On self-reciprocal functions under a class of integral transforms

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We use the fact that a rather general class of integral transforms—complex linear and radial canonical transforms—are equivalent to hyperdifferential operators, to formulate the problem of self-reciprocal functions under these transforms as an eigenvalue problem for (second-order) differential operators. We thus find the solution for Fourier, Hankel, bilateral Laplace, Bargmann, Weierstrass–Gauss and Barut–Girardello transforms. These involve the Schrödinger attractive and repulsive harmonic oscillator and/or centrifugal potential wavefunctions. A general concept of "self-reproducing" functions is introduced which includes all of the above plus linear potential wavefunctions. In particular, two new generalized bases for Bargmann's Hilbert space of analytic functions are found.

I. INTRODUCTION

A function $f(x)$ will be said to be self-reciprocal under an integral transform $\mathcal{T}_M$ [defined through integration over an interval $\mathbb{I}\subseteq\mathbb{R}$ with a kernel $A_M(x,x')$] when

$$(\mathcal{T}_M f)(x)=\int_{\mathbb{I}} dx' A_M(x,x')f(x')=\lambda f(x),\quad \lambda\in\mathbb{C}. \tag{1.1}$$

This corresponds to the eigenfunction problem for the operator $\mathcal{T}_M$. The cases we are interested in include the well-known cases of the Fourier[1] and Hankel[2] transforms, as well as the bilateral Laplace,[3] Bargmann,[4] Weierstrass–Gauss[5] (which represents the time evolution of the solutions of the heat equation), and Barut–Girardello[6] transforms. These constitute special cases of a class of integral transforms termed canonical transforms[7,8] which will be described in Sec. II.

The functions which are self-reciprocal under the Fourier transform are well known,[9] while some properties of functions self-reciprocal under Hankel transforms have been analyzed in the work of Hardy and Titchmarsh.[10] Further results on the Hankel self-reciprocal functions and the consideration of the (unilateral) Laplace transform and some of its variants has been presented in a series of papers by Indian mathematicians.[11] The solution we present to the problem (1.1) makes use of the observation that, for the class of canonical transforms and $f\in\mathcal{C}_M^\infty$ [the intersection of the space $\mathcal{C}^\infty$ and the space of functions for which the integral (1.1) exists at least in a generalized sense], one can realize $\mathcal{T}_M$ as a hyperdifferential operator

$$(\mathcal{T}_{M(\tau)}f)(x)=\exp(i\tau H^\omega)f(x), \tag{1.2}$$

where τ is a continuous parameter which for certain values yields the particular transforms mentioned above, and H^ω is a second-order differential operator, self-adjoint in $\mathcal{L}^2(\mathbb{I})$. Clearly, the solution of the eigenvalue equation

$$H^\omega\Phi_\mu^\omega(x)=\mu\Phi_\mu^\omega(x) \tag{1.3}$$

solves (1.1) with $\lambda=\exp(i\tau\mu)$.

When τ is a rational multiple of π and the spectrum $\Lambda^\omega=\{\mu\}$ of H^ω is discrete and integer-spaced (cases of Fourier and Hankel), the spectrum $\Lambda_M=\{\lambda\}$ of $\mathcal{T}_M$ will consist of a finite number N_M of values of unit modulus, and will divide the space $\mathcal{C}_M^\infty$ into subspaces $\mathcal{S}_\lambda$ of self-reciprocal functions labeled by the eigenvalue λ. The functions spanning each of these spaces will be the subsets of eigenfunctions Φ_μ^ω with $\mu\equiv\mu \bmod N_M$. When the usual closing procedure is implemented, the $\mathcal{S}_\lambda$ become Hilbert spaces. This is applied to the Fourier and Hankel transforms in Secs. III and IV. When the spectrum Λ^ω and hence Λ_M are continuous, the generalized eigenfunctions of H^ω will still provide the self-reciprocal functions of (1.1). This is the case of the bilateral Laplace and Bargmann transforms analyzed in Secs. V and VI respectively. The case of the Bargmann transform is particularly interesting since its generalized self-reciprocal functions (the repulsive quantum oscillator wavefunctions), being orthonormal (in the sense of Dirac) and complete in $\mathcal{L}^2(\mathbb{R})$ are so too, in the same sense, in the Bargmann–Hilbert space[4] $\mathcal{F}_B$ [of entire analytic functions of growth $(2, 1/2)$]. This generalized basis is new and adds to the known harmonic oscillator and coherent-state[4,12] bases of $\mathcal{F}_B$. The results for the Weierstrass–Gauss and Barut–Girardello transforms are sketched in Secs. VII and VIII. As for the general case of the complex linear[7] and radial[8] canonical transforms, we explore the generalized "self-reproducing" functions in Sec. IX. These include, beside the functions studied before, the Airy functions.

