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HARMONIC ANALYSIS ON BILATERAL CLASSES*

KURT BERNARDO WOLF† AND THOMAS H. SELIGMAN‡

Abstract. The theory of harmonic analysis over coset and conjugation class spaces in groups is generalized to functions over the space of *bilateral classes*. The latter are novel equivalence sets which include the above as particular cases. The relevant orthogonal function bases are *partial traces*. The standard harmonic functions and characters are recovered as special examples.

1. Introduction. Equivalence classes of group elements are among the main objects of study not only of group theory per se, but of any branch of mathematical physics which requires homogeneous spaces for group action. Closely related to these is the theory of group representations and the associated harmonic analysis. All textbooks on this matter introduce the concepts of cosets and of conjugation classes, and the ensuing developments of harmonic functions and characters are ubiquitous throughout the literature. It is thus perhaps surprising that a generalization of these two cases of equivalence classes can be defined rather naturally, and certain consequences drawn which somehow seem to have escaped notice by several generations of thorough workers in this field.

The need for a more general classification of group elements in equivalence classes exhibiting a certain correlation between the right and left group action arose originally in applied studies in quantum chemistry [3]. They concerned the classification of transition amplitudes between certain molecules called permutational isomers, which differ only in the way in which the ligands are distributed on the skeleton. The mathematical meaning and subsequent construction of what are now called *bilateral classes* were explored shortly thereafter, and appeared in condensed form in [4]. A more complete discussion is given in [5]; some of the results of this paper were briefly summarized in [7].

As the construction of bilateral classes is not yet widely known, we shall restate the relevant points in § 2, stressing certain particular cases. Section 3 sets up the notation and the needed subgroup reduction adapted to the most general bilateral class partition such that functions of this space can be subject to a reduced harmonic analysis. The complete and orthogonal basis function set, which we call *partial traces*, is then constructed in § 4. In the two traditional cases of cosets and conjugation classes they reduce to the well-known spherical functions and characters. In § 5 we offer some concluding remarks.

2. Short survey of bilateral classes. The elements of a group G can be partitioned into a complete family of disjoint sets through an equivalence relation. Equivalence relations with group theoretical significance which have been fruitfully exploited are those which lead to left, right or double cosets, or conjugation subclasses: if H and K are subgroups of G, $g \in G$, $h \in H$, $k \in K$, then the above relations are $g' \sim g$ iff there exist h, k, such that, respectively, $g' = hg$, $g' = gk$, $g' = hgk$ or $g' = hgh^{-1}$.

A generalization of the above relations with group-theoretical definition makes use of the following construction.

(a) Let $(g_1, g_2) \in G \times G$, and consider the action of this group on the elements $g \in G$

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† Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, México 20 D.F., Mexico.

‡ Instituto de Física, Universidad Nacional Autónoma de México, México 20 D.F., Mexico.

given by

$$g \xrightarrow{(g_1, g_2)} g' = g_1 g g_2^{-1}. \tag{1}$$

(b) Select a subgroup $P \subset G \times G$ and introduce the following equivalence relation between the elements of G:

$$g' \overset{P}{\sim} g \Leftrightarrow \exists (g_1, g_2) \in P | g' = g_1 g g_2^{-1}. \tag{2}$$

The properties of symmetry, reflexivity, and transitivity hold for $\overset{P}{\sim}$, since P is a group. The equivalence relation $\overset{P}{\sim}$ thus partitions G into a complete family of disjoint sets which we call bilateral classes or, more specifically, *P-bilateral classes*. The bilateral class containing an element $g_i \in G$ is the set

$$B_i^P = \{g_1 g_i g_2^{-1}, (g_1, g_2) \in P \subset G \times G\}. \tag{3}$$

The exploration of all possible subgroups P of $G \times G$ (sometimes called *subdirect* products of G with G) was undertaken by Goursat [1], who showed that the relevant structure is described by the quintuplet

$$P\{\hat{H}, H; \varphi; K, \hat{K}\},$$

where $\hat{H} \lhd H \subset G \supset K \rhd \hat{K}$, ($H$ and K are subgroups of G, and $\hat{H}$ and $\hat{K}$ are normal subgroups, respectively, of H and K), and where $\varphi : H/\hat{H} \to K/\hat{K}$ is an isomorphism which correlates the factor groups.

