Kinam, Vol 3, 323-346 (1981)

Equally-spaced energy spectra: the harmonic oscillator with a centrifugal barrier or with a centripetal well

Kurt Bernardo Wolf

Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas (IIMAS)
Universidad Nacional Autónoma de México, México DF, México

(Recibido el 6 de noviembre de 1981)

We examine the spectrum and eigenfunctions of the quantum harmonic oscillator with a centrifugal barrier and with a centripetal well. Three intervals in the centrifugal-centripetal strenght parameter range are relevant: in one the spectrum is unique, lower-bound, and equally-spaced, in the second it is lower-bound but not unique and –except for two cases– not equally spaced, while in the third it has neither of the three properties. An associated two-component Hamiltonian has a spectrum exhibiting equal spacing over the full parameter range.

I. INTRODUCTION

Whether or not one regards quantum mechanics as a mathematically complete and definitive physical theory of microscopic Nature, it is true that it may yield answers to problems posed outside the realm of actual physical systems. Most introductory books on the subject will present and solve the one-dimensional square barrier square, square well, and the quantum harmonic oscillator among other idealized models, finding their energy levels and their spacing patterns. The solution of eigenvalue equations with boundary and/or integrability conditions is required for solving the common Schrödinger equation eigenvalue problem

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + V(r)\right] \Psi_\mu(r) = 2\mu\Psi_\mu(r) \ , \qquad (1.1)$$

(where the mass and $\hbar$ are set equal to unity). In the real world, energy spectra must always be bounded from below.

To be sure, there is a wide mathematical literature on operator theory

[1] which deals with general questions which assume very little on the exact nature of $V(r)$. Bound eigenstates, we know, should be square-integrable so as to permit a probabilistic interpretation of the wavefunction. If the potential has certain types of singularities however, this may be impossible, or it may not lead to a unique quantization of energy levels. This latter situation is well understood by operator-theory specialists, but it may be perplexing to research workers in physics who have never dealt with fringe-case potentials.

The purpose of this article is to examine one such case: the one-dimensional half-harmonic oscillator [i.e. over $R^+ = (0, \infty)$] with a γ/r^2-singularity, namely

$$V(r) = \frac{1}{2}\left(\frac{\gamma}{r^2} + r^2\right), \quad r > 0 \tag{1.2}$$

in (1.1), and to obtain the corresponding wavefunctions $\Psi_\mu(r)$ and eigenvalues μ for γ positive as well as negative. We shall see that, in fact, a more meaningful division of the range of the singularity coefficient γ is in terms of three intervals: a) $[3/4, \infty)$, b) $(-1/4, 3/4)$, and c) $(-\infty, -1/4]$. This quantum system is rather well known for the first and second intervals, when it corresponds to the radial part of a multidimensional isotropic harmonic oscillator: it has a spectrum which is bounded from below, and the eigenvalues are equally spaced. In the second interval $(-1/4, 3/4)$, however, depending on the function domain of the Schrödinger Hamiltonian operator, nonequally-spaced spectra, also bounded from below, may be produced. This phenomenon (i.e., the spectrum not being unique unless a domain is specified) is due to the mathematical fact that there is a one-parameter *family of self-adjoint extensions* for the formal Hamiltonian operator.

The last interval $(-\infty, -1/4]$ corresponds to a harmonic oscillator with a strong centripetal well at the origin. Such a system is physically unrealistic, of course, the quantum spectrum is not bounded from below, nor is it equally spaced, nor again is it unique, unless a definite self-adjoint extension is chosen through appropiate specification of the function domain.

The study of this particular example of a singular potential with these properties would be only mildly interesting, were it not for the fact that equal spacing of the eigenvalues is an extremely important feature from the point of view of applied group theory. It tells us that raising and lowering operators may be built and this leads to a Lie algebra of which

the Hamiltonian is an element. It tells us that the system possesses a dynamical algebra and group, which in turn makes it a study favourite for geometrical quantization, special-function theory, and separation of variables. The dynamical algebra of this system is the three-dimensional Lorentz algebra $so(2, 1)$, and these lower-bound equally-spaced spectra correspond to the discrete-series representations [2]. The fact that the continuous series representations did not appear had been a source of puzzlement. These are recognized through equally-spaced unbounded spectra.

There are two main points to be made in this article. First, we examine the system in the said intervals as a problem in quantum mechanics, to show that the spectrum is not unique in $\gamma \in (-\infty, 3/4)$, and that this ambiguity does not seem to be resolved by physics. Secondly, we shall show how, through the consideration of two-component functions, the property of equal eigenvalue spacing is restored for all real values of γ.

Section II presents the Shrödinger equation and finds the square-integrable solutions for $\gamma \in R$. Sec III and IV then examine the eigenvalue problems, leaving for Sec V the "linearization" of the eigenvalue spacing for the $\gamma \in (-\infty, 3/4)$ class of cases. We offer some connections, in Sec VI, with current work which has used mathematical multicomponent operators which have an attractive quantum mechanical interpretation. The appendix gives the group theoretic framework of our interest in this problem, and settles thereby the phase convention for the wave functions.

II. SQUARE-INTEGRABLE EIGENFUNCTIONS

The Schrödinger equation (1.1) for the potential (1.2) will be written as

$$H\Psi_\mu(r) = 2\mu\Psi_\mu(r), \tag{2.1a}$$

$$H = \frac{1}{2}\left[-\frac{d^2}{dr^2} + \frac{\gamma}{r^2} + r^2\right], \tag{2.1b}$$

where 2μ is the energy eigenvalue used to label the corresponding eigenfunction $\Psi_\mu(r)$. The asymptotic behaviour of the solutions may be obtained through the Liouville-Green method [3] (closely related to the WKB method. it consists here in essentially disregarding γ/r^2 and μ as

vanishing in comparison with r^2) which yields one solution behaving as $r^{-1/2} e^{-r^2/2}$ and another as $r^{-1/2} e^{+r^2/2}$. Clearly, the former is square-integrable in (a, ∞), $a > 0$, while the latter is unacceptable as a quantum wave function.

Equation (2.1) is a Fuchsian differential equation with two singular points: a regular one at the origin and an irregular one at infinity. It is thus recognized as a confluent hypergeometric equation whose solutions are expressible —modulo changes of variables and multiplier functions— in terms of confluent hypergeometric functions ${}_1F_1\ (a;c;x)$. Indeed, they may be put in terms of Whittaker functions of the first and second type, $M_{\kappa\lambda}(x)$ and $W_{\kappa\lambda}\ (x)$. The defining property of $W_{\kappa\lambda}\ (x)$ is to have an exponentially vanishing behaviour at $x \to \infty$ [4, 5]. We may write the acceptable solutions of (2.1) as

$$\Psi_\mu^k(r) = c_\mu^k r^{-1/2} W_{\mu, k-1/2}(r^2)$$

$$= c_\mu^k e^{-r^2/2} \{[\Gamma(-2k+1)/\Gamma(1-k-\mu)] r^{2k-1/2}\, {}_1F_1(k-\mu; 2k; r^2)$$

$$+ [\Gamma(2k-1)/\Gamma(k-\mu)] r^{-2k+3/2}\, {}_1F_1(1-k-\mu; 2-2k; r^2)\}\ , \qquad (2.2)$$

where for convenience we have introduced a new parameter k related to the strenght γ of the centrifigal potential, through

$$\gamma = (2k-1)^2 - 1/4\ , \qquad k = \frac{1}{2}\,[1 \pm (\gamma + 1/4)^{1/2}]\ , \qquad (2.3)$$

and expressed the Whittaker function in terms of the confluent hypergeometric functions [6]. It should be noticed that (2.2) and (2.3) are invariant under the exchange $k \leftrightarrow 1-k$. When Eq. (2.1) is obtained as the radial equation for an N-dimensional harmonic oscillator with total angular momentum ℓ, then $k = \ell/2 + N/4$ ($\ell = 0, 1, 2, \ldots$). In group theory, k is the Bargmann label [7] which serves to specify the representations of the three-dimensional Lorentz group.

