Canonical transforms. IV. Hyperbolic transforms: Continuous series of SL(2,*R*) representations

Kurt Bernardo Wolf[a)]

Centre de Recherches Mathématiques, Université de Montréal, Montréal, Québec, Canada

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We consider the sl(2,*R*) Lie algebra of second-order differential operators given by the Schrödinger Hamiltonians of the harmonic, repulsive, and free particle, all with a strong centripedal core placing them in the C_q^ϵ continuous series of representations. The corresponding SL(2,*R*) Lie group is shown to be a group of integral transforms acting on a (two-component) space of square-integrable functions, with an integral (matrix) kernel involving Hankel and Macdonald functions. The subgroup bases for irreducible representations consist of Whittaker, power, Hankel, and Macdonald functions. We construct the operator which intertwines this realization of SL(2,*R*) with the more familiar Bargmann realization on functions on the unit circle. This operator implements the canonical transformation of the above Schrödinger systems to action and angle variables.

1. INTRODUCTION

The program to explore the role of canonical transformations in quantum mechanics followed by Moshinsky and collaborators[1,2] has lead to advances and applications in three related fields: (a) It has given a better understanding of the dynamical *groups* (as opposed to dynamical or similarity algebras) for quantum systems and partial differential equations,[3,4] (b) it has brought a significant unification into the theory of integral transforms,[5–7] and (c) it has complemented the study of the three-dimensional Lorentz group generated by algebras of second-order differential operators.[8–10] In this article, the fourth of a series,[5,6,11] we would like to explore the following territory: Consider the three operators

$$\mathbb{J}_1 = \frac{1}{4}\left(-\frac{d^2}{d\rho^2} - \frac{\mu}{\rho^2} - \rho^2\right), \tag{1.1a}$$

$$\mathbb{J}_2 = -\frac{i}{2}\left(\rho\frac{d}{d\rho} + \frac{1}{2}\right), \tag{1.1b}$$

$$\mathbb{J}_3 = \frac{1}{4}\left(-\frac{d^2}{d\rho^2} - \frac{\mu}{\rho^2} + \rho^2\right), \quad \mu > \frac{1}{4}, \tag{1.1c}$$

which form an sl(2,*R*)$\simeq$sp(2,*R*)$\simeq$so(2,1) Lie algebra, with the well-known commutation relations

$$[\mathbb{J}_1,\mathbb{J}_2] = -i\mathbb{J}_3, \quad [\mathbb{J}_2,\mathbb{J}_3] = i\mathbb{J}_1, \quad [\mathbb{J}_3,\mathbb{J}_1] = i\mathbb{J}_2. \tag{1.2}$$

Among the algebra elements we have the Schrödinger Hamiltonians corresponding to a strongly attractive centripedal well ($\mathbb{J}_1 + \mathbb{J}_3$), and similarly welled harmonic ($2\mathbb{J}_3$) and repulsive ($2\mathbb{J}_1$) oscillators. The algebra (1.1) constitutes the dynamical algebra for these systems. On calculating the value of the Casimir invariant of Eqs. (1.1), we find

$$\mathbb{Q} = \mathbb{J}_1^2 + \mathbb{J}_2^2 - \mathbb{J}_3^2 = q\mathbb{1}, \tag{1.3a}$$

$$q = \tfrac{1}{4}\mu + \tfrac{3}{16} = k(1-k) = \tfrac{1}{4}(1+\lambda^2) > \tfrac{1}{4}, \tag{1.3b}$$

$$k = \tfrac{1}{2}(1+i\lambda), \quad \lambda^2 = \mu - \tfrac{1}{4} > 0, \tag{1.3c}$$

i.e., this set of operators belongs to the continuous or principal series of representations C_q as defined by Bargmann.[12] In the proper function domain—so that Eqs. (1.1) be self-adjoint—their spectra will have no lower bound.[13] The potential singularity at the origin is indicative of the rather delicate domain problems we would find should we meet the problem starting from the algebra. This has been emphasized by Mukunda and Radhakrishnan,[10] who also considered this realization.

In Sec. 2, we shall embed the sl(2,*R*) algebra (1.1) as a subalgebra of sp(4,*R*), reduced with respect to a "hyperbolic" subalgebra so(1,1)$\oplus$sl(2,*R*). This chain is distinct from the "radial" so(2)$\oplus$sl(2,*R*) chain considered in Refs. 3, 6 (Appendix B), and 14. The parameterization of the plane in hyperbolic coordinate will lead to a two-component space $\mathscr{L}^2_{\rm II}(\mathscr{R}^+) = \mathscr{L}^2(\mathscr{R}^+) + \mathscr{L}^2(\mathscr{R}^+)$ of square-integrable functions on the half-line, as the appropriate domain for Eqs. (1.1), carrying both the C_q^0 and $C_q^{1/2}$ representations.

In Sec. 3 we consider the Lie group SL(2,*R*)$\simeq$Sp(2,*R*) generated by Eqs. (1.1), associated with the corresponding group of matrices through

$$\exp(i\alpha\mathbb{J}_1): \begin{pmatrix} \cosh(\alpha/2) & -\sinh(\alpha/2) \\ -\sinh(\alpha/2) & \cosh(\alpha/2) \end{pmatrix}, \tag{1.4a}$$

$$\exp(i\beta\mathbb{J}_2): \begin{pmatrix} \exp(-\beta/2) & 0 \\ 0 & \exp(\beta/2) \end{pmatrix}, \tag{1.4b}$$

$$\exp(i\gamma\mathbb{J}_3): \begin{pmatrix} \cos(\gamma/2) & -\sin(\gamma/2) \\ \sin(\gamma/2) & \cos(\gamma/2) \end{pmatrix}, \tag{1.4c}$$

whose adjoint action of the algebra—which is independent of the realization—is given by

$$\begin{pmatrix} \mathbb{J}_1 \\ \mathbb{J}_2 \\ \mathbb{J}_3 \end{pmatrix} \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} \tfrac{1}{2}(a^2 - b^2 - c^2 + d^2) & bd - ac & \tfrac{1}{2}(a^2 - b^2 + c^2 - d^2) \\ cd - ab & ad + bc & -cd - ab \\ \tfrac{1}{2}(a^2 + b^2 - c^2 - d^2) & -bd - ac & \tfrac{1}{2}(a^2 + b^2 + c^2 + d^2) \end{pmatrix} \begin{pmatrix} \mathbb{J}_1 \\ \mathbb{J}_2 \\ \mathbb{J}_3 \end{pmatrix}. \tag{1.5}$$

[a)] On sabbatical leave from IIMAS, Universidad Nacional Autónoma de México. Apdo. Postal 20-726, México 20, D.F.

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This group of automorphisms of the algebra will induce a corresponding group SL(2,R) of integral transforms of $\mathscr{L}^2_{\rm II}(\mathscr{R}^+)$. In the first paper of this series,[5] the algebra whose group of automorphisms was studied was the Heisenberg–Weyl algebra of quantum mechanics. The group turned out to be, as here, SL(2,R), but the integral transform carried the oscillator (or *metaplectic*) representation $D^+_{1/4} \dotplus D^+_{3/4}$. In the second paper[6] it was the sl(2,R) algebra—as here—which provided the "quantum mechanics" out of which we built the group of automorphisms (1.5) carrying the discrete D^+_k series of representations. The integral transform kernel consisted of a Gaussian times a Bessel function. Here, it will involve Gaussian functions times Hankel and Macdonald functions of imaginary index. In contradistinction with the previous cases,[5,6] this integral transform group does not allow a complex extension in the group parameters to a unitary semigroup of transforms.

In Sec. 4 we build the intertwining operator (i.e., the quantum mechanical canonical transform to action-and-angle variables) between the realization (1.1) of sl(2,R) and the well-known Bargmann realization[12] of the algebra in terms of first-order differential operators on the circle S_1:

$$\mathbb{J}^{\rm I}_1 = i\, e^{-i\epsilon\phi}\left(\cos\phi \frac{d}{d\phi} - k \sin\phi\right)e^{i\epsilon\phi}, \tag{1.6a}$$

$$\mathbb{J}^{\rm I}_2 = i\, e^{-i\epsilon\phi}\left(\sin\phi \frac{d}{d\phi} + k \cos\phi\right)e^{i\epsilon\phi}, \tag{1.6b}$$

$$\mathbb{J}^{\rm I}_3 = -i\, e^{-i\epsilon\phi} \frac{d}{d\phi} e^{i\epsilon\phi},\quad k = \tfrac{1}{2}(1 + i\lambda),\ \lambda\in\mathscr{R},\ \epsilon = 0,\tfrac{1}{2}, \tag{1.6c}$$

which also carry the C^ϵ_q representations of the continuous series. In the third paper of this series,[11] we solved the same problem for the D^+_k case, being faced with the construction of an appopiate inner product to define a Hilbert space[15] where the spectrum of Eq. (1.6c) has a lower bound,[16] leading to the definition of a nonlocal measure on S_1. Here the problem is simpler as the appropriate Hilbert space in plainly $\mathscr{L}^2(S_1)$.