II. CANONICAL TRANSFORMS AND THEIR HYPERDIFFERENTIAL REALIZATION

To every complex unimodular 2×2 matrix $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ we associate the integral transform $\mathcal{T}_M$ given as in (1.1) with the kernel

$$A_M(x,x')=(2\pi|b|)^{-1/2}\varphi_b\exp\left(\frac{i}{2b}(ax'^2-2x'x+dx^2)\right), \tag{2.1a}$$

$$\varphi_b=\exp[-(i/2)(\pi/2+\arg b)], \tag{2.1b}$$

which we call the canonical transform.[7]

When a,b,c,d are real,[13] $\mathcal{T}_M$ can be seen to be a unitary mapping from $\mathcal{L}^2(\mathbb{R})$ onto $\mathcal{L}^2(\mathbb{R})$, while if these parameters are complex, the resulting transform [when bounded: for $\mathrm{Im}(a/b)>0$ and b real if $a=0$] is a unitary mapping between $\mathcal{L}^2(\mathbb{R})$ and Hilbert spaces $\mathcal{F}_M$ of analytic functions defined through the scalar product over the complex plane, for $\bar f=\mathcal{T}_M f$ and $\bar g=\mathcal{T}_M g$,

$$(\bar f,\bar g)_M=\int_{\mathbb{C}} d\mu_M(x)\,\bar f(x)^*\,\bar g(x), \tag{2.2a}$$

$$d\mu_M(x) = 2(2\pi v)^{-1/2} \exp\left[\frac{1}{2v}(ux^2 - 2xx^* + u^*x^{*2})\right] d\,\mathrm{Re}\,x\; d\,\mathrm{Im}\,x, \tag{2.2b}$$

$$u = a^*d - b^*c, \quad v = 2\,\mathrm{Im}\,b^*a. \tag{2.2c}$$

The functions $\tilde{f}(x)$ in the spaces $\mathcal{F}_M$ are characterized as $\tilde{f}(x) = \exp(-ux^2/2v)\tilde{f}_B(x)$, where $\tilde{f}_B(x)$ are elements in the Bargmann–Hilbert space.[4] The transform inverse to (1.1) is given by

$$f(x) = (\mathcal{T}_M^{-1}\tilde{f})(x) = \int_{\mathbb{C}} d\mu_M(x') A_M(x', x)^* \tilde{f}(x'). \tag{2.3}$$

For the transforms we are interested in, we can identify the following:

$$(\text{Fourier}) \quad \mathcal{F} = \exp(i\pi/4)\,\mathcal{T}_F, \quad F = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right),$$

$$(\text{Bilateral Laplace}) \quad \mathcal{L} = i\sqrt{2\pi}\,\mathcal{T}_L, \quad L = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right), \tag{2.4b}$$

$$(\text{Bargmann}) \quad \mathcal{B} = (2\pi)^{1/4}\,\mathcal{T}_B, \quad B = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -i \\ -i & 0 \end{smallmatrix}\right), \tag{2.4c}$$

[with the specific choice of phase $\pm i = \exp(\pm i\pi/2)$]

$$(\text{Weierstrass–Gauss}) \quad \mathcal{W}_t = \mathcal{T}_{W_t}, \quad W_t = \left(\begin{smallmatrix} 1 & -2it \\ 0 & 1 \end{smallmatrix}\right). \tag{2.4d}$$

We will find it useful to define the geometric transform as

$$(\mathcal{T}_{G(\beta,c)}f)(x) = e^{\beta/2}\exp[\tfrac{1}{2}ie^{\beta}cx^2] f(e^{\beta}x),$$
$$G(\beta, c) = \begin{pmatrix} e^{-\beta} & 0 \\ c & e^{\beta} \end{pmatrix}, \tag{2.4e}$$

which can be obtained from the general case (2.1) letting[7] $b \to 0$.