The ranges which k is assigned to take in each of the three intervals mentioned in the introduction (due to the symmetry under $k \leftrightarrow 1-k$), are:

$$\gamma \geqslant 3/4 \Leftrightarrow k \geqslant 1 \qquad \textit{strong barrier} \text{ interval}\ . \qquad (2.4a)$$

$$-1/4 < \gamma < 3/4 \Leftrightarrow 1/2 < k < 1 \qquad \textit{exceptional} \text{ interval}\ . \qquad (2.4b)$$

$$\gamma \leqslant -1/4 \Leftrightarrow k = (1 + i\kappa)/2, \quad \kappa \geqslant 0 \qquad \textit{strong well} \text{ interval}\ . \qquad (2.4c)$$

As stated before, the square-integrability of $\Psi_\mu^k\ (r)$ on (a, ∞), $0 < a < \infty$ is due to the asymptotic exponential decrease of the Whittaker function $W_{\kappa\lambda}\ (x)$. The behaviour at the origin will determine whether or not $\Psi_\mu^k\ (r)$ is an element of $L^2\ (R^+)$; i.e., the space of square-integrable functions on the positive half-axis. This may be readily analyzed from the last expression in (2.2), since ${}_1F_1\ (a;c;x) \to 1$ when $x \to 0$. (We assume c is not a negative integer.)

Indeed,

$$\Psi_\mu^k(r) \underset{r \to 0^+}{\sim} c_\mu^k \left\{ [\Gamma(1-2k)/\Gamma(1-k-\mu)] r^{2k-1/2} \right.$$

$$\left. + [\Gamma(2k-1)/\Gamma(k-\mu)] r^{-2k+3/2} \right\}\ , \qquad (2.5)$$

where it should be noted that the power behaviour does not depend on μ, but only on k. The function r^α is square-integrable on $(0,a)$, $0 < a < \infty$ when

$$\int_0^a dr\, r^{2\alpha} = (2\alpha+1)^{-1} r^{2\alpha+1} \Big|_0^a < \infty\ ,$$

i.e., when $\mathrm{Re}(2\alpha+1) > 0$, or $\mathrm{Re}\,\alpha > -1/2$. The first summand in (2.5) is thus in $L^2(R^+)$ for $\mathrm{Re}\,k > 0$ while the second is in $L^2\ (R^+)$ for $\mathrm{Re}\,k < 1$. Hence, the strong-well and exceptional intervals (2.4b, c) possess $L^2\ (R^+)$-solutions for all values of k and μ. In the strong-barrier interval (2.4a), however, the second term in (2.2) – (2.5) prevents its square integrabiliy *unless* its coefficient vanishes. This happens when the Γ-function in the denominator is evaluated at zero or at a negative integer,

i.e., for $k - \mu = -n$, $n = 0, 1, 2, \ldots$ This feature yields the quantization of the energy levels of the harmonic oscillator with a strong barrier. For later use, we examine the square-integrability of the *derivative* $\Psi_\mu^{k\prime}$ of (2.5) at $r \to 0^+$, simply substracting 1 from the exponents, and in the case of exponent zero, seeing that $\Psi_\mu^k(0^+)$ is constant. The first summand in $\Psi_\mu^{k\prime}$ is thus in $L^2(R^+)$ for $\mathrm{Re}\, k > 1/2$ and $k = 1/4$, while the second summand has this property for $\mathrm{Re}\, k < 1/2$ and $k = 3/4$. The result is that Hamiltonians in the strong-well interval have *no* $L^2(R^+)$-derivative eigenfunctions, the strong-barrier interval has *only* $L^2(R^+)$-derivative eigenfunctions, while for each k in the exceptional interval ($k \neq 3/4$) it has *one* such eigenfunction, i.e., that which is obtained when $k - \mu = -m$, or $\mu = k + m$, $m = 0, 1, 2, \ldots$; for $k = 3/4$ a second $L^2(R^+)$-derivative eigenfunction exists when the first term vanishes (i.e., when $1 - k - \mu = -m$, or $\mu = 1 - k + m$, $m = 0, 1, 2, \ldots$). The case $k = 1/2$ will require further analysis.

Eigenfunctions of an operator H need not be square-integrable, but only satisfy $H\Psi = 2\mu\Psi$ for some fixed number μ; in that sense, the *second* solution of (2.1) –obtained replacing in (2.2) the $M_{\kappa\lambda}$-functions for the $W_{\kappa\lambda}$ ones– is as good as the first one. Square-integrability becomes important when we want to define a *domain*, a space of functions where H may be *self-adjoint* and there possess a spectrum. If the spectrum consists only of (an infinite number of) isolated points –a point spectrum– then the eigenfunctions will provide a denumerable basis for the domain. Self-adjointness of operators is required by the standard Dirac-von Neumann [8] formulation of quantum mechanics, if these are to be associated with physical observables, and the Hamiltonian H of a system is certainly one. The concept of adjunction of operators is defined when an inner product in the domain is given. The usual sesquilinear inner product in configuration space is the Lebesgue integral of the product of two functions (the first complex conjugate) over the range of the functions:

$$(f, g) = \int_0^\infty dr\, f(r)^*\, g(r) \ .$$

In terms of this inner product H is self-adjoint when $(f, Hg) = (H^\dagger f, g) = (Hf, g)$ and when the domain of $H^\dagger$ is equal to the domain of H. The mathematically rigourous proof of the latter condition is not easy, but the former one is constructive and rather direct for Hamiltonians (2.1b). A well-known consequence of the former condition is that the spectrum of the operator is real, and that eigenfunctions corresponding to different

eigenvalues are orthogonal [1, 8]. We have to examine, thus, the inner products (Ψ_ν^k, Ψ_μ^k) between the eigenfunctions (2.2); if these are not zero for some $\mu \neq \nu$, we must conclude that Ψ_ν^k and Ψ_μ^k do not belong to the same self-adjunction domain of H. The spectrum of H will be given thus by all μ such that $(\Psi_\nu^k, \Psi_\mu^k) = 0$ for some fixed ν, including ν itself. Whether or not the spectrum thus found is unique (i.e., independent of the choice of ν) is a question to which most physicists in the habit of solving physical quantum mechanical problems would answer yes. Perhaps so, but we will show that, for a range of values of the parameter γ in the model Hamiltonian (2.1), the answer is no. This range includes $[0, 3/4)$ which would not ordinarily be regarded as manifestly unphysical.

III. THE HAMILTONIAN SPECTRA

The inner product of two eigenfunctions Ψ_ν^k, Ψ_μ^k, $\mu \neq \nu$ the same operator H [Eqs. (1.1), (1.2) and (2.1)] may be obtained as follows: multiply (2.1) by $\Psi_\nu(r)^*$ and substract the same equation, complex conjugate, with μ and ν exchanged. The result is

$$(\mu - \nu)\Psi_\nu^k(r)^*\Psi_\mu^k(r) = -\frac{1}{2}\frac{d}{dr}\left[\Psi_\nu^k(r)^*\Psi_\mu^{k\prime}(r) - \Psi_\nu^{k\,*\prime}(r)\Psi_\mu^k(r)\right] , \qquad (3.1)$$

where we recall that the eigenvalues are real. Now integrate over R^+ to obtain the inner product (2.6) and express the right-hand member in terms of the Wronskian $W(f, g,) = fg' - f'g$ valuated at the interval endpoints

$$(\Psi_\nu^k, \Psi_\mu^k) = [2(\mu - \nu)]^{-1} W(\Psi_\nu^k(r)^*, \Psi_\mu^k(r))\Big|_{r \to 0^+} \qquad (3.2)$$

Since the Ψ's given in (2.2) vanish at $r \to \infty$, we need only evaluate the Wronskian in the zero limit. Since taking derivatives and the $r \to 0^+$ limit may be done here in either order here, the result (2.5) may be used.