From the point of view of the program on nonlinear canonical transformations outlined Ref. 17, our case presents a challenge which merits deeper study, since the classical canonical transformation to action-and-angle variables

$$p_\phi = J^c_3,\quad \phi = \arctan(J^c_2/J^c_1), \tag{1.7}$$

[where J^c_k is the "classical counterpart" ($-id/d\rho \mapsto p_\rho$) of Eqs. (1.1) and Poisson brackets replace commutators] has the same "ambiguity group"[16] for all $\mu > 0$. Moreover, the interval $\frac{1}{4} \geqslant \mu > -\frac{3}{4}$, $(0 < k < 1)$ is particularly troublesome, since various choices of boundary conditions[18] lead to representations which may belong to the lower-bounded discrete series[19] or to the unbounded supplementary series—a problem still to be solved for the algebra (1.1)—which are not quite apparent in the formal expressions in Eqs. (1.1), and invisible in the classical Poisson-bracket construct. In establishing our results from the point of view of groups of integral transforms, we hope to settle some of the uncertainties which may arise in the algebraic approach to canonical transformations in quantum mechanics. Finally in Sec. 5 we outline some applications and offer some concluding remarks.

2. THE CHAIN sp(4,R)⊃so(1,1)⊕sl(2,R) AND HYPERBOLIC COORDINATES

We consider the usual quantum mechanical operators of position and momentum in two dimensions $[\mathbb{Q}_m f(\mathbf{q}) = q_m f(\mathbf{q})$ and $\mathbb{P}_m f(\mathbf{q}) = -i\partial f(\mathbf{q})/\partial q_m, m = 1,2]$ and out of these we build the symmetrized quadratic expressions $\mathbb{Q}_m\mathbb{Q}_n$, $\mathbb{P}_m\mathbb{P}_n$, $\frac{1}{2}\{\mathbb{Q}_m,\mathbb{P}_n\}_+$. These ten operators span under Lie commutation the four-dimensional real symplectic algebra sp(4,R), isomorphic to the pseudo-orthogonal algebra so(3,2). Let us denote the latter's generators in the Cartesian basis by

$$\begin{aligned}
&\mathbb{M}_{12} = \tfrac{1}{2}(\mathbb{Q}_1\mathbb{P}_2 - \mathbb{Q}_2\mathbb{P}_1), \mathbb{M}_{13} = -\tfrac{1}{2}(\mathbb{P}_1\mathbb{P}_2 + \mathbb{Q}_1\mathbb{Q}_2),\\
&\mathbb{M}_{14} = -\tfrac{1}{2}(\mathbb{Q}_1\mathbb{P}_2 + \mathbb{Q}_2\mathbb{P}_1),\ \mathbb{M}_{15} = -\tfrac{1}{2}(\mathbb{P}_1\mathbb{P}_2 - \mathbb{Q}_1\mathbb{Q}_2),\\
&\mathbb{M}_{23} = \tfrac{1}{4}(\mathbb{P}_1^2 - \mathbb{P}_2^2 + \mathbb{Q}_1^2 - \mathbb{Q}_2^2),\ \mathbb{M}_{24} = \tfrac{1}{2}(\mathbb{Q}_1\mathbb{P}_1 - \mathbb{Q}_2\mathbb{P}_2),
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&\mathbb{M}_{25} = \tfrac{1}{4}(\mathbb{P}_1^2 - \mathbb{P}_2^2 - \mathbb{Q}_1^2 + \mathbb{Q}_2^2),\\
&\mathbb{M}_{34} = -\tfrac{1}{4}(\mathbb{P}_1^2 + \mathbb{P}_2^2 - \mathbb{Q}_1^2 - \mathbb{Q}_2^2),\\
&\mathbb{M}_{35} = \tfrac{1}{4}(\{\mathbb{Q}_1,\mathbb{P}_1\}_+ + \{\mathbb{Q}_2,\mathbb{P}_2\}),\\
&\mathbb{M}_{45} = \tfrac{1}{4}(\mathbb{P}_1^2 + \mathbb{P}_2^2 + \mathbb{Q}_1^2 + \mathbb{Q}_2^2),
\end{aligned}$$

where the metric is $(+ + + - -)$. The set of operators generating the compact subgroup SO(2) ⊗ SO(3) ⊂ SO(3,2) [i.e., those which have a discrete spectrum in $\mathscr{L}^2(\mathscr{R}^2)$] is $\{\mathbb{M}_{45};\mathbb{M}_{12},\mathbb{M}_{13},\mathbb{M}_{23}\}$. The set generating the "radial" subgroup SO(2) ⊗ SL(2,R) of Refs. 3 and 6 is $\{\mathbb{M}_{12};\mathbb{M}_{34},\mathbb{M}_{35},\mathbb{M}_{45}\}$. Here, we shall consider the set $\{\mathbb{M}_{14};\mathbb{M}_{23},\mathbb{M}_{25},\mathbb{M}_{35}\}$ generating the "hyperbolic" subgroup SO(1,1) ⊗ SL(2,R) ⊂ Sp(4,R). The so(1,1) element is the Lorentz boost generator in the plane, while the sl(2,R) elements are built out of the harmonic (h) and repulsive (r) one-dimensional Schrödinger Hamiltonians $\mathbb{H}^{(\cdot)}_k, k = 1,2$ as $\mathbb{M}_{23} = \frac{1}{2}(\mathbb{H}^{(h)}_1 - \mathbb{H}^{(h)}_2)$ and $\mathbb{M}_{25} = \frac{1}{2}(\mathbb{H}^{(r)}_1 - \mathbb{H}^{(r)}_2)$, [rather than $\mathbb{M}_{45} = \frac{1}{2}(\mathbb{H}^{(h)}_1 + \mathbb{H}^{(h)}_2)$ and $\mathbb{M}_{34} = -\frac{1}{2}(\mathbb{H}^{(r)}_1 + \mathbb{H}^{(r)}_2)$ as in the radial case]. The generator $\mathbb{M}_{35}$ is common to the hyperbolic and radial subgroups. In $\mathscr{L}^2(\mathscr{R}^2)$, thus, the eigenfunctions of $\mathbb{M}_{23}$ will be $\Psi_{n_1n_2}(\mathbf{q}) = \Psi^h_{n_1}(q_1)\Psi^h_{n_2}(q_2)$ [where $\Psi^h_n(q) = (-1)^n\Psi^h_n(-q)$ are the simple harmonic oscillator wavefunctions], and its spectrum will be given by $m = \frac{1}{2}(n_1 - n_2)$, $n_1,n_2 = 0,1,2,\cdots$. This set of functions will thus constitute a basis for the two continuous series representations of sl(2,R): C^0_q spanned by the subset with $n_1 + n_2$

even [so that m is integer $\Psi_{n_1 n_2}(-\mathbf{q}) = \Psi_{n_1 n_2}(\mathbf{q})$], and $C_q^{1/2}$ by the subset with $n_1 + n_2$ odd [m in half-integer and $\Psi_{n_1 n_2}(-\mathbf{q}) = -\Psi_{n_1 n_2}(\mathbf{q})$].

We shall now parametrize the plane in hyperbolic coordinates (ρ,ϕ,σ), dividing it into two regions labeled by σ as

$$\text{for } q_1^2 - q_2^2 > 0: \quad \sigma = +1, \quad q_1 = \rho \cosh\phi, \quad q_2 = \rho \sinh\phi, \quad \rho,\phi \in R; \tag{2.2a}$$

$$\text{for } q_1^2 - q_2^2 < 0: \quad \sigma = -1, \quad q_1 = \rho \sinh\phi, \quad q_2 = \rho \cosh\phi, \tag{2.2b}$$

and disregard the cone $q_1^2 - q_2^2 = 0$, as this is a submanifold of lower dimension. The elements $f(\mathbf{q})$ of the space of functions $\mathscr{L}^2(\mathscr{R}^2)$ on the plane will be correspondingly represented by *pairs* of functions $f_\sigma(\rho,\phi)$, $\sigma = \pm 1$, elements of a space $\mathscr{L}_1^2(\mathscr{R}) \dotplus \mathscr{L}_{-1}^2(\mathscr{R})$ which can be arranged as a two-component vector column

$$\mathbf{f}(\rho,\phi) = \begin{pmatrix} f_1(\rho,\phi) \\ f_{-1}(\rho,\phi) \end{pmatrix}, \quad f_\sigma(\rho,\phi) = f(\mathbf{q}(\rho,\phi,\sigma)). \tag{2.3}$$

