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THE CLEBSCH–GORDAN COEFFICIENTS FOR THE COVERING OF THE (2+1)-LORENTZ GROUP IN THE PARABOLIC BASIS

Debabrata BASU* and Kurt Bernardo WOLF

Instituto de Investigaciones en Matematicas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, Apartado Postal 20–726, México 20, DF, México

We report on work in which we build the *generalized oscillator algebra*, an SO (2,1) algebra of second-order differential operators with a specified domain, realizing all self-adjoint irreducible representations belonging to the discrete or the continuous series. The diagonal subalgebra of the direct sum of two such algebras leads to the definition of product and coupled states, whose inner product provides the Clebsch–Gordan coefficients. These are obtained as solutions to (multichart) Schrödinger equations for Pöschl–Teller potentials, which involve at most Gauss hypergeometric functions ${}_2F_1$.

1. Introduction

The Clebsch–Gordan series and coefficients for the noncompact semisimple algebra $so(2,1) \simeq su(1,1) \simeq sp(2,R) \simeq sl(2,R)$ has been considered by a number of authors, among them Pukánszky[1]), Ferreti and Verde[2]), Holman and Biedenharn[3]), and Mukunda and Radhakrishnan[4]). The last two references treat the Clebsch–Gordan coefficients in the elliptic $[so(2,1) \supset so(2)]$ and hyperbolic $[so(2,1) \supset so(1,1)]$ subalgebra bases. Here we treat the Clebsch–Gordan coefficients in the parabolic $[so(2,1) \supset iso(1)]$ basis. We claim that this chain leads to the simplest method of solution and simplest final expressions. It is unique in that the algebra realization, as shown in section 2, is of up-to-*second-order*[5]) differential operators with specific common irreducible domains. This realization exponentiates to a group of integral transforms[5]) which have permitted a unified evaluation of all unitary irreducible representations of the group[6]). Section 3 presents the main points of the programme; the details and list of results are given in ref. 7.

2. Operators and domains

Let us first introduce the set of so(2,1) generators as second-order differential operators, and specify their domain, so as to have the *generalized oscillator realization* which belongs to the most general self-adjoint irreduci-

* On leave from the Indian Institute of Technology, Kharagpur, India.

ble representation of so(2,1). Consider the space $\mathcal{S} = \{-1, 1\} \times \mathcal{R}^+$ of points (σ, r) where $\sigma \in \{-1, 1\}$ and $r > 0$, and the space of functions $\mathcal{L}^2_\epsilon(\mathcal{S})$, of elements $f_\sigma(r) = f(\sigma, r)$ with the natural inner product $(f, g) = \Sigma_\sigma \int_0^\infty f_\sigma(r)^* g_\sigma(r)\, \mathrm{d}r$. The label ϵ will specify an so(2, 1)-irreducible subspace.

The algebra generators are, formally

$$J_+^k = \frac{\sigma}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\gamma}{r^2}\right) = J_0^k + J_1^k, \tag{1a}$$

$$J_-^k = \frac{1}{2}\sigma r^2 = J_0^k - J_1^k, \tag{1b}$$

$$J_2^k = -\frac{\mathrm{i}}{2}\left(r\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{2}\right), \tag{1c}$$

where $\gamma = (2k-1)^2 - 1/4 \epsilon \mathcal{R}$. The Casimir operator built out of eqs. (1) is a multiple q of the identity, with $q = k(1-k) = -\gamma/4 + 3/16$, and we identify k with its homonymous Bargmann label of self-adjoint irreducible representations.

The operator J_0^k in $\mathcal{L}^2(\mathcal{S})$ has in general more than one self-adjoint extension; the nature of its family of extensions depends on γ (or on k), but it is fixed once a (possible) spectrum $\{\mu\}$ is specified. It turns out that all so(2,1) self-adjoint irreducible representation J_0-spectra are allowed; when the eigenvalues are $\mu = m + \epsilon$, for a range of integers m, $\mathcal{L}^2_\epsilon(\mathcal{S})$ is the closure of the span of the normalized functions

$$\psi_\mu^k(\sigma, r) = -(\sigma)^{\mu-\epsilon}[\tfrac{1}{2}\Gamma(k + \sigma\mu)\Gamma(1 - k + \sigma\mu)]^{-1/2} r^{-1/2} W_{\sigma\mu, k-1/2}(\gamma^2). \tag{2}$$

The spectrum of the two-chart operator J_0^k is equally spaced and $\mathcal{L}^2_\epsilon(\mathcal{S})$ thus defined serves as the common invariant domain for the three generators (1) of the algebra. We shall denote the operators with this domain by $J_\alpha^{\epsilon,k}$, where $\alpha = 0, 1, 2, +, -$.