Among the three intervals presented in the last section, consider first the interval $k > 1/2$ [cases (2.4a) and (2.4b)]. Whenever both $\nu = k + n$ and $\mu = k + m$, $n, m = 0, 1, 2, \ldots$, the functions Ψ_ν^k and Ψ_μ^k have the same behaviour $r^{2k - 1/2}$ near the origin [c.f., in Eq. (2.5)], so their Wronskian is zero at $r = 0$, and hence the two functions are orthogonal. The same

situation occurs in the interval $1/2 < k < 1$ [case (2.4b)] when both $\nu = 1 - k + n$ and $\mu = 1 - k + m$, $n, m = 0, 1, 2, \ldots$. The results are thus that:

a) For $k \geqslant 1$ the only square-integrable eigenfunctions of the harmonic oscillator Hamiltonian with a strong ($\gamma \geqslant 3/4$) centrifugal barrier are the $\Psi_\mu^k(r)$ given by the first summand in (2.2). (Normalization will be done in the next section.) The energy spectrum is unique:

$$\mu = k, k + 1, k + 2, \ldots . \tag{3.3a}$$

It is bounded from below and equally spaced.

b) for $1/2 < k < 1$, there exists a set of square-integrable functions $\Psi_\mu^k(r)$, which vanish at the origin, with square-integrable derivative, also given by the first summand in (2.2), whose energy spectrum is (3.3a). There *also* exists a second set of eigenfunctions, given by the *second* summand in (2.2),whose energy spectrum is

$$\mu = 1 - k, 2 - k, 3 - k, \ldots , \tag{3.3b}$$

also bounded from below and equally spaced. For $1/2 < k < 3/4$ they vanish at the origin, while for $k = 3/4$ the derivative vanishes there. With the exception of the case $k = 3/4$, the derivative of these eigenfunctions is *not* square-integrable. [As will be shown later, the value $k = 1/2$ may be included in this case, (3.3a) and (3.3b) being the same.]

The eigenfunction set associated to the spectrum (3.3a) is not orthogonal to the eigenset associated to (3.3b): the first one behaves near the origin as $r^{2k - \frac{1}{2}}$ and the second as $r^{-2k + 3/2}$. From (2.5) we find

$$W(\Psi_{k+n}^{k*}, \Psi_{1-k+m}^{k}) = -2(2k-1)\frac{\Gamma(1-2k)\Gamma(2k-1)}{\Gamma(1-2k-n)\Gamma(2k-1-m)} \tag{3.4}$$

which is different from zero for $1/2 < k < 1$. The spectrum of H, hence, may not be the union of two spectra. For $k = 3/4$, i.e., (2.1) with $\gamma = 0$, we note, we have a "half-harmonic oscillator". The eigenfunction set (3.3a) is the set of odd oscillator eigenstates which vanish at the origin, while (3.3b) is the set of even states whose derivative vanishes there. These are orthogonal on $L^2(R)$ due to parity, but not on $L^2(R^+)$.

When $\gamma < 3/4$ [cases (b) and (c)], we saw that both summands in (2.2)--(2.5) are square-integrable, so the condition (3.3) need not be imposed on the exceptional interval, nor can they be imposed in the strong-well case since μ is real while k is not.

In those cases we may directly evaluate the Wronksian in (3.2) at the $r \to 0^+$ limit using (2.5). The algebra differs somewhat for $k^* = k$ and for $k^* = 1 - k$, but the final result is valid for both and can be written as

$$W(\Psi_\nu^{k*}, \Psi_\nu^k)\Big|_{r \to 0^+} = c_\nu^{k*} c_\mu^k \pi \csc(2\pi k)$$

$$\times \{[\Gamma(k-\nu)\Gamma(1-k-\mu)]^{-1} - [\Gamma(1-k-\nu)\Gamma(k-\mu)]^{-1}\} . \tag{3.5}$$

Barring for the moment the special case $k = 1/2$, the right hand side will be zero only for μ's such that

$$f_k(\mu) = \frac{\Gamma(k-\mu)}{\Gamma(1-k-\mu)} = \frac{\Gamma(k-\nu)}{\Gamma(1-k-\nu)} = \text{constant.} \tag{3.6}$$

We may now pose concretely the question raised at the end of the last section, namely: given one fixed ν which we thus impose to be a point in the spectrum of H, which other μ's will belong to it so that the corresponding Ψ_μ^k's are orthogonal to Ψ_ν^k? Since for this set of μ's $f_k(\mu)$ in (3.6) has the same constant value, the set of Ψ_μ^k's --including Ψ_ν^k-- will be an orthogonal set of vectors, in effect an eigenbasis of H where it has spectrum $\{\mu\}$. As a continuation to point (b) above, we thus consider

b′) For $1/2 < k < 1$, $f_k(\mu)$ is a real function, it has simple poles at the point (3.3a). zeros at the points (3.3b), and is monotonically decreasing between any two poles, as shown in figure 1. A choice of ν yields a single value for $f_k(\nu)$ which, drawn as a line parallel to the μ–axis, gives the values of μ which satisfy (3.6) as the ordinate of its intersection with the function $f_k(\mu)$. Seen with the μ–axis pointing up, these points give the energy spectrum of H in the span of the function set $\{\Psi_\mu^k\}$. It can be shown [1, 9] that the closure of a function set so defined constitutes a Hilbert space, subspace of $L^2(R^+)$, and that the operator with that domain is a *self-adjoint extension* of the formal differential operator H as given in (2.1b). The spectrum has the following characteristics: any real ν which is chosen has its companion μ's. If this ν is different from $k + n$ or

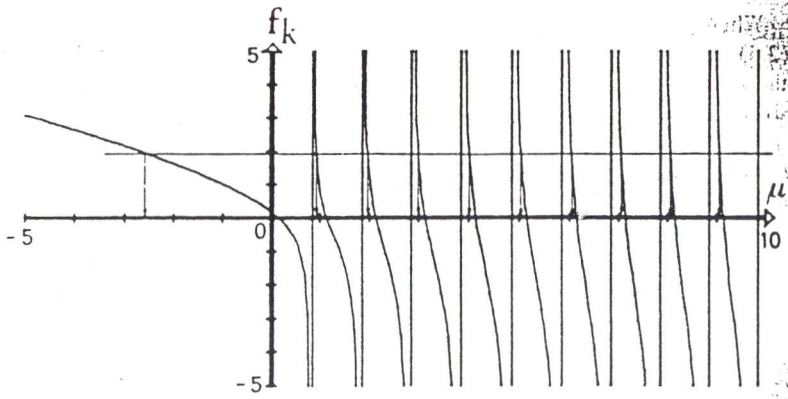

Figure 1 Spectrum of a half-oscillator with a weak centrifugal barrier γ/r^2, for $k = 0.85$, i.e., $\gamma = 0.24$. The self-adjoint extension is chosen through $f_k = 2$. The corresponding spectrum $\{\mu\}$ is found as the solutions of $f_k(\mu) = 2$, and indicated by means of the thin arrows.