The inner product in $\mathscr{L}^2(\mathscr{R}^2)$ becomes

$$(\mathbf{f},\mathbf{g})_2 = \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2 f(q_1,q_2)^* g(q_1,q_2) = \sum_{\sigma=\pm 1} \int_{-\infty}^{\infty} |\rho| d\rho \int_{-\infty}^{\infty} d\phi f_\sigma(\rho,\phi)^* g_\sigma(\rho,\phi), \tag{2.4}$$

in terms of the hyperbolic coordinates. Finally, the generators of $\mathrm{SO}(1,1) \otimes \mathrm{SL}(2,R)$ can be written as

$$\mathbb{K}_0 = -\mathbb{M}_{14} = -i\frac{1}{2}\frac{\partial}{\partial\phi}, \tag{2.5}$$

$$\mathbb{K}_1 = \mathbb{M}_{25} = \sigma\tfrac{1}{4}\rho^{-1/2}\left[-\frac{\partial^2}{\partial\rho^2} - \rho^{-2}\times\left(\frac{1}{4} - \frac{\partial^2}{\partial\phi^2}\right) - \rho^2\right]\rho^{1/2}, \tag{2.6a}$$

$$\mathbb{K}_2 = \mathbb{M}_{35} = -i\tfrac{1}{2}\rho^{-1/2}\left[\rho\frac{\partial}{\partial\rho} + \frac{1}{2}\right]\rho^{1/2}, \tag{2.6b}$$

$$\mathbb{K}_3 = \mathbb{M}_{23} = \sigma\tfrac{1}{4}\rho^{-1/2}\left[-\frac{\partial^2}{\partial\rho^2} - \rho^{-2}\times\left(\frac{1}{4} - \frac{\partial^2}{\partial\phi^2}\right) + \rho^2\right]\rho^{1/2}. \tag{2.6c}$$

The operators (2.6) exhibit commutation relations analogous to Eq. (1.2). Acting on the column-vector function (2.3), the generators above will be represented by 2×2 diagonal matrices with operator elements, which for Eqs. (2.6a) and (2.6c) have opposite signs. The adjoint action of the group generated by Eqs. (2.6) on themselves can be verified to be formally identical to Eq. (1.5), as it should be, since the latter is a relation independent of the particular operator realization. For the $\sigma = -1$ components, we have a reversal of the signs of α and γ in Eqs. (1.4), i.e., of b and c in the elements of the 2×2 matrix realization in Eq. (1.5). This leaves the 3×3 matrix in Eq. (1.5) invariant.

The subalgebras so(1,1) and sl($2,R$) generated by Eqs.(2.5) and (2.6) are *conjugate* in sp($4,R$); the reduction to an irreducible subspace (irrep) of the former leads to a corresponding irrep of the latter. Since for sp($4,R$) itself we do not have a single irrep space but a direct sum of two—those with a basis with integer and with half-integer eigenvalues m under $\mathbb{M}_{45}$ or $\mathbb{M}_{23}$—the corresponding reduction of the sl($2,R$) generators will be the direct sum of two irrep's C_q^0 and $C_q^{1/2}$, respectively. An irrep space for $\mathbb{K}_0$ within Eqs. (2.1) is provided by functions $f_\sigma^\lambda(\rho,\phi) = f_\sigma^\lambda(\rho)\exp(i\lambda\phi)$, $\lambda \in \mathscr{R}$. This will replace the operator $-\partial^2/\partial\phi^2$ in Eqs. (2.6) by λ^2 and bring the $\mathbb{K}_k$ to within a similarity transformation (by $\rho^{-1/2}$) of the forms (1.1).

In the following sections we shall be interested in certain discrete operations on the plane in Cartesian and hyperbolic coordinates which are, nevertheless, elements of the parent Sp($4,R$) group and which can be connected to the identity. These will be identified using the notation of Mukunda and Radhakrishnan[10]. First, we have the full space inversion

$$\mathbb{P}:(q_1,q_2) \to (-q_1,-q_2), \text{i.e.}, \mathbb{P}:(\rho,\phi,\sigma) \to (-\rho,\phi,\sigma), \tag{2.7a}$$

$$\mathbb{P}\mathbb{K}_\nu = \mathbb{K}_\nu\mathbb{P}, \quad \nu = 0,1,2,3, \tag{2.7b}$$

$$\mathbb{P} = \exp(2\pi i\mathbb{M}_{23}) = \exp(2\pi i\mathbb{M}_{45}), \tag{2.7c}$$

i.e., it is the rotation-by-2π element of SL($2,R$) which commutes with the algebra so$(2,1)\simeq$sl($2,R$) and which can be used to distinguish the vector and spinor constituent irrep's C_q^0 and $C_q^{1/2}$ by demanding that $\mathbb{P}$ be diagonal. We use its eigenvalues $p = \pm 1$ to distinguish the irrep spaces for C_q^ϵ through

$$\epsilon = \tfrac{1}{4}(1-p), \text{i.e., } \epsilon = 0(1/2) \text{ for } p = +1(-1). \tag{2.7d}$$

Second, we have the inversion of the second Cartesian coordinate

$$\mathbb{B}:(q_1,q_2) \to (q_1,-q_2), \text{i.e., } \mathbb{B}(\rho,\phi,\sigma) \to (\sigma\rho,-\phi,\sigma), \tag{2.8a}$$

$$\mathbb{B}\mathbb{K}_0 = -\mathbb{K}_0\mathbb{B}\,; \mathbb{B}\mathbb{K}_k\mathbb{B}, \quad k = 1,2,3, \tag{2.8b}$$

$$\mathbb{B} = \exp(i\pi[\mathbb{M}_{45} - \mathbb{M}_{23}]). \tag{2.8c}$$

This element commutes with the sl($2,R$) algebra and with $\mathbb{P}$, but will intertwine the λ and $-\lambda$ representations of so(1,1), and hence those of sl($2,R$). Its effect on the properly reduced irrep space C_q^ϵ will be to change the sign of the lower component of the $\epsilon = \frac{1}{2}$ function pair.

Third, we have the element $\mathbb{B}\mathbb{P}$, which will not interest us separately, and fourth, the operator

$$\mathbb{A}:(q_1,q_2) \to (q_2,q_1), \text{i.e., } \mathbb{A}:(\rho,\phi,\sigma) \to (\rho,\phi,-\sigma), \tag{2.9a}$$

$$\mathbb{A}\mathbb{K}_j = \mathbb{K}_j\mathbb{A}, \quad j = 0,2; \ \mathbb{A}\mathbb{K}_k = -\mathbb{K}_k\mathbb{A}, \quad k = 1,3, \tag{2.9b}$$

$$\mathbb{A} = \mathbb{B}\exp(i\pi\mathbb{M}_{12}).$$

This element does not commute with $\mathbb{B}$ (instead, $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{P}\mathbb{A}$), but it commutes with $\mathbb{P}$ and $\mathbb{K}_0$ and is thus representable as a unitary transformation in each C_q^ϵ irrep which reverses the sign of the $\mathbb{K}_3$ eigenvalues. Its own eigenvalues ($a = \pm 1$) will be used to classify the double-multiplicity $\mathbb{K}_2$ eigenfunctions. It is representable as a $\boldsymbol{\sigma}_1$ Pauli matrix in the two-component function space (2.3). The $\mathbb{A}$ and $\mathbb{B}$ automorphisms are outer to SL($2,R$), while $\mathbb{P}$ is inner.

3. THE INTEGRAL TRANSFORM GROUP

The integral transform action of the Sp(4,R) group generated by (2.1) on $\mathscr{L}^2(\mathscr{R}^2)$ is known[20,21]. In particular, for the SL(2,R) subgroup generated by Eqs. (2.6), represented by the matrices[22,23]

$$\mathbf{M}=\begin{pmatrix} a\mathbf{1} & b\boldsymbol{\sigma}_3 \\ c\boldsymbol{\sigma}_3^{-1} & d\mathbf{1}\end{pmatrix},\quad ad-bc=1,$$
$$\mathbf{1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \boldsymbol{\sigma}_3=\begin{pmatrix}1 & 0\\ 0 & e^{-i\pi}\end{pmatrix}, \tag{3.1}$$

it is

$$f(\mathbf{q})\xrightarrow{\mathbf{M}}[\mathbb{C}_{\mathbf{M}}\mathbf{f}](\mathbf{q})=\int_{R^2} d^2\mathbf{q}'C_{\mathbf{M}}(\mathbf{q},\mathbf{q}')f(\mathbf{q}'), \tag{3.2a}$$

where the integral kernel is, for $b\neq 0$,

$$C_{\mathbf{M}}(\mathbf{q},\mathbf{q}')=(2\pi|b|)^{-1}\exp[i(a\{q_1'^2-q_2'^2\} -2\{q_1'q_1-q_2'q_2\}+d\{q_1^2-q_2^2\})/2b], \tag{3.2b}$$

while for $b=0$, it is

$$C_{\mathbf{M}(b=0)}(\mathbf{q},\mathbf{q}')=a^{-1}\exp[ic(q_1^2-q_2^2)/2a] \times\delta^2(\mathbf{q}'-a^{-1}\mathbf{q}). \tag{3.2c}$$

For $\mathbf{M}=\mathbf{1}$ we have thus the reproducing kernel under Eq. (3.2a). This integral transform group provides a vector representation[24] of SL(2,R):

$$\int_{\mathscr{R}^2} d^2\mathbf{q}'C_{\mathbf{M}_1}(\mathbf{q},\mathbf{q}')C_{\mathbf{M}_2}(\mathbf{q}',\mathbf{q}'')=C_{\mathbf{M}_1\mathbf{M}_2}(\mathbf{q},\mathbf{q}''), \tag{3.3}$$

and the transforms are unitary in $\mathscr{L}^2(\mathscr{R}^2)$.