The elements of $H/\hat{H}$ and $K/\hat{K}$ are sets of elements in G. Out of each one of these we can choose a representative $\dot{h} \in \mathscr{H}$ and $\dot{k} \in \mathscr{K}$, where $\mathscr{H}$ and $\mathscr{K}$ are sets of elements in G which need not form a group. Different representatives $\dot{h}$, $\dot{k}$ of the same element of the factor groups may be obtained through multiplication by elements of the normal groups. We subduce from the isomorphism φ a one-to-one mapping $\varphi : \mathscr{H} \to \mathscr{K}$ which by abuse we denote through the same symbol as $\varphi(\dot{h}) = \dot{k}$. The equivalence relations (2) may then be presented in more detail as,

$$g' \overset{P}{\sim} g \Leftrightarrow \exists \hat{h} \in \hat{H},\ \dot{h} \in \mathscr{H},\ \hat{k} \in \hat{K} | g' = \hat{h}\dot{h} g \varphi(\dot{h})^{-1} \hat{k}. \tag{4}$$

The bilateral class (3) of an element g_i is then characterized as the set

$$B_i^P = \hat{H}\dot{h} g_i \varphi(\dot{h})^{-1} \hat{K}, \qquad \dot{h} \in \mathscr{H}. \tag{5}$$

The number of elements of P is $|P| = |\hat{H}|\,|\hat{K}|\,|H/\hat{H}|$. If P_i is the stability group P of a given element g_i, then the number of elements of B_i^P is $|B_i^P| = |P|/|P_i|$. Of course, $|P_i|$ divides $|P|$, and $|P|$ divides $|G|^2$.

As particular cases of the P-bilateral classes we have the following classical ones. Left cosets of G by H are determined by $P = H \times e \simeq H$ given through $P\{H, H; —; e, e\}$, where e is the group identity in G and the dash indicates that the correlation function is trivial. Right cosets by K are characterized by $P = e \times K \simeq K$ through $P\{e, e; —; K, K\}$, and double cosets by $P = H \times K$ through $P\{H, H; —; K, K\}$; i.e., the left and right factors in (1) are uncorrelated. Finally, conjugation subclasses of G by H correspond to $P = (H \times H)_D \simeq H$ given though $P\{e, H; \varphi_e; H, e\}$, where φ_e is the identity isomorphism in H; i.e., the right and left factors in (1) are totally correlated.

A particular case which is of practical importance occurs when the group H *splits*, i.e., when it is a semidirect product $H = \hat{H} \wedge \dot{H}$, $\hat{H} \lhd H \supset \dot{H}$. In that case $\mathscr{H}$ may be identified with $\dot{H}$, because a set of representatives exists whose elements form by themselves a subgroup of G. When $H = \hat{H} \wedge \dot{H}$ and $K = \hat{K} \wedge \dot{K}$, every element of these groups can be decomposed uniquely as $h = \hat{h}\dot{h} = \dot{h}\hat{h}'$ and $k = \hat{k}\dot{k} = \dot{k}\hat{k}'$. In this case $\varphi : \dot{H} \to \dot{K}$ is an isomorphism between subgroups of G.

Bilateral classes defined by $P\{\hat{H}, H; \varphi; K, \hat{K}\}$, where H and K split, will be related now to bilateral classes defined through $P\{e, \dot{H}; \varphi; \dot{K}, e\}$, where $\dot{K} = \varphi(\dot{H})$, and to double cosets defined through $P\{\hat{H}, \hat{H}; —; \hat{K}, \hat{K}\}$. For the former, $P \simeq \dot{H}$ and

(6) $$g' \overset{P}{\sim} g \Leftrightarrow \exists \dot{h} \in \dot{H} | g' = \dot{h} g \varphi(\dot{h})^{-1}.$$

The set containing g_i,

(7) $$C_i^\varphi = \dot{H} g_i \varphi(\dot{H})^{-1},$$

will be called a *φ-twisted subclass*. A bilateral class $P\{\hat{H}, H; \varphi; K, \hat{K}\}$ in this case is the union of an entire number of φ-twisted subclasses $P\{e, \dot{H}; \varphi; \dot{K}, e\}$ (as in (7)) whose representatives g_i are subject to the double-coset equivalence relation defined by $P\{\hat{H}, \hat{H}; —; \hat{K}, \hat{K}\}$. Conversely, the same bilateral class consists of an entire number of the latter double cosets whose representatives are subject to the φ-twisted subclass equivalence relation (6). Note that whereas in general a bilateral class always consists of a direct union of double cosets $P\{\hat{H}, \hat{H}; —; \hat{K}, \hat{K}\}$, the decomposition into φ-twisted subclasses occurs only when H and K split.