$1 - k + n$, $n = 0, 1, 2, \ldots$. Then the spectrum is *not* equally spaced [if μ is a solution to (3.6), $\mu \pm m$ are not solutions], but it does tend asymptotically to equal spacing as μ grows (i.e., the harmonic oscillator envelope remains as the dominating potential for high energies). The spectrum corresponding to any one self-adjoint extension is bounded from below (there are no zeros or finite poles to the left of $\mu = 1 - k$), but the lowest eigenvalue may be chosen as far negative as we please. This analysis includes the choices $\nu = 1 - k$ or $\nu = k$, corresponding to $f_k(\mu)$ zero or infinity.

c) For $k = (1 + i\kappa)/2$, $\kappa \in R^+$, since $k^* = 1 - k$, the range of the function (3.6) is the unit circle in the complex plane. We may nevertheless compute $f_k(\mu)$ and plot its argument modulo 2π as in figure 2. If we choose any fixed real ν, the companion μ's which determine the spectrum and eigenfunction set can be found as before. The spectrum $\{\mu\}$ is again *not* equally spaced, but it tends asymptotically to equal spacing as μ grows. This behaviour may be seen in the relation [10]

$$\arg f_k(\mu + 1) = \arg f_k(\mu) + 2 \arctan [\kappa/(2\mu + 1)] \tag{3.7}$$

which as $\mu \to \infty$ has a harmonically vanishing second summand. Inverting (3.7) for $f_k(\mu - 1)$ as $\mu \to -\infty$ the phase of $f_k(\mu)$ increases ever slower, as the increase of the harmonic series, and does *not* turn by 2π between

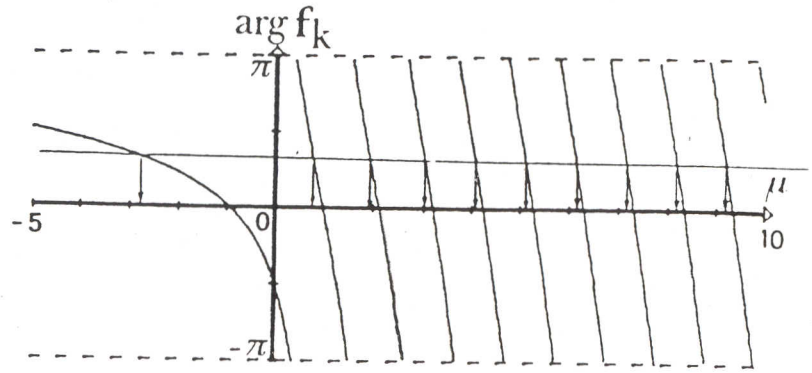

Figure 2 Spectrum of a half-oscillator with a strong centripetal well γ/r^2, for $k = (1 + i)/2$, i.e., $\gamma = -1.25$. Now $|f_k(\mu)| = 1$, and the self-adjoint extension is chosen for $\arg f_k = 60°$. The corresponding spectrum is found as in the first figure.

points separated by integers. For negative μ the spacing thus grows ever larger but the spectrum is not bounded from below.

For every chosen ν we thus have a corresponding spectrum $\{\mu\}$. Again, eigenfunctions belonging to a given spectrum span a space of functions whose closure determines a Hilbert space, and a corresponding self-adjoint extension of the formal differential operator H. Eigenfunctions of different self-adjoint extensions of H are not orthogonal.

IV. NORMALIZATION

We require the eigenfunction sets (2.2) to have unit norm [$\|\Psi_\nu^k\| = (\Psi_\nu^k, \Psi_\nu^k)^{1/2} = 1$] under the inner product (2.6) so that they represent proper quantum-mechanical states. The procedure we follow here avoids the use of integral tables, is apparently novel [11] and will be used again in the next section.

We find the normalization constant c_ν^k in (2.2) calculating the inner product (3.2)-(3.5), writing $\mu = \nu + \epsilon$, expanding all functions of ϵ in Taylor series and retaining terms up to first order. The limits $\epsilon \to 0^\pm$ and $r \to 0^+$ may be exchanged. We thus write

$$1 = \|\Psi_\nu^k\|^2 = \lim_{\epsilon \to 0} (\Psi_\nu^k, \Psi_{\nu+\epsilon}^k) = \lim_{\epsilon \to 0} (2\epsilon)^{-1} W(\Psi_\nu^{k*}, \Psi_{\nu+\epsilon}^k)\Big|_{r \to 0^+}$$

$$= \frac{1}{2}\frac{\partial}{\partial\mu} W(\Psi_\nu^{k*}, \Psi_\mu^k)\Big|_{\substack{r\to 0^+ \\ \mu\to\nu}}$$

$$= |c_\nu^k|^2 \frac{\pi}{2}\csc(2\pi k)\frac{\partial}{\partial\mu}\{[\Gamma(k-\nu)\Gamma(1-k-\mu)]^{-1}$$

$$-[\Gamma(1-k-\nu)\Gamma(k-\mu)]^{-1}\}\Big|_{\mu=\nu} . \quad (4.1)$$

We introduce the digamma or ψ-function defined through [12]

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \Gamma'(x)/\Gamma(x) , \quad (4.2)$$

obtaining

$$c_\nu^k = \phi_\nu^k\left[\frac{\pi}{2}\csc(2\pi k)\frac{\psi(1-k-\nu)-\psi(k-\nu)}{\Gamma(1-k-\nu)\Gamma(k-\nu)}\right]^{-1/2} , \ |\phi_\nu^k| = 1 \quad (4.3)$$

where we have left the possibility of fixing a phase ϕ_ν^k following some convention. The same result may be found through the standard integral tables [13]. The case $k = 1/2$ may be obtained letting $k = (1/2) + \Theta$ with $\Theta \to 0$ from positive real or imaginary values. The result is, for $\nu \neq (1/2) + n$, n integer.

$$c_\nu^{1/2} = \phi_\nu^{1/2}\{\tfrac{1}{2}\psi'(\tfrac{1}{2}-\nu)/[\Gamma(\tfrac{1}{2}-\nu)]^2\}^{-1/2} , \quad (4.4)$$

where the trigamma function $\psi'(x) = d\psi(x)/dx$ appears.

In the exceptional interval $1/2 < k < 1$ the same general expression (4.3) with a positive radicand applies. The special cases $\nu = k + n$, $1 - k + n$, $n = 0, 1, 2, \ldots$ are obtained using

$$\Gamma(-n-\epsilon) \underset{\epsilon\to 0}{\sim} (-1)^{n+1}/n!\epsilon \quad (4.5a)$$

$$\psi(-n-\epsilon) \underset{\epsilon\to 0}{\sim} 1/\epsilon . \quad (4.5b)$$

and the reflection formula [14]

$$\Gamma(z) = \pi\csc(\pi z)/\Gamma(1-z) , \quad (4.5c)$$

for $z = 2k + n$ and $2 - 2k + n$. They are given by

$$c_{k+n}^k = \phi_{k+n}^k[\tfrac{1}{2}n!\Gamma(2k+n)]^{-1/2} , \quad (4.6a)$$

$$c_{1-k+n}^k = \phi_{1-k+n}^k[\tfrac{1}{2}n!\Gamma(2-2k+n)]^{-1/2} , \quad (4.6b)$$

for $n = 0, 1, 2, \ldots$. The case $k = 1/2$ is then covered by

$$c_{1/2+n}^{k=1/2} = \phi_{1/2+n}^{1/2}\sqrt{2}/n! , \quad (4.7)$$

and can also be obtained from (4.4) using (4.5a) and the derivative of (4.5b).

