution operator in (5.2) leaves neither the p nor the q subspaces invariant, indicating that a polarization is not immediately obtained. The matrix of the time evolution subgroup, however, is diagonalizable in this case, and the diagonalizing matrix is

$$\mathbf{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{m\omega} & -i/\sqrt{m\omega} \\ -i\sqrt{m\omega} & 1/\sqrt{m\omega} \end{pmatrix}. \quad (5.4)$$

Again, this is an automorphism external to $\widetilde{G}_{(m,\omega)}$ but internal to HSp(2,ℂ). Once $\mathbf{B}$ has been applied to (5.2), the elements of the transformed $\widetilde{G}_{(m,\omega)}$ are written

$$g\left\{\begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix}, \ \sqrt{m/\omega}(iC^+, C), \ \theta\right\} = \mathbf{B}\tilde{g}\mathbf{B}^{-1}, \quad (5.5)$$

where $C = (1/m\sqrt{2})(m\omega q + ip)$ and the composition law is given by

$$\begin{aligned} C''^+ &= C'^+ e^{i\omega t} + C^+, \\ C'' &= C' e^{-i\omega t} + C, \\ \theta'' &= \theta' + \theta + \tfrac{1}{2}[iC'C^+ e^{-i\omega t} - iC'^+ C e^{i\omega t}], \\ t'' &= t' + t. \end{aligned} \quad (5.6)$$

It is now clear that X^L_t and X^L_C are appropriate to define the polarization, and since

$$\begin{aligned} X^L_t &= \frac{\partial}{\partial t} - i\omega C\frac{\partial}{\partial C} + i\omega C^+\frac{\partial}{\partial C^+}, \\ X^L_C &= \frac{\partial}{\partial C} - \frac{i}{2}C^+\Xi, \end{aligned} \quad (5.7)$$

the conditions $X^L_C\cdot\psi = 0$ and $X^L_t\cdot\psi = 0$ give

$$\psi = \zeta\varphi(C^+, t)e^{-C^+C/2} \quad (5.8a)$$

and

$$\frac{\partial\varphi}{\partial t} = -i\omega C^+\frac{\partial\varphi}{\partial t}, \quad (5.8b)$$

respectively. Again, X^L_t generates the characteristic module for this case; this may be seen by evaluating Θ from (5.7) [plus $X^L_{C^+}$ and X^L_ζ, cf. (4.5) and (4.11)], which turns out to be

$$\Theta = \tfrac{1}{2}i(C^+\,dC - C\,dC^+) - \omega CC^+\,dt + d\zeta/i\zeta. \quad (5.9)$$

The inner product for the ψ's now induces for the functions φ the following one:

$$\left(\frac{2}{\pi}\right)^{1/2}\int_{\mathbf{R}^2} d\,\mathrm{Re}\,C^+\,d\,\mathrm{Im}\,C^+\,\varphi_2^*(C^+,t)\varphi_1(C^+,t)e^{-|C|^2} \quad (5.10)$$

as obtained from (2.7) with $v = 0$, $\nu = 1$. This is the Bargmann scalar product,[14] as one should expect. The natural polarization leads thus to the BFS picture, and it may be checked that the evolution operator preserves the polarization X^L_C, i.e.,

$$[X^L_t, X^L_C] = i\omega X^L_C. \quad (5.11)$$

It should be noted that Eq. (5.8b) is not quite the Schrödinger equation for the harmonic oscillator. It corresponds to the Bohr–Wilson–Sommerfeld quantization, and the additive constant $\frac{1}{2}$ corresponding to the ground state energy is missing. In the usual approach to geometric quantization, this is remedied by looking to the transformation properties of the "half-forms" under the time evolution generator.[4] The result is that Eq. (5.8b) is corrected to

$$\frac{\partial\varphi}{\partial t} = -i\omega\left(C^+\frac{\partial}{\partial C} + \frac{1}{2}\right)\varphi, \quad (5.12)$$