We now introduce hyperbolic coordinates (ρ,ϕ,σ) as given by Eqs. (2.2). The kernel (3.2b) and (3.2c) then appears as

$$\begin{aligned}C_{\mathbf{M}}(\rho,\phi,\sigma;\rho',\phi',\sigma')&=C_{\mathbf{M}}(-\rho,\phi,\sigma;-\rho',\phi',\sigma')\\ &=C_{\mathbf{M}}(\rho,\phi-\phi',\sigma;\rho',0,\sigma')\\ &=(2\pi|b|)^{-1}\exp[i(a\sigma'\rho'^2-2\rho\rho'\operatorname{hyp}_{\sigma,\sigma'}(\phi'-\phi)\\ &\quad+d\sigma\rho^2)/2b],\end{aligned} \tag{3.4a}$$

$$\begin{aligned}\operatorname{hyp}_{1,1}(z)&=\cosh(z)=-\operatorname{hyp}_{-1,-1}(z),\\ \operatorname{hyp}_{1,-1}(z)&=\sinh(z)=-\operatorname{hyp}_{-1,1}(z),\end{aligned} \tag{3.4b}$$

and can be arranged into a 2×2 matrix with rows and columns as the functions are represented by Eq. (2.3). We can display the eigenspaces of $\mathbb{P}$ and $\mathbb{K}_0$ through the operator

$$\begin{aligned}f_\sigma^{p,\lambda}(\rho)&=(\mathbb{T}^{p,\lambda}\mathbf{f}_\sigma)(\rho)=pf_\sigma^{p,\lambda}(-\rho)\\ &=|\rho|^{1/2}(\mathbb{1}+p\mathbb{P})\int_{-\infty}^{\infty}d\phi f_\sigma(\rho,\phi)\exp(-i\lambda\phi),\end{aligned} \tag{3.5a}$$

thus allowing us to reduce the domain of the functions to the interval $\rho\geqslant 0$. Conversely,

$$\begin{aligned}f_\sigma(\rho,\phi)&=(\mathbb{T}^{\phi}\mathbf{f}_\sigma^{\cdot\cdot})(\rho,\phi)\\ &=\frac{1}{4\pi}|\rho|^{-1/2}\sum_{p=\pm1}\int_{-\infty}^{\infty}d\lambda f_\sigma^{p,\lambda}(\rho)\exp(i\lambda\phi).\end{aligned} \tag{3.5b}$$

We define an inner product in the (p,λ) subspace $\mathscr{L}^2_{\text{II}}(\mathscr{R}^+)_{p,\lambda}=\mathscr{L}^2_1(\mathscr{R}^+)\dotplus\mathscr{L}^2_{-1}(\mathscr{R}^+)$ as

$$(\mathbf{f},\mathbf{g})_{p,\lambda}=\sum_{\sigma=\pm1}\int_0^\infty d\rho f_\sigma^{p,\lambda}(\rho)^*g_\sigma^{p,\lambda}(\rho), \tag{3.6}$$

and note that it will relate to Eq. (2.4) through

$$(\mathbf{f},\mathbf{g})_2=\frac{1}{4\pi}\sum_{p=\pm1}\int_{-\infty}^{\infty}d\lambda\,(\mathbf{f},\mathbf{g})_{p,\lambda}. \tag{3.7}$$

The properties of $\mathbb{T}^{p,\lambda}$ are such that

$$\mathbb{T}^{p,\lambda}\mathbb{P}=p\mathbb{T}^{p,\lambda},\quad \mathbb{T}^{p,\lambda}\mathbb{K}_0=\tfrac{1}{2}\lambda\mathbb{T}^{p,\lambda}, \tag{3.8a}$$

$$(\mathbb{J}_1^{\text{II}},\mathbb{J}_2^{\text{II}},\mathbb{J}_3^{\text{II}})=\mathbb{T}^{p,\lambda}(\mathbb{K}_1,\mathbb{K}_2,\mathbb{K}_3)=(\sigma\mathbb{J}_1,\mathbb{J}_2,\sigma\mathbb{J}_3)\mathbb{T}^{p,\lambda}, \tag{3.8b}$$

$$\mathbb{T}^{p,\lambda}\mathbb{A}=\mathbb{A}\mathbb{T}^{p,\lambda},\quad \mathbb{T}^{p,\lambda}\mathbb{B}=\mathbb{T}^{p,-\lambda}. \tag{3.8c}$$

Equations (3.8a) only state that $\mathbb{T}^{p,\lambda}$ indeed projects out eigenspaces of $\mathbb{P}$ and $\mathbb{K}_0$, while Eqs. (3.8c) give relations which will be used later on. Equations (3.8b), finally, bring the three algebra generators (1.1) into the picture and, besides telling us that the special Sp(4,R) transform (3.2) leaves the (p,λ) subspace invariant, allows us to calculate the integral transform representing the operator $\mathbb{C}_{\mathbf{M}}^{p,\lambda}=\mathbb{T}^{p,\lambda}\mathbb{C}_{\mathbf{M}}$ which maps $\mathscr{L}^2_{\text{II}}(\mathscr{R}^+)_{p,\lambda}$ onto itself unitarily. Since the inner product (3.6) does not explicitly contain the labels p,λ, we shall henceforth drop them from specifying the space $\mathscr{L}^2_{\text{II}}(\mathscr{R}^+)$.

For functions $f_\sigma(\rho)\in\mathscr{L}^2_{\text{II}}(\mathscr{R}^+)$, thus, the SL(2,$R$) group generated by the operators $\mathbb{J}_k^{\text{II}}$, $k=1,2,3$, acts as

$$\begin{aligned}f_\sigma(\rho)&\xrightarrow{\mathbf{M}}[\mathbb{C}_{\mathbf{M}}^{p,\lambda}\mathbf{f}]_\sigma(\rho)\\ &=\sum_{\sigma'=\pm1}\int_0^\infty d\rho' C^{p,\lambda}_{\mathbf{M},\sigma,\sigma'}(\rho,\rho')f_{\sigma'}(\rho'),\end{aligned} \tag{3.9a}$$

with the integral kernel

$$\begin{aligned}C^{p,\lambda}_{\mathbf{M},\sigma,\sigma'}(\rho,\rho')&=\rho^{1/2}(\mathbb{T}^{p,\lambda}\mathbf{C}_{\mathbf{M},\sigma,\sigma'})(\rho,\rho')\\ &=(\rho\rho')^{1/2}\int_{-\infty}^{\infty}d\psi[C_{\mathbf{M},\sigma,\sigma'}(\rho,\psi;\rho',0)\\ &\quad+pC_{\mathbf{M},\sigma,\sigma'}(\rho,\psi;-\rho',0)]\exp(-i\lambda\psi)\\ &=G_{\mathbf{M},\sigma,\sigma'}(\rho,\rho')H^{p,\lambda}_{\sigma,\sigma'}(\rho\rho'/b),\end{aligned} \tag{3.9b}$$

where, on evaluating this expression from Eqs. (3.4) for $b\neq 0$, we find it to be a product of a Gaussian factor

$$G_{\mathbf{M},\sigma,\sigma'}(\rho,\rho')=(2\pi|b|)^{-1}(\rho\rho')^{1/2}\times\exp[i(d\sigma\rho^2+a\sigma'\rho'^2)/2b], \tag{3.10}$$

and a factor $H^{p,\lambda}_{\sigma,\sigma'}(z)$ which contains the integration over ψ and which can be performed in terms of Hankel and Macdonald[25] functions, yielding[26]

$$\begin{aligned}H^{p,\lambda}_{1,1}(z)&=pH^{p,\lambda}_{-1,-1}(z)\\ &=4p\int_0^\infty d\psi\operatorname{trig}_p(z\cosh\psi)\cos(\lambda\psi)\\ &=i\pi[pe^{-\lambda\pi/2}H^{(1)}_{i\lambda}(z)-e^{\lambda\pi/2}H^{(2)}_{i\lambda}(z)]\\ &=pH^{p,\lambda}_{1,1}(-z)=H^{p,-\lambda}_{1,1}(z),\end{aligned} \tag{3.11a}$$

$$\begin{aligned}H^{p,\lambda}_{1,-1}(z)&=pH^{p,\lambda}_{-1,1}(z)\\ &=4p\int_0^\infty d\psi\operatorname{trig}_p(z\sinh\psi)\operatorname{trig}_p(\lambda\psi)\\ &=4(\operatorname{sign}z)^{2\epsilon}\operatorname{hyp}_{1,p}(\lambda\pi/2)K_{i\lambda}(|z|)\\ &=pH^{p,\lambda}_{1,-1}(-z)=pH^{p,-\lambda}_{1,-1}(z),\end{aligned} \tag{3.11b}$$

$$\operatorname{trig}_{+1}(z)=\cos(z),\ \operatorname{trig}_{-1}(z)=i\sin(z). \tag{3.11c}$$

The case $b = 0$ may be obtained either from Eq. (3.11) for $b \to 0$ and the use of the asymptotic properties of the cylinder functions,[27] or directly from Eqs. (3.2c) and (3.9), as

$$C^{p,\lambda}_{\mathbf{M}(b=0),\sigma,\sigma'}(\rho,\rho') = |a|^{-1/2}(\text{sign}a)^{2\epsilon}\exp(i\sigma c\rho^2/2a)\,\delta_{\sigma,\sigma'}\delta(\rho' - \rho/|a|). \tag{3.12}$$

This integral transform is unitary[28] on $\mathscr{L}^2_{\text{II}}(\mathscr{R}^+)$ with the inner product (3.6).