In the $k \geqslant 1$ interval, corresponding to a strong centrifugal barrier, the reasoning (4.1) applies provided we let $r \to 0^+$ last, since $\Psi_{k+n+\epsilon}^k$ is not square-integrable for $\epsilon \neq 0$. The result, for $\nu = k + n$, $n = 0, 1, 2, \ldots$ is also given by (4.6a) –thereby valid for $k \geqslant 1/2^-$ whith a proper positive radicand. Equation (4.6b) does not hold in this range: eigenfunctions are not square-integrable, and this appears in (4.6b) through a negative radicand. When $\nu = k + n$ or $\nu = 1 - k + n$, $n = 0, 1, 2, \ldots$ (for the values of k for which this is applicable) the quantum-mechanical wavefunctions (2.2) may be written in terms of the well-known Laguerre polynomials. Indeed, substituting for the former case (4.6a) in (2.2), using [15] and performing a small amount of algebra we find explicitly

$$\Psi_{k+n}^k(r) = \phi_{k+n}^k(-1)^n[2n!/\Gamma(2k-n)]^{1/2}$$

$$\times e^{-r^2/2}r^{2k-1/2}L_n^{(2k-1)}(r^2) , \quad (4.8a)$$

which can be compared with the standard literature [8], we set thus

$$\phi_{k+n}^k = (-1)^n , \quad (4.8b)$$

for this case of lower bound spectra. As we shall see in the next section, this corresponds to Bargmann's phase convention. In the case $\nu = 1 - k + n$, a similar procedure leads to

$$\Psi^{k}_{1-k+n}(r) = \Psi^{1-k}_{1-k+n}(r) \ , \tag{4.9a}$$

with choice of phase

$$\phi^{k}_{1-k+n} = \phi^{k}_{k+n} = (-1)^n \ . \tag{4.9b}$$

This means that the form (4.8) is valid for $k > 0$ provided we understand that the new range of k, $0 < k < 1/2$, where H has spectrum $\{\mu\}$, $\mu = k + n$. For all other values of k and ν, from (4.3) the normalized wavefunctions are

$$\Psi^{k}_{\nu}(r) =$$

$$\phi^{k}_{\nu}\left[\frac{\pi}{2}\csc(2\pi k)\frac{\psi(1-k-\nu)-\psi(k-\nu)}{\Gamma(1-k-\nu)\Gamma(k-\nu)}\right]^{-1/2} r^{-1/2}W_{\nu,k-1/2}(r^2) \ , \tag{4.10}$$

with the spectrum $\{\nu\}$.

The developments of the preceding sections constitute a straightforward application of the quantum mechanical formalism to a family of potentials $\gamma/r^2 + r^2$ (which is at least partially unphysical for $\gamma < 0$) which presents us with a freedom of choice in the spectrum for $\gamma < 3/4$, for which there is apparently no selection rule. Probably most physicists would select the Hamiltonian self-adjoint extension where the spectrum is $\mu = k + n$, $n = 0, 1, 2, \ldots$ on the grounds that the kinetic energy has then a finite expectation value:

$$(\Psi^{k}_{\mu}, \tfrac{1}{2}P^2\Psi^{k}_{\mu}) = \tfrac{1}{2}(P\Psi^{k}_{\mu}, P\Psi^{k}_{\mu}) < \infty, \quad P = -id/dr \tag{4.11}$$

since —as was observed in section II— these functions have square-integrable derivative. The extension of the operator thus defined is the *Friedrichs* extension [1]. The choice exists for $0 < \gamma < 3/4$ ($3/4 < k < 1$) while for $\gamma = 0$ [$k = 3/4$ and $1/4$ according to the identification (4.9)], 'common sense' would say that to restrict a one-dimensional quantum problem to the $r > 0$ half-axis requires an infinite potential barrier at $r < 0$, so that the eigenfunctions which vanish at the origin [$k = 3/4$,

$\mu = k + n$] constitute the correct choice of self-adjoint extension for the Hamiltonian operator.

V. EQUALLY-SPACED SPECTRA

We would like to present now a two-chart operator closely related to H in (2.1), whose eigenfunctions can be represented by two-component functions, each component given by (2.2), which includes the equally-spaced spectra seen in the last sections for $k \geqslant 1/2$, and whose spectra —for all k— are equally spaced.

The interest in this operator arose in the study of the coupling of two harmonic oscillator Hamiltonians H_1 and H_2 (each with $\gamma = 0$) to a total $H' = H_1 - H_2$. This has been used to find the continuous-series representations [17-19] and the Clebsch-Gordan coefficients of the three-dimensional Lorentz group [20]. It is

$$H' = H_1 - H_2 = \frac{1}{2}\left[-\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + x_1^2 - x_2^2\right] \ , \quad x_1, x_2 \in R \ . \tag{5.1}$$

The minus sign between H_1 and H_2 replaces the plus sign commonly used to construct a two-dimensional oscillator which may be separated in polar coordinates. In the latter the radial part (2.1) ($\gamma = m^2 - 1/4$, $m = 0, 1, 2, \ldots$) is obtained projecting everything on the subspace of states with a definite angular momentum m. In the present case, we reparametrize the R^2 plane x_1, x_2) in two charts of *hyperbolic* coordinates (σ, r, θ):

$$\text{for } x_1 > x_2 \ , \ \sigma = +1, \ x_1 = r\cosh\theta, \ x_2 = r\sinh\theta, \ r, \theta \in R \tag{5.2a}$$

$$\text{for } x_1 < x_2 \ , \quad \sigma = -1, \ x_1 = r\sinh\theta, \ x_2 = r\cosh\theta \ . \tag{5.2b}$$

There are two more one-dimensional charts for $x_1 = x_2$ and $x_1 = -x_2$, but these may be disregarded here. In the $\sigma = +1$ chart, the r = constant line parametrized by θ is one branch of an equilateral hyperbola which crosses the x_1-axis at r. For $r \in R$ this covers the 'inside' of the cone $|x_1| > |x_2|$. In the $\sigma = -1$ chart, r = constant lines are hyperbolae crossing the x_2 axis. The two charts are needed to cover the R^2-plane (excepting the $|x_1| = |x_2|$ lines). The surface element is $dx_1\,dx_2 = r\,dr\,d\theta$ in both charts.

Put in terms of hyperbolic coordinates, the operator (5.1) appears as

$$H' = \frac{\sigma}{2}\left[-\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+r^2\right]$$

$$= r^{1/2}\frac{\sigma}{2}\left[-\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}-\frac{1}{4}\right)+r^{-1/2}\right]. \qquad (5.3)$$

A similarity transformation of all operators and eigenfunctions by $r^{-1/2}$ followed by a projection on the space of eigenfunctions of $-i\partial/\partial\theta$ with eigenvalue λ [i.e., of those functions $f(x_1, x_2) = f[\sigma, r, \theta] = \phi_\sigma(r)\, e^{i\lambda\theta}$ with fixed $\lambda \in R$] produces a Hamiltonian operator of the form (2.1b) with $\gamma = -\lambda^2 - 1/4 \leqslant -1/4$. The resulting operator,

$$\mathrm{H} = r^{-1/2} H' r^{1/2}|_\lambda = \frac{\sigma}{2}\left[-\frac{d^2}{dr^2}+\frac{\gamma}{r^2}+r^2\right] = \sigma H \qquad (5.4)$$

appears to be closely related to H in (2.1b), except for:

a) H is a *two-chart* operator, i.e., $\sigma = +1$ gives its form when acting on functions in the $|x_1| > |x_2|$ hyperbolic coordinate chart, and $\sigma = -1$ when acting in the $|x_1| < |x_2|$ chart. Both forms constitute a single operator which can be equivalently written as a diagonal 2×2 matrix operator acting on two-component functions $\phi_\sigma(r)$. The eigenvalue problem (2.1) thus reads here

$$\mathrm{H}\Psi^k_\mu(r) = 2\mu\Psi^k_\mu(r)\ , \qquad (5.5a)$$

$$\mathrm{H} = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \quad \Psi^k_\mu(r) = \begin{pmatrix} \Psi^k_{+1,\mu}(r) \\ \Psi^k_{-1,\mu}(r) \end{pmatrix}\ . \qquad (5.5b)$$

b) The domain of each function component in [17] was over $r \in R$ rather than $r \in R^+$ as here. In Ref. 17 we further decomposed this function according to parity and restricted thus r to R^+. The price paid was to obtain only spectra where the eigenvalues are integers (even functions) or half-integers (odd functions). In starting with (5.5) on R^+ instead of (5.1) we do away with this restriction.

c) As we saw in (5.4), γ is restricted to $(-\infty, -1/4]$ (the value $\gamma = 1/4$ is absent if the eigenvalues μ are half-integers). The "coupling" construction (5.1)-(5.5) does not yield values of γ in the exceptional interval. Again, we may pose (5.5) *ab initio* with no restriction on γ.