There are two main types of φ-twisted subclasses $P\{e, \dot{H}; \varphi; \dot{K}, e\}$: those which are defined by conjugation automorphisms $\varphi_l(\dot{h}) = l\dot{h}l^{-1}$, and those which are not. The former may be extended from $\dot{H}$ to the whole of G, and further classified into those which are inner to $\dot{H}$ (i.e., $l \in \dot{H}$), and those which are outer to $\dot{H}$ but inner to G (i.e., $l \notin \dot{H}$ but $l \in G$). The following result allows us to recognize φ-twisted subclasses defined by conjugation automorphisms, and to determine all the possible extensions to G:

A φ-twisted subclass partition of G by a subgroup $\dot{H}$ stems from a conjugation automorphism if and only if there exist classes consisting of a single group element. The φ-twist is then induced by an element $l \in G$, where l^{-1} is any of the one-element classes.

In order to prove this statement, consider the centralizer Z of $\dot{H}$ in G, i.e., $z\dot{h} = \dot{h}z$ for all $z \in Z$, $\dot{h} \in \dot{H}$. The set Z contains at least e. Then, every element $z \in Z$ is an (untwisted φ_e) conjugation subclass of G by $\dot{H}$, and every zl^{-1} (for fixed $l \in G$) will be a φ_l-twisted conjugation subclass. Conversely, let $l^{-1} \in G$ be a single-element φ-twisted subclass of G by $\dot{H}$, i.e., $l^{-1} = \dot{h}l^{-1}\varphi(\dot{h})^{-1}$ for all $\dot{h} \in \dot{H}$; it follows that the automorphism φ is given by $\varphi(\dot{h}) = l\dot{h}l^{-1}$. If we replace l by any lz, $z \in Z$, the new automorphism will coincide on $\dot{H}$ with the original one, but its extension to G will be different for elements of Z which are not in the center of the latter.

It is also evident that if $C_i^{\varphi_e}$ is an ordinary (i.e., untwisted) subclass, then for any conjugation automirphism φ_l the corresponding φ_l-twisted subclass will be $C_i^{\varphi_l} = C_i^{\varphi_e} l^{-1}$; i.e., $g \overset{P(\varphi_l)}{\sim} g' \Leftrightarrow gl \overset{P(\varphi_e)}{\sim} g'l$. The φ_l-twisted partition of G is thus simply a right translate by l^{-1} of the untwisted conjugation class partition.

If the automorphism φ defining $P \subset G \times G$ is outer to G, the above arguments no longer hold.

3. Representations and subgroup adaptation. In describing harmonic analysis in this article, we shall consider only the case when G is a discrete, finite group of $|G|$ elements. This is done in order to keep our considerations as simple as possible, without involving ourselves with the definitions of Haar and Plancherel measures. The structure of the results in this and the following section, however, will point to a straightforward generalization to compact Lie groups and, provided sufficient knowledge is available about the Plancherel measures, to locally compact Lie groups as well.

Let the Unitary Irreducible Representations (UIR's) of G be labelled by γ, $\gamma \in \tilde{G}$, and let $D^{\gamma}_{\rho\rho'}(g)$ be the UIR matrix elements with row and column labels ρ and ρ'. Let the dimension of $\mathbf{D}^{\gamma}$ be $d(\gamma)$. Then, it is known [2] that the UIR matrix elements form an orthogonal and complete set of functions over G.

We can also define *skew* UIR's [6] by choosing two unitary matrices $\mathbf{U}$ and $\mathbf{V}$, and writing

$$\boldsymbol{\Delta}^{\gamma}(g) = \mathbf{U}^{\dagger}\mathbf{D}^{\gamma}(g)\mathbf{V}. \tag{8}$$

The representation properties then require the use of a metric $\mathbf{V}^{\dagger}\mathbf{U}$ as

$$\boldsymbol{\Delta}^{\gamma}(g'g) = \boldsymbol{\Delta}^{\gamma}(g')\mathbf{V}^{\dagger}\mathbf{U}\boldsymbol{\Delta}^{\gamma}(g). \tag{9}$$