where the functions $\varphi(C^+,t)$ belong to the Bargmann–Segal space $\mathscr{B}_{\mathbf{B}}$, with $\mathbf{B}$ given by (5.4). The solutions to (5.12) which separate into the product of C^+ times a function of t are, up to a multiplicative constant a_ν,

$$a_\nu(C^+)^\nu e^{-i(\nu+1/2)t} \quad (5.13)$$

for $\nu\in\mathbb{C}$. The condition that the solutions of (5.12) belong to $\mathscr{B}_{\mathbf{B}}$ (entire analytic functions with mild decrease conditions[14]) restricts the range of ν to the nonnegative integers, $\nu\in\{0,1,2,\cdots\}$. The normalization factor in (5.13) is found to be $a_n = (2\pi)^{-1/4}\cdot(n!)^{-1/2}$.[18] The set of eigenstates of the system is thus

$$u_n(C^+,t) = (2\pi)^{-1/4}(n!)^{-1/2}(C^+)^n e^{-i(n+1/2)t}, \quad n = 0,1,2,\cdots. \quad (5.14)$$

in BSF space. The eigenstates in the configuration space may be recovered through an inverse Bargmann transform [cf. (2.5)] on (5.14) yielding the familiar harmonic oscillator wave functions

$$\psi_n(q,t) = \int d^2\mu_{\mathbf{B}}(C^+,C)u_n(C^+,t)C_{\mathbf{B}}(C^+,q)^* \quad (5.15a)$$

$$= 2^{-n/2}\pi^{-1/4}(n!)^{-1/2}\exp(-\tfrac{1}{2}m\omega q^2) \times H_n(\sqrt{m\omega}q)e^{-i(n+1/2)t}. \quad (5.15b)$$

In essence, the Bargmann transform (for $m = \omega = 1$) is a group automorphism which "rotates" the usual harmonic oscillator Hamiltonian—the algebra generator $\frac{1}{4}(\mathbb{P}^2 + \mathbb{Q}^2)$—which appears in (5.1) onto the change of scale generator $\frac{1}{2}\{\mathbb{P},\mathbb{Q}\}_+$. As already mentioned, these two algebra elements belong to different orbits in sp(2,ℝ) under the adjoint action of Sp(2,ℝ), but may be brought into similarity through $g_{\mathbf{B}}$, a transformation in HSp(2,ℂ), obtained from (2.5) with **B** given by (5.4). This transformation corresponds to a rotation by $i\pi/4$ around the $\frac{1}{4}(\mathbb{P}^2 - \mathbb{Q}^2)$ axis (see Ref. 10, Sec. 9.2). The eigenfunctions of the change-of-scale generator are the (complex) power functions, i.e., essentially (5.13).

6. THE REPULSIVE "OSCILLATOR"

We complete our study of the one-dimensional potentials by considering the somewhat unphysical potential $-\frac{1}{2}m\omega^2\mathbb{Q}^2$, the repulsive "oscillator." For it we find

$$\exp\{-it\,[(1/2m)\mathbb{P}^2 - \tfrac{1}{2}m\omega^2\mathbb{Q}^2]\}$$
$$\times\exp[i(p\mathbb{Q} + q\mathbb{P} + \theta\mathbb{I})]$$
$$= g\left\{\begin{pmatrix}\cosh\omega t & \sinh\omega t/m\omega\\ m\omega\sinh\omega t & \cosh\omega t\end{pmatrix},\ (p,q),\ \theta\right\} \qquad (6.1)$$

and

$$\begin{aligned} t'' &= t' + t,\\ q'' &= q + q'\cosh\omega t + \frac{p'}{m\omega}\sinh\omega t,\\ p'' &= p + p'\cosh\omega t + m\omega q'\sinh\omega t,\\ \theta'' &= \theta' + \theta + \tfrac{1}{2}[(pq' - p'q)\cosh\omega t\\ &\quad + (pp'/m\omega - qq'm\omega)\sinh\omega t\,] \end{aligned} \qquad (6.2)$$

for the group law.