To conform with (a) and (b), the inner product between the two-component functions $\mathrm{f}(r) = \{f_{+1}(r), f_{-1}(r)\}$ is defined as

$$(\mathrm{f}, \mathrm{g}) = \int_0^\infty dr\, \mathrm{f}(r)^\dagger\, \mathrm{g}(r) = \sum_{\sigma=\pm1}\int_0^\infty dr\, f_\sigma(r)^*\, g_\sigma(r)\ . \qquad (5.6)$$

The inner product, in turn, defines a Hilbert space where we require the eigenfunctions of H to lie, and H to have one or more self-adjoint extensions.

To this end, most of the groundwork has been developed in the former sections. The solutions of the eigenfunction equation (5.5) are the same as those for (2.1) –except for the minus sign in the lower component. Thus

$$\Psi^k_{\sigma,\mu}(r) = c^k_{\sigma,\mu} r^{-1/2} W_{\sigma\mu,\, k-1/2}(r^2)\ , \qquad (5.7)$$

where the search for the new constants $c^k_{\sigma,\mu}$ will be done below. The analysis of the square-integrability properties under (5.6) proceeds as before. For $\gamma \geqslant 3/4$ there is one family of square-integrable solutions with positive spectrum: $\mu = k + n$, $n = 0, 1, 2, \ldots$ for $k \geqslant 1$ which *has only an upper component*, so $c^k_{-1,\mu} = 0$, since the lower component is not square-integrable. There is also one family of negative-spectrum solutions $\mu = -k - n$, $n = 0, 1, 2, \ldots$ with $c^k_{+1,\mu} = 0$ which *has only a lower component*. These are the only spectra which depend on square-integrability properties.

When $\gamma < 3/4$ we must use the analogue of Eqs. (3.2)-(3.5) for the new inner product (5.6), i.e., we must additionally sum over σ noting that the $\sigma = +1$ and $\sigma = -1$ components have different eigenvalue signs:

$$(\Psi^k_\nu, \Psi^k_\mu) = [2(\mu-\nu)]^{-1}\left[W(\Psi^{k*}_{+1,\nu}, \Psi^k_{+1,\mu}) - W(\Psi^{k*}_{-1,\nu}, \Psi^k_{-1,\mu})\right]\Big|_{r\to 0^+}$$

$$= [2(\mu-\nu)]^{-1}\pi\csc(2\pi k)\left[\frac{c^{k*}_{+1,\nu}c^k_{+1,\mu}}{\Gamma(k-\nu)\Gamma(1-k-\mu)} - \frac{c^{k*}_{-1,\nu}c^k_{-1,\mu}}{\Gamma(k+\nu)\Gamma(1-k+\mu)}\right.$$

$$\left. - \frac{c^{k*}_{+1,\nu}c^k_{+1,\mu}}{\Gamma(1-k-\nu)\Gamma(k-\mu)} + \frac{c^{k*}_{-1,\nu}c^k_{-1,\mu}}{\Gamma(1-k+\nu)\Gamma(k+\mu)}\right] \qquad (5.8)$$

The spectrum for the one-component operator H was found cancelling the first and third terms. Here we have more possibilities, chief among which is the cancellation of the first with the second, and the third with the fourth terms. The two resulting equalities can be made to have one common term involving the c^k's:

$$\frac{c^{*k}_{+1,\nu}c^{k}_{+1,\mu}}{c^{*k}_{-1,\nu}c^{k}_{-1,\mu}} = \frac{\Gamma(k-\nu)\Gamma(1-k-\mu)}{\Gamma(k+\nu)\Gamma(1-k+\mu)} = \frac{\Gamma(1-k-\nu)\Gamma(k-\mu)}{\Gamma(1-k+\nu)\Gamma(k+\mu)} \quad . \tag{5.9a}$$

The second equality in the above formula will be the condition which the μ's should satisfy –given a fixed ν– in order that (5.8) vanish, and the corresponding $\{\Psi^k_\mu\}$ form an orthogonal eigenfunction set. This transcendental equation is rather simple, however: using the reflection formula (4.5c) it may be brought to the form

$$\frac{\sin\pi(k-\mu)}{\sin\pi(k+\mu)} = \frac{\sin\pi(k-\nu)}{\sin\pi(k+\nu)} \tag{5.9b}$$

whose roots are given by

$$\mu = \nu + n \ , \qquad n \text{ integer}, \tag{5.10}$$

i.e., they yield an equally-spaced, unbounded spectrum. The square root of the product of the two last members of (5.9a) factors into two identical functions, one of ν and the other of μ. This yields the ratio

$$\frac{c^k_{+1,\mu}}{c^k_{-1,\mu}} = \left[\frac{\Gamma(k-\mu)\Gamma(1-k-\mu)}{\Gamma(k+\mu)\Gamma(1-k+\mu)}\right]^{1/2} \tag{5.11}$$

–up to a phase– between the coefficients of the two components of Ψ^k_μ. This ratio is here obtained as an output, a consequence of assuming that the two neighbouring pairs of terms in (5.8) cancel. One may conversely assume the ratio $c^k_{+1,\mu}/c^k_{-1,\mu}$ to be some function $\rho(k,\mu)$, extracting thus $c^{*k}_{+1,\nu}c^k_{+1,\mu}$ from the brackets in (5.8) and, requiring this bracket to vanish for fixed ν, thence find the values of ν which constitute the spectrum. The two-component Hamiltonian operator (5.5) is thus considerably richer in self-adjoint extensions than its one-component

counterpart (2.1) examined in sections II-IV. Our interest, however, lies in the equally-spaced spectrum (5.10) derived from the ratio function (5.11). We may thus set

$$c^k_{\sigma,\mu} = \phi^k_{\sigma,\mu}[\Gamma(k-\sigma\mu)\Gamma(1-k-\sigma\mu)]^{1/2}d^k_\mu \ , |\phi_{\sigma,\mu}| = 1 \tag{5.12}$$

where as before we leave the possibility of a phase $\phi^k_{\sigma,\mu}$. We now inquire into the proper normalization constant d^k_μ. This may be obtained through the same process as that followed in section IV, replacing the inner product in (4.1) by that in (5.8)-(5.12). We use (4.2) to differentiate the Γ-functions with respect to μ; the four distinct ψ functions which appear combine, under the logarithmic derivative of (4.5c)

$$\psi(1-z) - \psi(z) = \pi\cot\pi z \ , \tag{5.13}$$

into trigonometric functions. The result is

$$d^k_\nu = \pi^{-1}[2\sin\pi(k-\nu)\sin\pi(k+\nu)]^{1/2} \ . \tag{5.14}$$

The normalized components of the square-integrable eigenfunctions Ψ^k_μ of the operator **H**, in the domain where it has the equally-spaced spectrum (5.10), may be thus written from (4.5c)-(5.7), (5.12), and (5.14), as

$$\Psi^{k,\epsilon}_{\sigma,\nu}(r) = \phi^k_{\sigma,\nu}[\tfrac{1}{2}\Gamma(k+\sigma\nu)\Gamma(1-k+\sigma\nu)]^{-1/2}r^{-1/2}W_{\sigma\nu,k-1/2}(r^2) \ , \tag{5.15a}$$

$$\nu = \epsilon + n \ , \quad -1/2 < \epsilon \leqslant 1/2 \ , \quad n \text{ integer} \ . \tag{5.15b}$$

The parameter $\epsilon \equiv \nu \pmod 1 \in (-1/2, 1/2]$ is used to label the self-adjoint extension of the operator **H**. As it appears here, it is identical with Bargmann's label ϵ which distinguishes between single ($\epsilon = 0$) and double-valued ($\epsilon = 1/2$) representations of the three-dimensional Lorentz group, and whose extended range $\epsilon \in (-1/2, 1/2]$ labels representations of the covering of that group. The "proper" choice of phase $\phi^k_{\sigma,\nu}$ will be made in the appendix. See Eq. (A.7).