The group properties of this matrix kernel are directly inherited from Eq. (3.3) via Eqs. (3.5a), namely,

$$\sum_{\sigma' = \pm 1}\int_0^\infty d\rho' C^{p,\lambda}_{\mathbf{M}_1,\sigma,\sigma'}(\rho,\rho')C^{p,\lambda}_{\mathbf{M}_2,\sigma',\sigma''}(\rho',\rho'') = C^{p,\lambda}_{\mathbf{M}_1\mathbf{M}_2,\sigma,\sigma''}(\rho,\rho''). \tag{3.13}$$

We should point out that the property which distinguishes the C^0_q and $C^{1/2}_q$ representations is clearly displayed:

$$C^{p,\lambda}_{-\mathbf{M},\sigma,\sigma'}(\rho,\rho') = (-1)^{2\epsilon}C^{p,\lambda}_{\mathbf{M},\sigma,\sigma'}(\rho,\rho'). \tag{3.14}$$

This is a consequence of Eqs. (3.12) and (3.13) which can also be seen from the explicit expressions (3.9)–(3.11), noting that the Gaussian factor is the same for $\mathbf{M}$ and $-\mathbf{M}$, while $H^{p,\lambda}_{\sigma,\sigma'}(-z) = pH^{p,\lambda}_{\sigma,\sigma'}(z)$.

Regarding the operator $\mathbb{B}$ defined in Eqs. (2.8), the last equality in Eqs. (3.12a) and (3.12b) shows that the $\sigma = -1$ components of the $\epsilon = 1/2$ irrep functions indeed consistently invert their signs and that this inversion is thus representable by a $\boldsymbol{\sigma}_3$ Pauli matrix in the two-component $\mathscr{L}^2_{\text{II}}(\mathscr{R}^+)$ space, intertwining the $k = \frac{1}{2}(1 + i\lambda)$ and $k^* = \frac{1}{2}(1 - i\lambda)$ representations $C^{1/2}_q$. Eigenfunctions of $\mathbb{B}$ with eigenvalue $b = \pm 1$ in $C^{1/2}_q$ can be built as functions with only a nonzero upper ($b = +1$) or a lower ($b = -1$) component. In Sec. 4, however, we shall gloss over this classification scheme in favor of others. In C^0_q, $\mathbb{B}$ is equivalent to the identity transformation.

Having given the kernel for the general hyperbolic canonical transforms, we would like to present a peculiarity of the "hyperbolic Fourier transform," i.e., the transform $\mathbb{C}^{p,\lambda}_{\mathbf{F}}$ corresponding to the matrix $\mathbf{M}$ given by $\mathbf{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ which from Eq. (1.4c) is $\mathbb{C}^{p,\lambda}_{\mathbf{F}} = \exp(-i\pi\mathbb{J}^{\text{II}}_3)$. In the case of linear canonical transforms[5,7] this is $e^{-i\pi/4}$ times the ordinary Fourier transform. For the radial case[6], it is the Hankel transform. Here, as $a = 0 = d$, the Gaussian factor (3.11) is simply $(\rho\rho')^{1/2}/2\pi$ and hence

$$\left[\mathbb{C}^{p,\lambda}_{\mathbf{F}}\begin{pmatrix}\mathbf{f}_1 \\ \mathbf{f}_{-1}\end{pmatrix}\right](\rho) = p\left[\mathbb{C}^{p,\lambda}_{\mathbf{F}^{-1}}\begin{pmatrix}\mathbf{f}_1 \\ \mathbf{f}_{-1}\end{pmatrix}\right](\rho)$$

$$= \frac{1}{2\pi}\int_0^\infty d\rho'(\rho\rho')^{1/2}\begin{pmatrix} H^{p,\lambda}_{1,1}(\rho\rho') & H^{p,\lambda}_{1,-1}(\rho\rho') \\ pH^{p,\lambda}_{1,-1}(\rho\rho') & pH^{p,\lambda}_{1,1}(\rho\rho')\end{pmatrix} \times \begin{pmatrix} f_1(\rho') \\ f_{-1}(\rho')\end{pmatrix}. \tag{3.15}$$

The inverse hyperbolic Fourier transform is thus identical to the direct one for the C^0_q irrep, while it differs by a minus sign for the $C^{1/2}_q$ irrep[29]. The origin of this property is the behavior of the sl(2,R) algebra under the $\mathbb{A}$ operator in Eq. (2.9b): $\mathbb{C}^{p,\lambda}_{\mathbf{F}}\mathbb{A} = \mathbb{A}\mathbb{C}^{p,\lambda}_{\mathbf{F}^{-1}} = p\mathbb{C}^{p,\lambda}_{\mathbf{F}}$. For $p = +1$ ($\epsilon = 0$) we may thus construct eigenspaces of $\mathbb{A}$ consisting of functions of even and odd parity under this operator: It suffices to apply the unimodular matrix $2^{-1/2}\begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix}$ to Eq. (3.15) in order to obtain through similarity a diagonal transformation matrix kernel with elements $(H^{1,\lambda}_{1,1} + H^{1,\lambda}_{1,-1})(\rho\rho')$ and $(H^{1,\lambda}_{1,1} - H^{1,\lambda}_{1,-1})(\rho\rho')$, corresponding to eigenvalues $a = +1$ and $a = -1$, respectively, under $\mathbb{A}$. Each $\mathbb{A}$-classified subspace is then transformed into itself under $\mathbb{C}^{1,\lambda}_{\mathbf{F}}$ and consists of functions such that $f_\sigma(\rho) = af_{-\sigma}(\rho)$. Besides providing a distinguishing label for the two $\mathbb{J}^{\text{II}}_2$ eigenfunctions (Sec. 4), the operator $\mathbb{A}$ may be thus used to construct and distinguish between these two $p = +1$ Fourier transforms in the space where—as for the Hankel transform—the square of the transform is the identity. A similar construction for the $C^{1/2}_q$ irrep yields an antidiagonal matrix kernel.

In closing this section, it should be noted that the analytic continuation in the group parameters of Eqs. (3.1)—so fruitfully exploited in Ref. 14—turns out to be impossible here: If one applies the criterion of Ref. 20 to this matrix, one sees that for no complex values of the parameters does one have a Hilbert–Schmidt operator. This becomes intuitively clear as the analog of the heat diffusion transform[4] ($a = 1 = d, c = 0, b = -2it$) is forward in time t for the first Cartesian coordinate, but backward in the second one. Examination of the kernel in Eqs. (3.9) or introduction of complex hyperbolic coordinates in Ref. 5 (Appendix B) corroborates this conclusion. This seems to be thus a major and inescapable distinction between the discrete and continuous SL(2,R) representation series.

4. THE INTERTWINING OPERATOR

In this section we shall build the operator which intertwines the two algebra realizations (1.1) and (1.6) or, more precisely, the unitary transform kernel mapping the space $\mathscr{L}^2_{\text{II}}(\mathscr{R}^+)$ described in Sec. 3 onto the more usual $\mathscr{L}^2(S_1)$ space, in such a way that the second-order differential operators $\mathbb{J}^{\text{II}}_k$ defined in Eqs. (3.8b) map onto the first-order ones $\mathbb{J}^{\text{I}}_k$ given in Eqs. (1.6). This is the proper quantum analog of the canonical transformation to action-and-angle variables (1.7)[30].