To define the polarization adequate for this problem, we may follow the same pattern as for the harmonic oscillator. The time evolution operator $\exp(-iH_{r_0}t)$ is not diagonal, but diagonalizable by the Sp(2,ℝ) transformation

$$\mathbf{R} = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{m\omega} & 1/\sqrt{m\omega}\\ -\sqrt{m\omega} & 1/\sqrt{m\omega}\end{pmatrix}, \qquad (6.3a)$$

which defines the new group variables

$$\alpha = \frac{1}{\sqrt{m\omega}}\frac{m\omega q + p}{\sqrt{2}},\quad \nu = \frac{1}{\sqrt{m\omega}}\frac{m\omega q - p}{\sqrt{2}}, \qquad (6.3b)$$

in terms of which the Hamiltonian is written $H = -\omega\alpha\nu$. In terms of these variables (6.1) reads

$$g\left\{\begin{pmatrix}e^{\omega t} & 0\\ 0 & e^{-\omega t}\end{pmatrix},\ (\alpha,\nu),\ \theta\right\}, \qquad (6.4a)$$

where the time evolution part has been diagonalized, and (6.2) is given by

$$\begin{aligned} t'' &= t' + t,\\ \alpha'' &= \alpha + \alpha' e^{\omega t},\\ \nu'' &= \nu + \nu' e^{-\omega t},\\ \theta'' &= \theta + \theta' + \tfrac{1}{2}[\alpha\nu' e^{-\omega t} - \alpha'\nu e^{\omega t}]. \end{aligned} \qquad (6.4b)$$

We now may proceed to evaluate the left-invariant vector fields, which are

$$\begin{aligned} X_t^L &= \frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial\alpha} - \nu\frac{\partial}{\partial\nu},\\ X_\alpha^L &= \frac{\partial}{\partial\alpha} + \frac{\nu}{2}\Xi,\\ X_\nu^L &= \frac{\partial}{\partial\nu} - \frac{\alpha}{2}\Xi,\\ X_\zeta^L &= \Xi \end{aligned} \qquad (6.5)$$

and the vertical component of the canonical 1-form,

$$\Theta = \tfrac{1}{2}(\alpha\,d\nu - \nu\,d\alpha) + \omega\alpha\nu\,dt + d\zeta/i\zeta. \qquad (6.6)$$

It is now easily seen that the vector fields defining a polarization are given by X_t^L and X_α^L and that, as it should, X_t^L is the characteristic vector field of Θ. The conditions $\Xi\cdot\psi = i\psi$ and $X_\alpha^L\cdot\psi = 0$ give

$$\psi = \zeta\varphi(\nu,t)e^{i\nu\alpha/2} \qquad (6.7)$$

and $X_t^L\cdot\psi = 0$ finally yields

$$\frac{\partial\varphi}{\partial t} - \omega\nu\frac{\partial\varphi}{\partial\nu} = 0. \qquad (6.8)$$

By a reasoning analogous to that leading to (5.12), the true Schrödinger equation is written

$$\frac{\partial\varphi}{\partial t} = \omega\left(\nu\frac{\partial}{\partial\nu} + \frac{1}{2}\right)\varphi. \qquad (6.9)$$

In fact, the presence of the $\frac{1}{2}$ term can also be justified by unitarity considerations: The transformation **R** [(6.3a)] maps $\mathscr{L}^2(\mathbb{R})$ unitarily on $\mathscr{L}^2(\mathbb{R})$ in ν. The separated solutions of (6.9) are, again up to a constant a_μ,

$$a_\mu\nu^\mu e^{(\mu + 1/2)t} \qquad (6.10)$$

for μ complex. Although these functions do not belong to $\mathscr{L}^2(\mathbb{R})$, they constitute a generalized basis for $\mathscr{L}^2(\mathbb{R})$ eigenbasis for $\frac{1}{2}\{\mathbb{Q},\mathbb{P}\}_+$. This is the bilateral Mellin transform basis (Ref. 10, Sec. 8.2),

$$(2\pi)^{-1/2}\nu_\pm^{-1/2 + i\lambda},\quad \lambda\in\mathbb{R}, \qquad (6.11a)$$