We now proceed to examine the information contained in the normalization factor over the full range of γ or k. The radicand is real, notice, for

either k real or $k^* = 1 - k$; in the latter case it is always positive, although in the former it may be negative.

a) For $\gamma \geq 3/4$ ($k \geq 1$) square-integrability restricted the spectrum to $\nu = k + n$ or to $\nu = -k - n$, n integer. In the former case $\Psi^k_{-1,\nu}(r) = 0$ and Ψ^k_ν has only an upper component; in the latter it has only a lower component. This was observed before.

b) For $-1/4 < \gamma < 3/4$ ($1/2 < k < 1$), when $\nu = k + n$, $1 - k + n$ ($|\epsilon| = 1 - k$) or $\nu = -k - n$, $-(1 - k) - n$, $n = 0, 1, 2$ ... the same statements as in (a) hold true, including the case $\gamma = -1/4$ ($k = 1/2$). In the general case, however, the radicand in (5.15) may be negative. Since the sign of $\Gamma(x)$ for $-N < x < -N + 1$ ($N = 1, 2, ...$), is $(-1)^N$, one can show that the radicand will be positive for all integer n *only* when $|\epsilon| < 1 - k$. If this condition is not satisfied, then there will be values $|\nu| \in (1 - k, k)$ for which the radicand is negative. The latter present in principle no problem since wavefunctions are defined up to a phase. In group representation theory, however, (see the appendix) this means that the raising or lowering operators [(A.4), (A.5)] cannot be expected to be representable in the H eigenbasis by matrices with purely real entries. The restriction $|\epsilon| < 1 - k$ for self-adjoint (unitary) representations of the Lorentz algebra (group) is well known [7]. The $|\epsilon| = 1 - k$ boundary of this region corresponds to the one-component functions seen above.

c) For $\gamma \leq 1/4$ [$k = (1 + i\kappa)/2$, $\kappa \in R^+$] the radicand is always real and positive. The only special case to consider is $k = 1/2$, $\epsilon = 1/2$, where we fall back on the one-component function case: only upper component for $\nu = 1/2, 3/2, ...$ and only lower component for $\nu = -1/2, -3/2, ...$

VI. CONCLUSION

We have examined in some detail the harmonic oscillator with a centrifugal barrier or with a centripetal well, and seen that the spectrum is not unique in the latter case, as well as for an exceptional interval which covers part of the parameter range of the former. Equal spacing of eigenvalues exists for the range of the centrifugal strength parameter which includes weakly atractive wells. The consideration of a closely related two-component equation has produced equally-spaced spectra for the full range of the barrier/well parameter. Such two-or more-component eigenfunction equations have a richer set of solutions. One finds these equations when performing changes of variables through transformations which are not global and where one must not disconnect the charts.

Another family of analogous cases which the author has considered elsewhere [19] has been in finding the Clebsch-Gordan coefficients for the

Lorentz group. There, the formal operators H are the Schrödinger Hamiltonians for the Pöschl-Teller potentials of the first type [$V_1(r) \sim \gamma_1 \csc^2\theta + \gamma_2 \sec^2\theta$, $\theta \in (0, \pi/2)$] and/or of the second type [$V_2(r) \sim \gamma_1 \operatorname{csch}^2\theta + \gamma_2 \operatorname{sech}^2\theta$, $\theta \in (0, \infty)$]. These were combined into one, two or three-chart operators with a corresponding inner product in suitable domains. The eigenvalue set yields essentially the Clebsch-Gordan series, while the wave functions are closely related to the Clebsch-Gordan coefficients.

In these examples, multicomponent Schrödinger equations are not unphysical; rather, they solve a mathematical problem in terms of quantities which may be associated in the physicist's interpretative mind with quantum mechanical observables or other familiar objects.

APPENDIX

The choice of the phase $\phi^k_{\sigma,\nu}$ for the wavefunction components (5.15) may be considered as irrelevant for quantum mechanics. In group theory, however, phase conventions should not to be disregarded for bases of irreducible representations. What follows is the explicit presentation of the group theoretic framework and of Bargmann's phase convention [16]. On the basis of the operators (2.1)–(5.5) we build

$$\mathbf{J}_0 = \begin{pmatrix} J_0 & 0 \\ 0 & -J_0 \end{pmatrix}, \quad J_0 = \frac{1}{2}H = \frac{1}{4}\left[-\frac{d^2}{dr^2} + \frac{\gamma}{r^2} + r^2\right], \tag{A.1a}$$

$$J_1 = \begin{pmatrix} J_1 & 0 \\ 0 & -J_1 \end{pmatrix}, \quad J_1 = J_0 - \frac{1}{2}r^2 = \frac{1}{4}\left[-\frac{d^2}{dr^2} + \frac{\gamma}{r^2} - r^2\right], \tag{A.1b}$$

$$\mathbf{J}_2 = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}, \quad J_2 = -\frac{i}{2}\left[r\frac{d}{dr} + 1/2\right]. \tag{A.1c}$$

These formal operators exhibit the comutation relations

$$[J_0, J_1] = iJ_2, \quad [J_1, J_2] = -iJ_0, \quad [J_2, J_0] = iJ_1 \tag{A.2}$$

valid for the diagonal entries as well as for the full $\mathbf{J}_k$, which define the $SO(2, 1)$ Lorentz algebra.

The Casimir invariant is a multiple of the unit operator:

$$Q = (J_1)^2 + (J_2)^2 - (J_0)^2 = q1 \ , \tag{A.3a}$$

$$q = -\frac{i}{4}\gamma + \frac{3}{16} = k(1-k) \ . \tag{A.3b}$$

The *raising and lowering operators* are defined as

$$J_+ = J_1 + iJ_2 = \begin{pmatrix} J_+ & 0 \\ 0 & -J_- \end{pmatrix} .$$

$$J_- = J_1 - iJ_2 = \begin{pmatrix} J_- & 0 \\ 0 & J_+ \end{pmatrix} , \quad J_\pm = J_1 \pm iJ_2 \ . \tag{A.4}$$

One verifies that, since $[J_0, J_\pm] = \pm J_\pm$ from (A.2), $J_\pm$ will raise or lower to $\Psi^k_{\mu\pm1}$ or $\Psi^k_{\mu-1}$, the eigenvector Ψ^k_μ of J_0 corresponding to eigenvalue μ. Through the usual Racah algebra arguments [21], one shows that if a given Ψ^k is a normalized function under inner product [(5.6) in our case], and if $\Psi^k_{\mu\pm1}$ are normalized too, then

$$J_+\Psi^k_\mu = \theta_\mu[k(1-k) + \mu(\mu+1)]^{1/2}\Psi^k_{\mu+1} \ , \quad |\theta_\mu| = 1 \ , \tag{A.5a}$$

$$J_-\Psi^k_\mu = \theta^{-1}_{\mu-1}[k(1-k) + \mu(\mu-1)]^{1/2}\Psi^k_{\mu-1} \ , \tag{A.5b}$$

up an indetermined phase θ_μ. Bargmann's phase convention [16] is $\theta_\mu = 1$, in order that $J_\pm$ have only nonnegative matrix elements.

Equations (A.5) display the essence of our interest in equally-spaced spectra. When the function domain of J_0 is such that its spectrum is equally spaced and simple, this domain is invariant and irreducible under $J_\pm$, and hence under the three $so(2,1)$ algebra generators (A.1). This domain then defines a common self-adjoint extension *for all algebra generators, and serves as an irreducible representation space of the* $so(2,1)$ algebra where the three generators (A.1) are self-adjoint. Through the exponential map, the same domain serves as a representation space for a unitary irreducible representation of the universal covering group $SO(2,1)$.