For $1/2 \leq k$, $\mu = k, k+1, \ldots$, and recalling the $k \leftrightarrow 1-k$ equivalence, $1/2 < k < 1$, $\mu = 1-k, 2-k, \ldots, \epsilon \equiv k \bmod 1$, the discrete series of self-adjoint irreducible representations, D_k^+, are realized for $k > 0$ (and with $\mu \leftrightarrow -\mu$, the D_k^- series). The functions in these spaces have a $\sigma = +1$ support only ($\sigma = -1$ support for D_k^-). For $k = (1 + \mathrm{i}\kappa)/2$, $\kappa \geq 0$, $\mu = \epsilon +$ an integer, $-1/2 < \epsilon \leq 1/2$ (excluding $k = 1/2$, $\epsilon = 1/2$) we have the nonexceptional continuous series self-adjoint irreducible representations C_q^ϵ while for $1/2 < k < 1$, $\mu = \epsilon +$ integer, $|\epsilon| < 1 - k$ the exceptional continuous series of self-adjoint irreducible representations C_q^ϵ are obtained.

Our favoured generator of the set (1) is $J_-^{\epsilon,k}$, whose generalized eigenfunctions are

$$\psi_{\tau,\rho}(\sigma, r) = \rho^{-1/2}\delta_{\sigma,\tau}\delta(r-\rho) = \delta_{\sigma,\tau}\delta(\tfrac{1}{2}r^2 - \tfrac{1}{2}\rho^2) \tag{3}$$

corresponding to the simple eigenvalue $\frac{1}{2}\tau\rho^2 \in \mathscr{R}$, $\tau \in \{-1, 1\}$, $\rho \in \mathscr{R}^+$.

3. Product and coupled states

In order to construct the Clebsch–Gordan coefficients we consider two sets of operators (1) in two independent variable sets $\{\sigma_j r_j\}$ where $j = 1, 2$ and build the 'coupled' so(2, 1) generators

$$J^{\epsilon}_{\alpha} = J^{\epsilon_1,k_1}_{(1)\alpha} + J^{\epsilon_2,k_2}_{(2)\alpha}, \quad \alpha = 0, 1, 2, +, -, \quad \epsilon \equiv \epsilon_1 + \epsilon_2 \bmod 1, \tag{4}$$

in a Hilbert space $\mathscr{L}^2_{\epsilon_1}(\mathscr{S}_{(1)}) \times \mathscr{L}^2_{\epsilon_2}(\mathscr{S}_{(2)})$. The representation realized by eq. (4) is not irreducible. In fact, the coupled Casimir operator is

$$Q = -J_0^2 + J_1^2 + J_2^2 = \frac{1}{4}\left[\sigma_1\sigma_2\left(r_1\frac{\partial}{\partial r_2} - \sigma_1\sigma_2 r_2\frac{\partial}{\partial r_1}\right)^2 - \gamma_1(1 + \sigma_1\sigma_2 r_2^2/r_1^2) - \gamma_2(1 + \sigma_1\sigma_2 r_1^2/r_2^2) + 1\right]. \tag{5}$$

Product states $\psi^{k_1,k_2}_{\tau_1,\rho_1,\tau_2,\rho_2}(\sigma_1, r_1, \sigma_2, r_2)$ are built as eigenstates of $Q_{(1)}$, $Q_{(2)}$, $J_{(1)-}$ and $J_{(2)-}$ with eigenvalues $q_1 = k_1(1-k_1)$, $q_2 = k_2(1-k_2)$, $\tau_1\rho_1^2/2$ and $\tau_2\rho_2^2/2$, respectively, and consist of products of two states given by eq. (3). *Coupled* states $\Psi^{k_1k_2}_{k,\tau,\rho}(\sigma_1, r_1, \sigma_2, r_2)$ are built as eigenstates of $Q_{(1)}$, $Q_{(2)}$, Q and J_- with eigenvalues q_1, q_2, $q = k(1-k)$ and $\tau\rho^2/2$, respectively. The last operator is, from eq. (4), $\sigma_1 r_1^2/2 + \sigma_2 r_2^2/2$. It is thus convenient to introduce a reparametrization of $(\sigma_1, r_1, \sigma_2, r_2)$ in $\mathscr{S}_{(1)} \times \mathscr{S}_{(2)}$ into six charts $(C; \sigma, r, \theta)$ labelled by C:

(i) Polar charts $C = P^{\pm}$, for $\sigma_1 = \pm 1 = \sigma_2$: $\sigma = \pm 1$, $r_1 = r\cos\theta$, $r_2 = r\sin\theta$, $r \in \mathscr{R}^+$, $\theta \in [0, \pi/2]$.

(ii) Hyperbolic charts $C = H^{\pm}_{>}$ for $\sigma_1 = \pm 1 = -\sigma_2$, $r_1 > r_2$: $\sigma = \pm 1$, $r_1 = r\cosh\theta$, $r_2 = r\sinh\theta$, $r \in \mathscr{R}^+$, $\theta \in [0, \infty)$.