This set of skew UIR matrix elements has the same orthogonality and completeness relations as the ordinary UIR's:

$$\sum_{g \in G} \Delta^{\gamma'}_{\rho'\mu'}(g)^{*}\Delta^{\gamma}_{\rho\mu}(g) = \delta_{\gamma',\gamma}\delta_{\rho',\rho}\delta_{\mu',\mu}|G|/d(\gamma), \tag{10a}$$

$$\sum_{\gamma \in \tilde{G}} \frac{d(\gamma)}{|G|} \sum_{\rho,\mu} \Delta^{\gamma}_{\rho\mu}(g')^{*}\,\Delta^{\gamma}_{\rho\mu}(g) = \delta_{g',g}, \tag{10b}$$

where the δ's are Kronecker symbols over $\tilde{G}$, G and the $d(\gamma)$-dimensional space of rows and columns, as implied by the context. Any complex-valued function $A(g)$ with domain on G can be thus expanded in the basis afforded by the skew UIR matrix elements as

$$A(g) = \sum_{\gamma \in \tilde{G}} \frac{d(\gamma)}{|G|} \sum_{\rho,\mu} A^{\gamma}_{\rho\mu}\Delta^{\gamma}_{\rho\mu}(g)^{*}. \tag{11a}$$

The generalized Fourier coefficients $\mathbf{A}^{\gamma}$ are matrix-valued functions on $\tilde{G}$ which can be determined through

$$A^{\gamma}_{\rho\mu} = \sum_{g \in G} A(g)\Delta^{\gamma}_{\rho\mu}(g). \tag{11b}$$

The unitary transformation matrices $\mathbf{U}$ and $\mathbf{V}$ may be chosen to symmetry-adapt the row and column labels to different chains of subgroups: we may choose $\rho = (p, \eta, r)$ where η labels the UIR's of $H \subset G$, p resolves the multiplicities in the subduction from γ to η, and r is some column label for $\mathbf{D}^{\eta}(h)$, the UIR's of H. Similarly, we choose $\mu = (q, \kappa, s)$, where κ labels the UIR's of $K \subset G$.

This sequence adaptation allows for the relation

$$\Delta^{\gamma}_{p\eta r,q\kappa s}(hgk^{-1}) = \sum_{r's'} D^{\eta}_{rr'}(h)\Delta^{\gamma}_{p\eta r',q\kappa s'}(g)D^{\kappa}_{s's}(k^{-1}). \tag{12}$$

We shall now consider the sequence adaptation to the chains of subgroups which are relevant for the decomposition into P-bilateral classes. In descending along the chains $\hat{H} \lhd H \subset G$ and $G \supset K \rhd \hat{K}$, we are interested in those representations of H and K which contain the trivial (unit) representation of the normal subgroups. We denote these by η_0 and κ_0. These are the most general UIR's of the factor groups $H/\hat{H}$ and $K/\hat{K}$. (This fact may be most familar to the reader in the case of the Poincaré group, where the representations containing the null momentum are labelled by the UIR's of Lorentz group.) Finally, because the elements of the factor groups $H/\hat{H}$ and $K/\hat{K}$ are related through the isomorphism φ, we may choose the row-and-column labels of their UIR's such that

$$D^{\eta_0(\tau)}_{\bar{r}\bar{r}'}(h_f) = D^{\kappa_0(\tau)}_{\bar{r}\bar{r}'}(\varphi(h_f)) = D^{\tau}_{\bar{r}\bar{r}'}(h_f), \tag{13}$$

where $h_f \in H/\hat{H}$ and τ labels the UIR's of the factor group.

Having made the above considerations on the UIR row and column indices following the structure of $p\{\hat{H}, H; \varphi; K, \hat{K}\}$, we shall now relate the space of bilateral classes to a remainder of these indices.

4. Functions over the space of bilateral classes and partial traces. We shall consider functions A on G which are constant over bilateral classes B_i, i.e., $g' \overset{P}{\sim} g \Rightarrow A(g') = A(g)$, and which thus may depend only on the bilateral class to which g belongs. We will express them as $A(g_i) = A(B_i)$, where $B_i \in \Gamma$ and Γ is the space of P-bilateral classes. The Fourier coefficients of such functions will have corresponding restrictions and independences, as we shall now see. The sum over the group G in the Fourier analysis formula (11b) can be split into a sum over the $|B_i|$ elements $g \in B_i$, times a sum over B_i, $B_i \in \Gamma$. The former, in turn, will be expressed as sums (due to (4)) over the group elements of $\hat{H}$, $\hat{K}$ and some of representatives $\dot{h}$ in $\mathscr{H}$, that is,

$$A^{\gamma}_{p\eta r, q\kappa s} = \sum_{B_i \in \Gamma} A(B_i) \sum_{g \in B_i} \Delta^{\gamma}_{p\eta r, q\kappa s}(g). \tag{14}$$