The application of differential operators $J_\pm$ [Eqs. (A.1)-(A.4)] to $\Psi^{k,\epsilon}_{\sigma,\mu}(r)$ [Eq. (5.15)] is easily accomplished using the raising and lowering relations for the Whittaker functions [22]:

$$(z\frac{d}{dz} - \frac{1}{2}z + \mu)W_{\mu,k-1/2}(z) = -W_{\mu+1,k-1/2}(z) \quad z = r^2 \tag{A.6a}$$

$$(z\frac{d}{dz} + \frac{1}{2}z - \mu)W_{\mu,k-1/2}(z) = -(k-\mu)(k+\mu-1)W_{\mu-1,k-1/2}(z) \ . \tag{A.6b}$$

The resulting normalization coefficient ratios $c^k_{+1,\mu}/c^k_{+1,\mu+1}$ and $-(k+\mu)(k-\mu-1)\,c^k_{-1,\mu}/c^k_{-1,\mu+1}$ match the coefficient in (A.5a) with $-\phi^k_{+1,\mu}/\phi^k_{+1,\mu+1} = \theta_\mu = \phi^k_{-1,\mu}/\phi^k_{-1,\mu+1}$, the second equality being a condition on the relative phases of the two components of Ψ^k_μ. Similarly, the ratios $(k-\mu)(k+\mu-1)c^k_{+1,\mu}/c^k_{+1,\mu-1}$ and $c^k_{-1,\mu}/c^k_{-1,\mu-1}$ match the coefficient in (A.5b) with $-\phi^k_{+1,\mu}/\phi^k_{+1,\mu-1} = \theta^{-1}_{\mu-1} = \phi^k_{-1,\mu}/\phi^k_{-1,\mu-1}$, which is satisfied if the first relation is. Bargmann's convention sets $\theta_\mu = 1$. So (up to a truly irrelevant overall phase ϕ) we choose

$$\phi^{k,\epsilon}_{+1,\mu} = \phi(-1)^{\mu-\epsilon}, \ \phi^{k,\epsilon}_{-1,\mu} = \phi \ , \tag{A.7}$$

and we can set $\phi = 1$.

This choice of phases satisfies (A.5) for the unbounded-spectrum cases, and includes the lower-bound spectrum cases seen in section IV contained in the upper component of the two-component function Ψ^k_μ. Indeed, the label ϵ in the lower-bound cases is given the meaning of the lowest eigenvalue, i.e., $\epsilon = k$ in (4.8) and $\epsilon = 1-k$ in (4.9), the two phase choices are contained in (A.7).

Acknowledgements Much of the material presented here was found and used in conjunction with Dr Debabrata Basu, from the Indian Institute of Technology (Kharagpur, India), during his stay in Mexico under CONACyT contract No. ICCBIND 790470, in the course of developing the results of Ref. [19]. The author would also like to thank Professor José Adolfo de Azcárraga for the hospitality extended at the Departamento de Física Teórica, Universidad de Valencia, Burjasot, Valencia, where this work was written.

REFERENCES

1. See e.g., the series of volumes by M Reed and B Simon, *Methods of Modern Mathematical Physics* (Academic Press, New York, 1972).
2. BG Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974), Chapters 18 and 20.
3. FWJ Olver *Asymptotics and Special Functions* (Academic Press, New York, 1974), Sects. 6.1 and 7.10.
4. A Erdélyi, W Magnus, F Oberhettinger, and FG Tricomi, *Higher Transcendental Functions*, Vol I (McGraw-Hill, New York, 1953), Sects. 61.-6.9.

5. M Abramowitz and IA Stegun, (Eds) *Handbook of Mathematical Functions* (Dover Publ., New York, 1974). Sect. 13.1
6. Ref. 5, Eqs. 13.1.32 and 13.1.34.
7. V Bargmann, *Ann Math* 48, 568 (1947).
8. A Messiah, *Quantum Mechanics* (Wiley, New York, 1962).
9. K Meetz, *Nuovo Cimento* 34, 690 (1964).
10. Ref. 5, Eq. 6.1.24.
11. The author is indebted to Dr Debabrata Basu for bringing this method to his attention.
12. Ref. 5, Eq. 6.3.1.
13. IS Gradshteyn and IMRyzhik, *Tables of Integrals, Sums and Products* (Academic Press, New York, 1965), Eq. 7.611.4
14. Ref. 5, Eq. 6.1.17.
15. Ref. 5, Eq. 13.6.9.
16. Ref. 7, Eqs. (6.23) and (7.10). The phase factors are chosen so that the raising and lowering operators, Eq. (6.26), have only nonnegative matrix elements.
17. N Mukunda and B Radhakrishnan, *J Math Phys* 13, 254 (1973).
18. KB Wolf, *J Math Phys* 21, 680 (1980), Eqs. (2.6c) and (4.13).
19. D Basu and KB Wolf, J Math. Phys. 23, 189 (1982). Eqs. (2.16c) and (4.5a).
20. D Basu and KB Wolf "The Clebsch-Gordan Coefficients of the Three-Dimensional Lorentz Algebra in the Parabolic Basis", *Comunicaciones Técnicas IIMAS* No. 268 (1981), to appear in *J Math Pys*, Eqs. (4.14c) and (5.7). to *J Math Phys*, Eqs. (4.14c) and (5.7).
21. See e.g., KB Wolf, "Topics in Noncompact Group Representations", in *Group Theoretical Methods in Physics, Latin American School of Physics, 1980*, TH Seligman (Ed.) (American Institute of Physics Lecture Notes No. 75, 1981).
22. Ref. 5, Eqs. 13.4.31 and 13.4.33.

RESUMEN*

Examinamos el espectro y las funciones propias del oscilador armónico cuántico en una barrera centrífuga y con un pozo centrípeto. Hay tres intervalos en el rango de intensidad del parámetro centrífugo-centrípeto que sean relevantes: en uno el espectro es único, acotado por debajo, e igualmente espaciado, en el segundo esté acotado por debajo pero no es único y --excepto por dos casos-- no está igualmente espaciado, mientras que en el tercero no tiene ninguna de las tres propiedades. Un hamiltoniano asociado de dos componentes tiene un espectro que presenta un espaciamiento igual en todo el rango del parámetro.

* Traducido del inglés por la Redacción.

Kinam, Vol 3, 347-363 (1981)

Simple variational approach to the one-dimensional Schrödinger equation

G Bozzolo* and A Plastino**

Facultad de Ciencias Exactas
Universidad Nacional de La Plata
C.C. 67, La Plata (1900), Argentina

(Recibido el 1 de octubre de 1981)

An approximate, variational method for the study of the eigensolutions of the Hamiltonian $H = p^2/2m + U(x)$ is presented, $U(x)$ representing a broad class of even functions of the coordinate which admits a series expansion. The idea of the method is to relate in a simple way the eigenstates of H to those of the harmonic oscillator, by means of a unitary operator $\exp(iF)$ which is determined by the variational principle. Several applications are discussed.

I. INTRODUCTION

It is well known that the Schrödinger equation, even in the one-particle, one-dimensional case, rarely possesses an exact solution. A tremendous amount of work has consequently been devoted, over several decades, in order to develop, analyze, and apply different approximation schemes.

Investigation of the eigensolutions of even the simplest (one-dimensional) Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + U(\hat{x}) \tag{1.1}$$

may prove quite interesting, due to the prevailing belief that, in many instances, the nature of those solutions may lead to a fuller understanding of equivalent models in field theory [1]. In other cases, the eigensolutions of (1.1) may be of particular relevance in molecular physics or in quantum chemistry [2].

*Fellow of CICBA, Argentina
**Member of CONICET, Argentina