$$\nu_+ = \begin{cases}\nu & \nu\geqslant 0,\\ 0, & \nu<0,\end{cases}\quad \nu_- = \begin{cases}0, & \nu\geqslant 0,\\ -\nu, & \nu<0,\end{cases} \qquad (6.11b)$$

i.e., (6.10) with $a_\mu = (2\pi)^{-1/2}$ and $\mu = -\frac{1}{2} + i\lambda$. The Mellin basis is Dirac-orthonormal and complete, and its eigenvalues under $\frac{1}{2}\{\mathbb{Q},\mathbb{P}\}_+$ cover twice the real line. Hence,

$$\omega_{\sigma,\lambda}(\nu,t) = (2\pi)^{-1/2}\nu^{-1/2 + i\lambda}e^{i\lambda t},$$
$$\sigma = \pm 1, \lambda\in\mathbb{R}, \qquad (6.12)$$

is the $\mathscr{L}^2$-complete Dirac-orthonormal set of solutions of (6.9). [The importance of the $\frac{1}{2}$ additive term in the Schrödinger equation—which merely shifts the whole spectrum for the case of the harmonic oscillator—is seen in this example since, otherwise, the time part in (6.10) would adopt the unsuitable form $e^{(-1/2 + i\lambda)t}$ instead of $e^{i\lambda t}$.]

In order to recover the wave functions in configuration space, we apply the inverse of the transform **R** of (6.3a) (for $m = \omega = 1$). This is a rotation by $\pi/4$ around the $\frac{1}{4}(\mathbb{P}^2 + \mathbb{Q}^2)$ axis—in fact, the square root of the Fourier transform—which through its action on the algebra, brings the repulsive oscillator $\frac{1}{2}(\mathbb{P}^2 - \mathbb{Q}^2)$ onto the change of scale operator

$\frac{1}{2}\{\mathbb{Q},\mathbb{P}\}_+$ appearing on the right-hand side of (6.9).

The configuration space eigenfunctions are thus the repulsive oscillator wavefunctions

$$\begin{aligned}\Upsilon_{\sigma,\lambda}(q,t) &= \int_{-\infty}^{\infty} d\nu\, \omega_{\sigma,\lambda}(\nu,t) C_{\mathrm{M}}(\nu,q)^* \\ &= \exp[i\pi(\tfrac{1}{2} - i\lambda)] 2^{-3/4}\pi^{-1}\Gamma(\tfrac{1}{2} - i\lambda) \\ &\quad \times D_{-1/2+i\lambda}(-\sigma 2^{1/2} e^{3i\pi/4} q) e^{i\lambda t}, \end{aligned} \qquad (6.13)$$

where $D_\tau(x)$ is the parabolic cylinder function (see Ref. 19; Chap. 19 and Ref. 10, Secs. 7.5 and 8.2).

7. CONCLUSIONS

The above study of all inequivalent one-dimensional quantum systems with up-to-quandratic potentials, which admit a finite-dimensional group,[20] has shown how a general definition for a quantum manifold (in the sense of the geometric quantization) can be given based on the group. It has also been shown that for each case there exists in our formalism a natural polarization and that the role of the Blattner–Konstant polarization changing transformation (see, e.g., Ref. 4) is played by integral transforms of the group WSp(2,$\mathbb{R}$) or of the semigroup HS(2,$\mathbb{C}$), which include the four one-dimensional systems considered in this paper.

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[1]V. Aldaya and J. A. de Azcárraga, J. Math. Phys. **23**, 1297 (1982).

[2]This method assumes the fruitful belief that symmetry—and, in general, geometric—considerations are very important in constraining a theory, almost to the point of determining it completely. Nevertheless, and as pointed out by Yang,[3] the fact that the geometric approach to physical problems has been up to now a very sucessful one does not imply that every geometrical concept has its physical counterpart; the history of physics is littered with many useless geometries.

[3]C. N. Yang, talk given at the Chern Symposium, Berkeley, Calif., June 1979 and in *To Fulfill a Vision*, Proc. of the 1979 Jerusalem Einstein Centennial Symposium, edited by Y. Ne'eman (Addison-Wesley, Reading, MA, 1981).