We shall now calculate the last sum using (a) the decomposition (12)–(13) of the last section; (b) the orthogonality relation (10a) for each of the subgroups in question, noting that the one-dimensional trivial representation appears for $\hat{H}$ and $\hat{K}$; and (c) the fact that the stability group P_i of any one element g_i in B_i, has $|P_i| = |P|/|B_i| = |\hat{H}||\hat{K}||H/\hat{H}|/|B_i|$ elements. As a matter of notation, we shall indicate by a bar (as $\bar{r}$ and $\bar{s}$) the row indices of the representations (as r and s) of $H/\hat{H} \simeq K/\hat{K}$.

We can thus write:

$$\begin{aligned}
&\sum_{g \in B_i} \Delta^{\gamma}_{p\eta r, q\kappa s}(g) \\
&\quad = \frac{1}{|P_i|} \sum_{r's'} \sum_{\dot{h} \in \mathscr{H}} \sum_{\hat{h} \in \hat{H}} \sum_{\hat{k} \in \hat{K}} D^{\eta}_{rr'}(\hat{h}\dot{h}) \Delta^{\gamma}_{p\eta r', q\kappa s'}(g_i) D^{\kappa}_{s's}(\varphi(\dot{h})^{-1}\hat{k}) \\
&\quad = \delta_{\eta, \eta_0(\tau)} \delta_{\kappa, \kappa_0(\tau')} \frac{|\hat{H}||\hat{K}|}{|P_i|} \sum_{\bar{r}'\bar{s}'} \sum_{h_f \in H/\hat{H}} D^{\tau}_{\bar{r}\bar{r}'}(h_f) \\
&\qquad \cdot \Delta^{\gamma}_{p\eta_0(\tau)\bar{r}', q\kappa_0(\tau')\bar{s}'}(g_i) D^{\tau'}_{\bar{s}'\bar{s}}(h_f^{-1}) \\
&\quad = \delta_{\eta, \eta_0(\tau)} \delta_{\kappa, \kappa_0(\tau')} \delta_{\tau, \tau'} \delta_{\bar{r}, \bar{s}} \frac{|B_i|}{d(\tau)} \sum_{\bar{t}} \Delta^{\gamma}_{p\eta_0(\tau)\bar{t}, q\kappa_0(\tau)\bar{t}}(g_i).
\end{aligned} \tag{15}$$

The first expression is thus diagonal in, and independent of, the row and column labels of the H and K UIR's, namely r and s. It depends only on the G UIR index γ, the $H/\hat{H}$ UIR index τ and the possible multiplicity indices p and q. This implies that the index dependence of the Fourier coefficients (14) will be restricted likewise to

$$A^{\gamma}_{p\eta r, q\kappa s} = \delta_{\eta, \eta_0(\tau)} \delta_{\kappa, \kappa_0(\tau')} \delta_{\tau, \tau'} \delta_{rs} A^{\gamma}_{p\tau q}. \tag{16}$$

We define the *partial traces* associated to the bilateral class partition $P\{\hat{H}, H; \varphi; K, \hat{K}\}$ as

$$\chi^{\gamma}_{p\tau q}(B_i) = \sum_{\bar{i}} \Delta^{\gamma}_{p\eta_0(\tau)\bar{i}, q\kappa_0(\tau)\bar{i}}(g_i)$$

(17)

$$= \frac{d(\tau)}{|B_i|} \sum_{g\in B_i} \Delta^{\gamma}_{p\eta_0(\tau)\bar{i}, q\kappa_0(\tau)\bar{i}}(g).$$

These will be an orthogonal and complete set of functions on the space Γ of P-bilateral classes since, from (10) and through steps analogous to (15) and tracing, we obtain

(18a) $$\sum_{B_i\in\Gamma} \frac{|B_i|}{d(\tau)} \chi^{\gamma'}_{p'\tau'q'}(B_i)^* \chi^{\gamma}_{p\tau q}(B_i) = \delta_{\gamma',\gamma}\delta_{\tau',\tau}\delta_{p',p}\delta_{q',q}|G|/d(\gamma),$$

(18b) $$\sum_{\gamma\in\hat{G}} \frac{d(\gamma)}{|G|} \sum_{p,\tau,q} \frac{|B_i|}{d(\tau)} \chi^{\gamma}_{p\tau q}(B_i)^* \chi^{\gamma}_{p\tau q}(B_j) = \delta_{i,j}.$$