Let $\Psi^{p,\lambda}_{m,\sigma}(\rho)$ and $X^{p,\lambda}_{\nu,\sigma}(\rho)$ be the (proper or generalized) eigenfunctions of $\mathbb{P}$ and two operators in the set $\mathbb{J}^{\text{II}}_k$, and $\psi^{p,\lambda}_m(\phi)$ and $\chi^{p,\lambda}_\nu(\phi)$ for the corresponding operators in the set $\mathbb{J}^{\text{I}}_k$. We can choose the first operator to be elliptic, specifically $\mathbb{J}^\bullet_3$, and the second to be either hyperbolic[31] ($\mathbb{J}^\bullet_1$ or $\mathbb{J}^\bullet_2$), or parabolic ($\mathbb{J}^\bullet_3 \pm \mathbb{J}^\bullet_1$)—specifically, we shall employ $\mathbb{J}^\bullet_3 - \mathbb{J}^\bullet_1$. The last choice will be followed, as it is the simplest: The generalized eigenfunctions are Dirac δ's while we are assured that the spectrum of this operator covers the real line *once*[32]. The intertwining integral kernel will then be computable as the generating function

$$K^{p,\lambda}_\sigma(\phi,\rho) = \sum_m \Psi^{p,\lambda}_{m,\sigma}(\rho)^*\psi^{p,\lambda}_m(\phi)\exp[i\Phi^\psi(p,\lambda,m,\sigma)]$$

$$= \int_{-\infty}^\infty d\nu X^{p,\lambda}_{\nu,\sigma}(\rho)^*\chi^{p,\lambda}_\nu(\phi)\exp[i\Phi^X(p,\lambda,\nu,\sigma)]. \tag{4.1}$$

The correct choice of phase[33] for Φ^{ψ} and Φ^{χ} is nontrivial for two reasons. First, it actually may change the generating function: Assume we apply $\mathbb{C}^{p,\lambda}_{\mathbb{M}(\gamma)} = \exp[i\gamma(\mathbb{J}_3^{\rm II} - \mathbb{J}_1^{\rm II})]$ to Eq. (4.1)[34], multiplying the integrand by $e^{i\gamma\nu}$ and thus producing a new generating function which, as a sum, will consist of eigenfunctions of[35] $\mathbb{C}^{p,\lambda}_{\mathbb{M}(\gamma)}\mathbb{J}_3^{\rm II}\mathbb{C}^{p,\lambda}_{\mathbb{M}(-\gamma)} \neq \mathbb{J}_3^{\rm II}$. Second, certain phase requirements exist, notably Bargmann's convention[36] for the $\mathbb{J}_3^{\rm I}$ eigenbasis, which involves definite transformation phases under the operator $\mathbb{A}$ in Eqs. (2.9). However, once we have used two generators [algebraic basis for sl(2,R)] to determine the phases for the intertwining kernel, no further requirement is imposed by the third (vector basis) generator, as its matrix elements are fixed by the first two.

It should be clear, however, that independent of the appropriate choice of phases, the kernel (4.1) will intertwine $\mathscr{L}^2_{\rm II}(\mathscr{R}^+)$ and $\mathscr{L}^2(S_1)$ as

$$f^{\rm I}(\phi) = \sum_{\sigma=\pm 1}\int_0^\infty d\rho K^{p,\lambda}_\sigma(\phi,\rho) f^{\rm II}_\sigma(\rho), \qquad (4.2a)$$

$$f^{\rm II}_\sigma(\rho) = \int_{-\pi}^{\pi} d\phi f^{\rm I}(\phi) K^{p,\lambda}_\sigma(\phi,\rho)^*, \qquad (4.2b)$$

for $f^{\rm II}_\sigma(\rho)$ and $f^{\rm I}(\phi)$ in the two spaces, respectively. The unitarity of the transformation is guaranteed by the assumed Dirac orthonormality and completeness of the two eigenbases—including any similarity transformation as mentioned above—which, from Eq. (4.1) alone, implies

$$\int_{-\pi}^{\pi} d\phi K^{p,\lambda}_\sigma(\phi,\rho) K^{p,\lambda}_{\sigma'}(\phi,\rho')^* = \delta_{\sigma\sigma'}\delta(\rho-\rho'), \qquad (4.3a)$$

$$\sum_{\sigma=\pm 1}\int_0^\infty d\rho K^{p,\lambda}_\sigma(\phi,\rho) K^{p,\lambda}_\sigma(\phi',\rho)^* = \delta(\phi-\phi'). \qquad (4.3b)$$

The phase definition we shall impose will stem from the requirement that if $f^{\rm II}_\sigma(\rho)$ is the $K^{p\lambda}$ transform of $f^{\rm I}(\phi)$ then the $K^{p,\lambda}$ transform of $(\mathbb{J}^{\rm I}_k f^{\rm I})(\phi)$ should be $(\mathbb{J}^{\rm II}_k f^{\rm II})_\sigma(\rho)$, with $\mathbb{J}^{\rm II}_k$ and $\mathbb{J}^{\rm I}_k$ given precisely by Eqs. (3.8b) and (1.6b), respectively, supplemented by the discrete transformation $\mathbb{A}$, as imposed by Bargmann's convention[36].

The $\mathbb{J}^{\rm I}_k$-basis eigenvectors are easy to obtain as they are solutions of first-order differential equations, and Fourier anaysis techniques allow us to find the correct constants for ordinary or Dirac orthonormality in $\mathscr{L}^2(S_1)$. This is more difficult for those of $\mathbb{J}^{\rm II}_k$ since, as will be borne out below, these are two-component, in general, Whittaker functions whose orthonormality and completeness relations certainly imply a careful analysis. For the parabolic operator in sl(2,R), the simplest one we can choose is

$$\mathbb{J}_3^{\rm II} - \mathbb{J}_1^{\rm II} = \tfrac{1}{2}\sigma\rho^2, \qquad (4.4)$$

since the set of generalized eigenfunctions is readily found as

$$X^{p,\lambda}_{\nu,\sigma}(\rho) = (2|\nu|)^{-1/4}\delta(\rho - [2|\nu|]^{1/2})\delta_{\sigma,{\rm sign}\nu}\exp(i\Phi^{\chi})$$
$$= \rho^{1/2}\delta(|\nu| - \tfrac{1}{2}\rho^2)\delta_{\sigma,{\rm sign}\nu}\exp(i\Phi^{\chi}), \quad \nu\in\mathscr{R}, \qquad (4.5)$$

where we have left a phase factor to be determined later on.

Note that $X^{p\lambda}_\nu(\rho)$ is a two-component function which has only an upper component for $\nu > 0$ and only a lower one for $\nu < 0$. As they stand, these functions may only involve the representation indices (p,λ), if at all[37], in the phase factor $\Phi^{\chi}(p,\lambda,\nu,\sigma)$.

Now, in $\mathscr{L}^2(S_1)$ the operator corresponding to Eqs. (4.2) for C^ϵ_q is

$$\mathbb{J}_3^{\rm I} - \mathbb{J}_1^{\rm I} = -i e^{-i\epsilon\phi}\left[(1+\cos\phi)\frac{d}{d\phi} - k\sin\phi\right]e^{i\epsilon\phi}, \quad (4.6)$$

with q, k, and λ related as in Eqs. (1.3), and p and ϵ as in Eq. (2.7d). Through the change of variables $\xi = \tan(\phi/2)$ we can find the generalized Dirac-normalized eigenfunctions to be[38]

$$\chi^{p,\lambda}_\nu(\phi) = (4\pi)^{-1/2}(\cos(\phi/2))^{-2k}e^{-i\epsilon\phi}\exp[i\nu\tan(\phi/2)]. \qquad (4.7)$$

The generating function (4.1) is thus readily calculated from the integral as

$$K^{p,\lambda}_\sigma(\phi,\rho) = \rho^{1/2}\chi^{p,\lambda}_{\sigma\rho^2/2}(\phi)\exp[i\Phi^{\chi}(p,\lambda,\sigma\rho^2/2,\sigma)]. \qquad (4.8)$$

In order to determine the phase function, consider the orthonormal $\mathscr{L}^2(S_1)$ eigenbasis for $\mathbb{J}^{\rm I}_3$ in C^ϵ_q:

$$\psi^{p,\lambda}_m(\phi) = [\eta^{p,\lambda}_m]^{-1}(2\pi)^{-1/2}\exp[i(m-\epsilon)\phi], \quad (4.9)$$

where m is the integer for $\epsilon = 0$ ($p = +1$) and half-integer for $\epsilon = 1/2$ ($p = -1$). The phase factors $\eta^{p,\lambda}_m$ will be those of Bargmann[36]:

$$\eta^{1,\lambda}_0 = 1 = \eta^{-1,\lambda}_{1/2}, \qquad (4.10a)$$

$$\eta^{p,\lambda}_m = (-1)^{m-\epsilon} \times \prod_{l=\epsilon+1/2}^{m-1/2}[(l-i\lambda/2)/(l+i\lambda/2)]^{1/2}, \quad m\geqslant 1, \qquad (4.10b)$$

$$\eta^{p,\lambda}_{-m} = (-1)^{m+\epsilon}(i\lambda/|\lambda|)^{2\epsilon}\eta^{p,\lambda}_m, \quad m\geqslant 1/2, \qquad (4.10c)$$

where the running index in Eq. (4.10b) takes the $m-\epsilon$ values $l = \epsilon+1/2, \epsilon+3/2, \ldots, m-1/2$. The basis vectors (4.9) of $\mathscr{L}^2(S_1)$ should, upon their transformation to $\mathscr{L}^2_{\rm II}(\mathscr{R}^+)$, provide the properly normalized eigenbasis for $\mathbb{J}^{\rm II}_3$. Thus, introducing Eq. (4.9) in (4.2b) with the intertwining (4.8) (with the as yet undetermined phase), we find, under a division of the integration range in two, trigonometric identities and an integration[39], that

$$\Psi^{p,\lambda}_{m,\sigma}(\rho) = \int_{-\pi}^{\pi} d\phi\,\psi^{p,\lambda}_m(\phi) K^{p,\lambda}_\sigma(\phi,\rho)^*$$
$$= [\eta^{p,\lambda}_m]^{-1}2^{1/2-i\lambda}[\Gamma(k+\sigma m)]^{-1}\rho^{-1/2+i\lambda} \times W_{\sigma m,-i\lambda/2}(\rho^2)\exp[-i\Phi^{\chi}(p,\lambda,\sigma\rho^2/2,\sigma)], \qquad (4.11)$$

which is valid for integer as well as half-integer values of m.