[4]J. M. Souriau, *Structure des systèmes dynamiques* (Dunod, Paris, 1970); B. Kostant, *Quantization and Unitary Representations*, Lecture Notes in Mathematics **170** (Springer-Verlag, New York, 1970), pp. 87–208; see also: D. J.Simms and N. M. J. Woodhouse, *Lecture Notes in Geometric Quantization* (Springer-Verlag, Berlin, 1976); J. Śniatycki, *Geometric Quantization and Quantum Mechanics* (Springer-Verlag, New York, 1980).

[5]The dynamical algebra of a given classical (quantum) system is a Lie algebra under the Poisson (commutator) bracket such that the system Hamiltonian is an element of the algebra *not* belonging to the center. For quantum systems with a discrete energy spectrum this means that raising and lowering operators are to be found in the dynamical algebra. The dynamical algbra is in general not unique (in the harmonic oscillaor system, for instance, the Hamiltonian, creation, annihilation, and unit operators constitute a minimal four-dimensional algebra). The conditions that the dynamical algebra be of finite dimension and that it contain a subalgebra contractible to the Galilei algebra determine that the appropriate algebra for the four systems studied here be wsp(2,$\mathbb{R}$). The dynamical group is the exponentiation of the dynamical algebra.

[6]The characteristic module $\mathscr{C}_\Theta$ of a form Θ is the space of vector fields X such that $i_X\Theta = 0$ and $i_X\, d\Theta = 0$, where i_X indicates inner product.

[7]See, e.g., K. B. Wolf, "The Heisenberg-Weyl Ring in Quantum Mechanics," in *Group Theoretical Methods and Its Applications*, edited by E. M. Loebl (Academic, New York, 1975), Vol. III.

[8]A. Weil, Acta Math. **11**, 143 (1963).

[9]K. B. Wolf, SIAM J. Appl. Math. **40**, 419 (1981).

[10]K. B. Wolf, *Integral Transformations in Science and Engineering* (Plenum, New York, 1979), Sec. 10.2.

[11]A. Joseph, J. Math. Phys. **13**, 351 (1972).

[12]Ref. 10, Sec. 9.1.

[13]M. Moshinsky and C. Quesne, J. Math. Phys. **12**, 1772, 1780 (1971); M. Moshinsky, SIAM J. Appl. Math. **25**, 193 (1973).

[14]V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961); **20**, 1 (1967).

[15]V. Bargmann, Ann. Math. **59**, 1 (1954).

[16]Condition (3.5), which constitutes the expression that ψ is a U(1) (equivariant) function may require some explanation. Let $H \xrightarrow{\tau} S$ be a vector bundle of base S and fiber F associated with a principal bundle $P(\mathfrak{G},X)$. Then, the modulus $\Gamma(H)$ of differentiable cross sections of $H \xrightarrow{\tau} S$ is isomorphic to the vector space of differentiable F-valued $\mathfrak{G}$ functions on P. In our case P is, as a manifold, $\widetilde{G}_{(m)}$; the structure group $\mathfrak{G}$ is U(1) and S is the Galilei group G. Thus, the space of the wave functions (cross sections of $H \xrightarrow{\tau} S$, and thus complex-valued functions of p, q, and t) may be defined as the space of $\mathbb{C}$-valued U(1) functions on $\widetilde{G}_{(m)}$.

[17]Note that the invariance condition for polarization is here oversimplified because of the monodimensional nature of the problem. When $\widetilde{G}_{(m)}$ is the eleven-dimensional extended Galilei group, X_t^L and X_q^L generate an invariant abelian subgroup of G including $\mathscr{C}_\Theta$.

[18]The normalization constants employed here are those of Ref. 10 and differ from those originally defined by Bargmann.[14] We follow that choice here since it assures us that the composition of transforms and the $\mathrm{M}\to 1$ limit are obtained without extra compensating factors.

[19]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).

[20]Only in dimension one is it possible to establish a one-to-one correspondence between the dynamical system and the dynamical group.