Thus any function on Γ may be expanded as

(19a) $$A(B_i) = \sum_{\gamma\in\hat{G}} \frac{d(\gamma)}{|G|} \sum_{p\tau q} A^{\gamma}_{p\tau q} \chi^{\gamma}_{p\tau q}(B_i)^*,$$

(19b) $$A^{\gamma}_{p\tau q} = \sum_{B_i\in\Gamma} \frac{|B_i|}{d(\tau)} A(B_i)\chi^{\gamma}_{p\tau q}(B_i).$$

Since the bilateral class partition generalizes the coset and conjugation subclass partitions the partial traces (17) will generalize harmonic functions and characters over the group. Thus, for left cosets $P\{H, H; —; e, e\}$, the sum over the row indices $\bar{i}$ of $H/H = \{e\}$ disappears. The partial traces become the harmonic functions $D^{\gamma}_{p0,q}(B_i)$ over the manifold of left cosets, where the row index is specified by an appropriate multiplicity label for the subduction $G \supset H$ which contains the trivial representation of H. The column index q is fully determined by a complete subgroup chain of G. Similarly, right cosets $P\{e, e; —; K, K\}$ lead to "spherical harmonics" $D^{\gamma}_{p,q0}(B_i)$, and double cosets $P\{H, H; —; K, K\}$ to "diamond Wigner d-functions" $D^{\gamma}_{p0,q0}(B_i)$. For ordinary conjugation classes $P\{e, G; \varphi_e; G, e\}$ we have the usual characters $\chi^{g}(B_i) = \sum_p D^{\gamma}_{pp}(B_i)$, while for conjugation subclasses $P\{e, H; \varphi_e; H, e\}$, the $\chi^{\gamma}_{p\tau q}(B_i)$ are as given by (17) for the row and column labels referring to the same subgroup chain.

For the case of split subgroups and φ-twisted subclasses defined through conjugation automorphisms φ_l as $P\{e, H; \varphi_l; K, e\}$, $K = lHl^{-1}$. The group subgroup chain for the column indices of the UIR matrices is thus obtained from the row indices through a transformation by l as

(20) $$\Delta^{\gamma}_{\rho\sigma}(g) = \sum_{\tau} D^{\gamma}_{\rho\tau}(g) D^{\gamma}_{\tau\sigma}(l),$$

where by $\mathbf{D}^{\gamma}(\cdot)$ we indicate UIR matrices whose rows and columns are classified by the same subgroup chain. The values of the partial traces for a φ-twist will thus be equal to the ordinary partial traces mentioned above, but valued at the class $C^{\varphi}_i = C^{\varphi_e}_i l^{-1}$. The Fourier coefficients of functions constant on C^{φ}_i can be referred to the same subgroup chain, in place of (16)–(19b), as

(21a) $${}^{\varphi}A^{\gamma}_{p\tau r, q\tau' s} = \sum_{p'} {}^{e}A^{\gamma}_{p\tau p'} D^{\gamma}_{p'\tau r, q\tau' s}(l^{-1}),$$

(21b) $${}^{e}A^{\gamma}_{p\tau p'} = \sum_{C^{\varphi}_i\in\Gamma} \frac{|C^{\varphi}_i|}{d(\tau)} A(C^{\varphi}_i)\, {}^{e}\chi^{\gamma}_{p\tau p'}(C^{\varphi}_i l).$$

In the above, τ refers to the UIR label of H. If we now let $A(C_i^\varphi)$ be constant over double cosets by $\hat{H}$ and $\hat{K}$, i.e., $A(C_i^\varphi) = A(\hat{h}C_i^\varphi\hat{k})$ the multiplicity indices p and q must be replaced by (p, η) and (q, κ), while constancy over these sets now imposes δ_{η,η_0} and δ_{κ,κ_0} factors, reducing the number of independent Fourier coefficients in (21) to those containing the trivial representations of $\hat{H}$ and $\hat{K}$.

5. Conclusion. We have extended the classical results of harmonic analysis on cosets and conjugation classes to bilateral classes. This unifies their treatment as well as that of a number of other special cases. From a practical point of view this result may also be quite useful, e.g., if we keep in mind the original applications [3]. Assume we wish to describe functions valued over the different transitions; these can certainly be expanded and analyzed with respect to their harmonic components. Such a procedure has proved useful in many applications and the possibility of performing it in this new situation seems relevant.

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