Notice that the phase factor cannot depend on m. Now, the (unnormalized) solutions of $\mathbb{J}_3\Psi(\rho) = m\Psi(\rho)$ which are bounded at infinity are of the form[40] $\rho^{-1/2}W_{m,\pm i\lambda/2}(\rho^2)$; the phase factor is thus constrained to be $\rho^{-i\lambda}$ times any other ρ-independent phase. We can set

$$\Phi^{\chi}(p,\lambda,\sigma\rho^2/2,\sigma) = \lambda\ln(\rho/2) \qquad (4.12)$$

and declare the proper eigenfunctions of $\mathbb{J}^{\rm II}_3$ corresponding to the eigenvalue m (integer or half-integer) to be

$$\Psi^{p\lambda}_{m,\sigma}(\rho) = [\eta^{p,\lambda}_m\Gamma(k+\sigma m)]^{-1} \times (\rho/2)^{-1/2}W_{\sigma m,-i\lambda/2}(\rho^2) \qquad (4.13)$$

spanning the C^ϵ_q irrep for sl(2,R). On Eq. (4.13) we can verify immediately that we have an eigenfunction of $\mathbb{J}_3^{\text{II}}$, as this operator acts as $\sigma\mathbb{J}_3$ [see Eq. (1.1c)] on the two $\sigma = +1$ and $\sigma = -1$ components. Hance the eigenvalue is indeed m. Normalization under the inner product (3.6) carr be checked straightforwardly[41]. In order to support our claim that Eq. (4.12) is indeed an appropriate phase, we may verify that the action of $\mathbb{J}_{\pm}^{\text{I}} = \mathbb{J}_1^{\text{I}} \pm i\mathbb{J}_2^{\text{I}}$ on the simple functions $\psi_m^{p,\lambda}(\phi)$, namely, $(\eta_m^{p\lambda}/\eta_{m\pm1}^{p\lambda})(k \pm m)\psi_{m\pm1}^{p,\lambda}(\phi)$ is the same as that of $\mathbb{J}_{\pm}^{\text{II}}$ on $\Psi_{m,\sigma}^{p,\lambda}(\rho)$. This has to be done separately on the upper and lower components as $\mathbb{J}_{\pm}^{\text{II}} = \sigma\mathbb{J}_1 \pm i\mathbb{J}_2$ and yields the same result through the recurrence realtions for Whittaker functions[42]. Finally, the transformation properties under $\mathbb{A}$ in Eqs. (2.9) can be defined explicitly, as their action on $\mathscr{L}_{\text{II}}^2(\mathscr{R}^+)$ is to exchange the two component functions

$$\mathbb{A}:\Psi_{m,\sigma}^{p,\lambda}(\rho) = \Psi_{m,-\sigma}^{p,\lambda}(\rho) = (\eta_{-m}^{p,\lambda}/\eta_m^{p,\lambda})\Psi_{-m,\sigma}^{p,\lambda}(\rho), \quad (4.14)$$

and it can be readily seen that $\mathbb{A}^2 = \mathbb{1}$. From Eqs. (4.10), the factor in Eq. (4.14) is $(-1)^m$ for $\epsilon = 0$ and [43] $i(-1)^{m+\epsilon} \times \text{sign}\lambda\,\text{sign}m$ for $\epsilon = 1/2$. The intertwining kernel can thus be written as

$$K_\sigma^{p,\lambda}(\phi,\rho) = (2\pi)^{-1/2}e^{-i\epsilon\phi}(\rho/2)^{1/2+i\lambda}(\cos(\phi/2))^{-1-i\lambda} \times \exp(i\tfrac{1}{2}\sigma\rho^2\tan(\phi/2)). \quad (4.15)$$

The generalized eigenfunctions of the parabolic generator $\mathbb{J}_3^{\text{II}} + \mathbb{J}_1^{\text{II}}$ can be found from those of $\mathbb{J}_3^{\text{II}} - \mathbb{J}_1^{\text{II}}$ in Eq. (4.5) through the Fourier transformation (3.15) representing a rotation by $-\pi$ around the 3-axis. Since Eqs. (4.5) are essentially Dirac δ's in ρ, the $\mathbb{J}_3^{\text{II}} + \mathbb{J}_1^{\text{II}}$ eigenfunctions will include $H_{\sigma,\sigma'}^{p,\lambda}([2|\nu|]^{1/2}\rho)$—Hankel and Macdonald functions of imaginary index—times $\rho^{1/2}$. The corresponding $\mathbb{J}_3^{\text{I}} + \mathbb{J}_1^{\text{I}}$ generalized eigenfunctions are obtained from Eq. (4.7) simply by a rotation of π in the argument. These basis functions and their transformation properties are particularly interesting, since from Eqs. (2.1) it can be seen that $\mathbb{M}_{23} + \mathbb{M}_{25}$ is the *Klein-Gordon* operator in a two-dimensional space–time. We reserve some observations pertaining to this subject and the Kontorovich–Lebedev transform for future development.

As a final calculation, let us use the preceding information in order to find the generalized eigenfunctions of the hyperbolic operators $\mathbb{J}_2^\bullet$ and $\mathbb{J}_1^\bullet$ in $\mathscr{L}_{\text{II}}^2(\mathscr{R}^+)$ and $\mathscr{L}^2(S_1)$. The four functions are related by pairs by a rotation by $\pi/2$ over the 3-axis (i.e., the square root of the hyperbolic Fourier transform) and by the intertwining operator. The simplest of the four are the $\mathscr{L}_{\text{II}}^2(\mathscr{R}^+)$ generalized eigenfunctions of $\mathbb{J}_2^{\text{II}}$ and $\mathbb{A}$:

$$\Upsilon_{\tau,a,\sigma}^{p,\lambda}(\rho) = (2\pi)^{-1/2}(\delta_{\sigma,1} + a\delta_{\sigma,-1})\rho^{-1/2+2i\tau}, \quad (4.16)$$

with eigenvalues $\tau \in \mathscr{R}$ and $a = \pm 1$, respectively. The spectrum of this hyperbolic operator thus covers the real line twice[44]. The functions (4.16) are Dirac orthonormal and complete with respect to Eq. (3.6) as can be ascertained through bilateral Mellin transformation[45]. The corresponding $\mathbb{J}_2^{\text{I}}$ eigenfunctions can be found through Eqs. (4.2a) and (4.15) using the Fourier transform of the complex power functions[46]. Defining the "cut" functions

$$x_+ = \begin{cases} x, & x \geqslant 0 \\ 0 & x < 0 \end{cases}, \quad x_- = \begin{cases} 0, & x \geqslant 0, \\ -x, & x < 0, \end{cases} \quad (4.17a)$$

we find

$$\begin{aligned} v_{\tau,a}^{p,\lambda}(\phi) &= (4\pi)^{-1}2^{i(\tau-\lambda/2)}\Gamma(\eta)e^{-i\epsilon\phi}(\cos(\phi/2))^{-1-i\lambda} \\ &\quad \times (e^{i\pi\eta/2} + ae^{-\pi\eta/2}) \\ &\quad \times [(\tan(\phi/2))_+^{-\eta} + a(\tan(\phi/2))_-^{-\eta}] \\ &= \pi^{-1}2^{-1/2+i\tau}\Gamma(\eta)e^{-i\epsilon\phi}\text{trig}_a(\pi\eta/2) \\ &\quad \times (\delta_{1,\text{sign}\phi} + a\delta_{-1,\text{sign}\phi})|\sin\phi|^{-k}|\tan(\phi/2)|^{-i\tau}, \end{aligned} \quad (4.17b)$$

where $\eta = k + i\tau = \frac{1}{2} + i(\tau + \lambda/2)$. The eigenfunctions of $\mathbb{J}_1^\bullet$ can be now found from Eqs. (4.16) and (4.17), as $\mathbb{J}_1^\bullet = \exp(i\frac{1}{2}\pi\mathbb{J}_3^\bullet)\mathbb{J}_2^\bullet\exp(-i\frac{1}{2}\pi\mathbb{J}_3^\bullet)$. This amounts to a rotation by $\pi/2$ in S_1:

$$\omega_{\tau,a}^{p,\lambda}(\phi) = v_{\tau,a}^{p,\lambda}(\phi + \pi/2). \quad (4.18)$$