(iii) Hyperbolic charts $C = H^{\pm}_{<}$ for $\sigma_1 = \mp 1 = -\sigma_2$, $r_1 < r_2$: $\sigma = \pm 1$, $r_1 = r\sinh\theta$, $r_2 = r\cosh\theta$, $r \in \mathscr{R}^+$, $\theta \in [0,\infty)$.

The eigenvalue equation for J_- thus implies that the coupled state has a factor $\delta_{\sigma\tau}\delta(r^2/2 - \rho^2/2)$ (and has thus support on a line passing through three of the six charts, times a θ-dependent function $F^C_{k,\tau}(\theta)$. This last function is found through an eigenvalue equation for Q in each chart C, which takes the form of Pöschl–Teller potential Schrödinger equations:

$$\left[-\frac{d^2}{d\theta^2} + \gamma_1\sec^2\theta + \gamma_2\,\mathrm{cosec}^2\,\theta\right]F^P_{k,\tau}(\theta) = (2k-1)^2 F^P_{k,\tau}(\theta), \tag{6a}$$

$$\left[-\frac{d^2}{d\theta^2} - \gamma_1 \operatorname{sech}^2\theta + \gamma_2 \operatorname{cosech}^2\theta\right] F^{H_>}_{k,\tau}(\theta) = -(2k-1)^2 F^{H_>}_{k,\tau}(\theta), \tag{6b}$$

$$\left[-\frac{d^2}{d\theta^2} + \gamma_1 \operatorname{cosech}^2\theta - \gamma_2 \operatorname{sech}^2\theta\right] F^{H_<}_{k,\tau}(\theta) = -(2k-1)^2 F^{H_<}_{k,\tau}(\theta). \tag{6c}$$

The solutions of eqs. (6) in $\mathscr{L}^2_\epsilon(\mathscr{S}_{(1)} \times \mathscr{S}_{(2)})$, $\epsilon \equiv \epsilon_1 + \epsilon_2 \bmod 1$, are given in terms of hypergeometric ${}_2F_1$ functions in θ, in each chart. Their proper normalization over (a subset of one, two or three of) the six charts is the main calculational problem which has been solved in ref. 7. From there, the Clebsch–Gordan coefficients may be found through an inner product over $\mathscr{S}_{(1)} + \mathscr{S}_{(2)}$

$$C\begin{pmatrix}\epsilon_1, k_1, & \epsilon_2, k_2; & \epsilon, k\\ \tau_1, \rho_1, & \tau_2, \rho_2; & \tau, \rho\end{pmatrix} = (\psi^{k_1,k_2}_{\tau_1,\rho_1,\tau_2,\rho_2}, \Psi^{k_1,k_2}_{k,\tau,\rho})_{\mathscr{S}^2}$$
$$= \delta(\tfrac{1}{2}\tau_1\rho_1^2 + \tfrac{1}{2}\tau_2\rho_2^2 - \tfrac{1}{2}\tau\rho^2)(\rho_1\rho_2)^{-1/2} F^C_{k,\tau}(\operatorname{arctrh}(\rho_2/\rho_1)), \tag{7}$$

where trh = sin, sinh and cosh, in the $C = P^\pm$, $H^\pm_>$ and $H^\pm_<$ charts, respectively. The inner product (7) does not involve integration since the support of the product state is on a single point in the $\mathscr{S}_{(1)} \times \mathscr{S}_{(2)}$ space, lying in one of the six charts seen above. The location of this support depends on the values of τ_1, τ_2, ρ_1 and ρ_2, and must fall on the line support of the coupled state. This gives rise to the 'magnetic number' $(\tau\rho^2/2)$ selection rule embodied in the Dirac δ-factors above. The general structure of the result takes the form

$$C\begin{pmatrix}\epsilon_1, k_1, & \epsilon_2, k_2; & \epsilon, k\\ \tau_1, \rho_1, & \tau_2, \rho_2; & \tau, \rho\end{pmatrix} = c\delta(\tau_1\rho_1^2/2 + \tau_2\rho_2^2/2 - \tau\rho^2/2)$$
$$\times \rho_1^a \rho_2^b \rho^d {}_2F_1(\alpha, \beta; \delta; \pm\rho_j^2/\rho^2), \quad j = 1 \text{ or } 2, \tag{8}$$

where C is the proper normalization constant, a, b, d, α, β and δ depend linearly on k_1, k_2, k, and the argument of the Gauss function is never on the branch cut. The normalization process involves solving a sum of up to three integrals in θ which may be evaluated by a common method. The Clebsch–Gordan series changes nontrivially for the $D^+ \times D^+$, $D^+ \times D^-$, $D \times C$ and $C \times C$ cases; in the latter cases, discrete as well as continuous series of self-adjoint irreducible representations appear in the decomposition and mutual orthogonality must be verified or imposed. Details and results of this calculation have been given in ref. 7.

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