In $\mathscr{L}_{\text{II}}^2(\mathscr{R}^+)$ this is the $2^{-1/2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ transform (3.9) of the chosen $(\mathbb{J}_2^{\text{II}},\mathbb{A})$ eigenfunctions (4.16). The hyperbolic canonical transform involves three integrals for each component[47]. After several cancellations and factorizations, we obtain

$$\Omega_{\tau,a,\sigma}^{p,\lambda}(\rho) = C_\tau^\lambda V_{a,\sigma}^{p,\lambda}e^{-\sigma\pi\tau/2}\rho^{-1/2}W_{-i\tau,i\lambda/2}(i\sigma\rho^2), \quad (4.19a)$$

$$C_\tau^\lambda = 2^{1/2+i\tau}(2\pi)^{-3/2}\Gamma(\tfrac{1}{2} + i\tau + i\tfrac{1}{2}\lambda)\Gamma(\tfrac{1}{2} + i\tau - i\tfrac{1}{2}\lambda), \quad (4.19b)$$

$$V_{a,\sigma}^{p,\lambda} = (\delta_{\sigma,1} + ap\delta_{\sigma,-1})[2a\text{hyp}_{1,p}(\lambda\pi/2) - i\tfrac{1}{2}(p-1)], \quad (4.19c)$$

which are indeed eigenfunctions of $\mathbb{J}_1^{\text{II}}$ with eigenvalue τ. They are not eigenfunctions of $\mathbb{A}$, or course; rather, $\mathbb{A}$ can be seen to map Eq. (4.19a) through $\sigma \to -\sigma$ into an eigenfunction of $\mathbb{J}_1^{\text{II}}$ with eigenvalue $-\tau$. As $W_{\mu,\nu}(z)$ and $W_{-\mu,\nu}(-z)$ are independent solutions to the Whittaker equation, their relation is not simple. In fact,

$$\begin{aligned} \mathbb{A}\mathbf{\Omega}_{\tau,a}^{p,\lambda} &= \mathbb{A}\exp(i\tfrac{1}{2}\pi\mathbb{J}_3^{\text{II}})\mathbf{\Upsilon}_{\tau,a}^{p,\lambda} \\ &= \exp(-i\tfrac{1}{2}\pi\mathbb{J}_3^{\text{II}})\mathbb{A}\mathbf{\Upsilon}_{\tau,a}^{p,\lambda} = a\mathbb{C}_{\text{F}}^{p,\lambda}\mathbf{\Omega}_{\tau,a}^{p,\lambda}, \end{aligned} \quad (4.20)$$

where $\mathbb{C}_{\text{F}}^{p,\lambda}$ is the hyperbolic Fourier transform as given by Eq. (3.15).

5. APPLICATIONS AND CONCLUSION

The analytic properties of the basis functions and transformations belonging to the continuous series of the SL(2,R) group generated by Eqs. (1.1) have been seen to be rather arduous. Their group-theoretic properties are, however, as simple as that of any other realization, and herein lies the advantage of using the latter to derive relations for the former. These relations take the form of integral identities involving Hankel, Macdonald, Whittaker, power, and exponential functions, some with imaginary indices and parameters, which are now endowed with a group-theoretic interpretation. In what follows, we outline five examples of applications of these concepts.

First, of course, we have the Hankel and Macdonald function integral relations implicit in the kernel composition (3.13). Second, Whittaker functions of the kind (4.13) and (4.19) are displayed as being *self-reciprocating*[48] under hyperbolic canonical transforms. This can be seen in the following way: Consider the matrix identity

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad a \neq 0, \tag{5.1a}$$

$$\alpha = (a^2 + b^2)^{1/2}, \ \gamma = (ac + bd)/\alpha, \ \tan t = -b/a. \tag{5.1b}$$

The integral transforms associated to these matrices will follow suit through Eq. (3.3). Now, apply these transforms to the $\mathbb{J}_3^{\text{II}}$ eigenfunctions $\Psi_m^{\rho\lambda}(\rho)$ in Eq. (4.13), noting that the rightmost transform will multiply the functions by $\exp(2imt)$, while the second transform is purely geometric and given by Eq. (3.12). Their composition thus leads to the integral relation

$$\sum_{\sigma' = \pm 1} \int_0^\infty d\rho' C_{\mathbf{M},\sigma,\sigma'}^{\rho,\lambda}(\rho,\rho') \Psi_{m,\sigma'}^{\rho,\lambda}(\rho') = |\alpha|^{-1/2} (\text{sign}\alpha)^{2\epsilon} \times e^{2imt} \exp(i\sigma\gamma\rho^2/2\alpha) \Psi_{m,\sigma}^{\rho,\lambda}(\rho/|\alpha|), \tag{5.2}$$

which, if written out explicitly [Eqs. (2.7d), (3.9)–(3.11), (4.13), and (5.1)], is rather difficult to solve by elementary methods. Decompositions analogous to Eqs. (5.1) can be made for the parabolic- and hyperbolic-operator eigenfunctions seen in Sec. 4.

Third, the intertwining operator (4.2) can be used to "close the fourth side of a rectangle" in applying a hyperbolic canonical transform to a given function in $\mathscr{L}^2_{\text{II}}(\mathscr{R}^+)$: we pass to $\mathscr{L}^2(S_1)$, transform the function there [this is an easy task since the group SL(2,R) in that space acts geometrically as its generators are of first order], and transform back to $\mathscr{L}^2_{\text{II}}(\mathscr{R}^+)$. Fourth, the intertwining integral may be solved if the functions involved are recognized to be canonical transforms of eigenfunctions of SL(2,R) generators. We use formulas such as Eq. (5.2) in order to transform them to the simplest eigenfunction of the orbit such as Eq. (4.5) for the parabolic and Eq. (4.16) for the hyperbolic cases, intertwine the resulting simpler function with the aid of the results of Sec. 4, and transform back in $\mathscr{L}^2(S_1)$. Fifth, $\mathscr{L}^2_{\text{II}}(\mathscr{R}^+)$ inner products between basis functions such as the right-hand side of Eq. (5.2) may be intertwined to their $\mathscr{L}^2(S_1)$ counterparts and the simpler ϕ — integral solved. The latter is nothing more than a SL(2,R) representation matrix element (same or mixed basis) and thus expressible in terms of ${}_2F_1$ hypergeometric functions[49].

From the point of view of canonical transformations in quantum mechanics, we have been occupied with potentials which are not realistic. Our approach, however, suggests that any other classical-quantum correspondence method of solution[17] tackling Eq. (1.7) should, when extended to strongly centripetal potentials, lead to the results in this article.

As regards SL(2,R) representation theory, only the supplementary series ($0 < q < \frac{1}{4}$) remains to be worked out, in particular, the peculiar properties of the representations at the values $q = 0$ and $\frac{1}{4}$ of the Casimir operator.

Finally, on the terrain of the integral transform theory, we have previously shown that[5–7] Fourier and Hankel transforms are particular cases of real linear and radial canonical transforms and that, through complex extension, one can reach the bilateral Laplace, Gauss–Weierstrass, Bargmann, and Barut–Girardello transforms. Hyperbolic canonical transforms do not seem to include any well-known particular cases, yet they come within close range: The Meijer-K, Kontorovich–Lebedev, and Neumann transforms[50]. The first ones, involving kernels with Macdonald functions of real index and related to the Laplace transform, may be reached if a valid analytic continuation of the kernel can be implemented. This may require nonunitary SO(1,1) representations in Eq. (2.5). The second transform involves Hankel functions of imaginary index, where the integrations take place on the argument and on the index. This seems to require either a different subgroup reduction of Sp(4,R) or operators other than Eq. (3.5) in the representation decomposition. As both of these cases involve single-component functions, we surmise that they correspond to the A-diagonal Fourier transform (3.15). Lastly, Neumann transforms—and, indeed, Hankel transforms as well—are suggested by the analytic continuation in λ of the kernel elements (3.11a), as even the Struve function contained in the inverse Neumann transform appears to be closely related[51] to the use of the representations of a compact subgroup. It is our intention to address these extensions and further the study of the Klein–Gordon operator elsewhere.

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[22]These results can be obtained from the analysis of $SL(2,R)_1 \otimes SL(2,R)_2$ on the two Cartesian coordinates. Our introduction of $Sp(4,R)$ is primarily for the purpose of exhibiting its various subgroup reductions, related integral transforms and automorphisms of $SL(2,R)$ which are inner in $Sp(4,R)$.
[23]The specification of the phase of the 2–2 element of the Pauli matrix insures that the q_2 factor in the transform has the appropriate phase. It is otherwise imposed by the inversion of (1.4a) and (1.4b) subgroups and the phase consistency of this operation. See Ref. 7, p. 389, where negative-b elements of a real 2×2 matrix must be set to have phase $-\pi$.
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[33]Mello and Moshinsky (Ref. 17) also comment on the definition of a phase for their canonical transformation kernel.
[34]This corresponds to the subgroup of lower-triangular matrices in Eq. (3.12) with $c = \gamma$ and $a = 1 = d, b = 0$.
[35]The new diagonal operator can be found from Eq. (1.5) with the parameters of Ref. 34.
[36]V. Bargmann, Ref. 12, Eqs. (6.23) and (7.10). The phase factors are chosen so that the raising operator, his Eq. (6.26), has only positive matrix elements.
[37]This is the case for the J_2^{11} eigenfunctions. Mukunda and Radhakrishnan devote a lengthier commentary on this fact; see Ref. 10, p. 1323.
[38]In writing Eq. (4.7) we are proposing a definite choice of phase. As far as the construction of the intertwining kernel is concerned, all phases may be ascribed to eigenfunctions (4.5).
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