These results and their derivation are found in Ref. 7, where we also analyze the behavior of the measure (2.2) when $v \to 0$. We should point out the novelty that in our treatment the Weierstrass–Gauss transform[5] becomes a unitary transformation between Hilbert spaces with a conserved scalar product and a proper inversion. One more result which we can extract from Ref. 7 is the fact that, when two transforms $\mathcal{T}_{M_1}$ and $\mathcal{T}_{M_2}$ are bounded, their composition (through integration over $\mathbb{R}$) follows the product of the matrices $M_1 M_2 = M_3$, so that $\mathcal{T}_{M_1} \circ \mathcal{T}_{M_2} = \varphi \mathcal{T}_{M_3}$, where φ is a phase (± 1) depending on 1–2 matrix elements of the M_i's. Notice, however, that it is not necessary that a transform $\mathcal{F}_M$ be bounded in order to have a nonvanishing domain $\mathcal{C}_M^\infty$ dense in $\mathcal{L}^2(\mathbb{I})$. This remark applies to the case of the bilateral Laplace transform, which is unbounded. Finally, we should stress that, while all (bounded) transforms are unitary mappings between $\mathcal{L}^2(\mathbb{I})$ and $\mathcal{F}_M$, when seen as mappings between $\mathcal{L}^2(\mathbb{I})$ and $\mathcal{L}^2(\mathbb{I})$, the transforms $\mathcal{T}_M$ with complex M are nonunitary.

In Ref. 7 it was shown that for $f \in \mathcal{C}_M^\infty$, the integral transforms (1.1)–(2.1) are equivalent to the action of the hyperdifferential operators (1.2). Specifically, for the one-parameter subsets which contain the transforms (2.4),

$$H^h = \tfrac{1}{2}(-\Delta + x^2) \quad \text{generating} \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}, \tag{2.5a}$$

$$H^r = \tfrac{1}{2}(-\Delta - x^2) \quad \text{generating} \begin{pmatrix} \cosh\tau & -\sinh\tau \\ -\sinh\tau & \cosh\tau \end{pmatrix}, \tag{2.5b}$$

$$H^f = -\tfrac{1}{2}\Delta \quad \text{generating} \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix}, \tag{2.5c}$$

where

$$\Delta = \frac{d^2}{dx^2}. \tag{2.5d}$$

Thus, the Fourier transform can be written in hyperdifferential form (1.2) with H^h as given by (2.4a) and $\tau = -\pi/2$, the Laplace and Bargmann transforms with H^r as (2.5b) and $\tau = -i\pi/2$ and $i\pi/4$, respectively, while the Weierstrass–Gauss transform appears with H^f as (2.5c) and $\tau = 2it$. This last case is commonly known.[5] Finally, the generators of geometric transforms (2.4d) are first-order differential operators which can be seen to be, for the β parameter,

$$H^d = -i\left(x\frac{d}{dx} + \frac{1}{2}\right) \quad \text{generating} \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix}, \tag{2.6a}$$

while for the γ parameter it is simply

$$\tfrac{1}{2}x^2 \quad \text{generating} \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}. \tag{2.6b}$$

In Ref. 8 we considered the "radial part" of a special n-dimensional version of the transform (2.1). This class of integral transforms have the form (1.1) over the interval $\mathbb{I} = \mathbb{R}^+$ (the positive half-axis), while the kernel, instead of (2.1), turns out to be

$$A_M^{[k]}(x, x') = b^{-1}\exp(-ik\pi)(xx')^{1/2} \exp\left[\frac{i}{2b}(ax'^2 + dx^2)\right] J_{2k-1}(xx'/b). \tag{2.7}$$

As before, for a, b, c, d real,[14] (1.1)–(2.7) is a unitary mapping from $\mathcal{L}^2(\mathbb{R}^+)$ onto $\mathcal{L}^2(\mathbb{R}^+)$. As particular cases we have the following transforms:

$$(\text{Hankel}) \quad \mathcal{H}^{[k]} = \exp(ik\pi)\mathcal{T}_F^{[k]}, \quad F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.8a}$$

$$(\text{Barut–Girardello}^{15}) \quad \mathcal{G}^{[k]} = \mathcal{T}_B^{[k]}, \quad B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{2.8b}$$

These transforms can also be put in hyperdifferential form (1.2) for functions in $\mathcal{C}_{M[k]}^\infty$. For the Hankel transform, the operator $H^{h[k]}$ has the form (2.5a) with $\tau = -\pi/2$ and the Barut–Girardello transform, (2.5b) with $\tau = i\pi/4$ but, instead of (2.6), the "radial" operator Δ is in these cases

$$\Delta^{[k]} = \frac{d^2}{dx^2} - \frac{(2k-1)^2 - 1/4}{x^2}. \tag{2.9}$$

The parameters of the transform kernel (2.7), when extended to complex values, define a unitary map from $\mathcal{L}^2(\mathbb{R}^+)$ to spaces $\mathcal{F}_M^{[k]}$ with a scalar product which corresponds basically to the radial part of the kth spherical harmonic part of (2.2). Thus, the "radial part" of an n-dimensional Bargmann transform is the Barut–Girardello transform and similar "radial Weierstrass–Gauss" transforms, for example, can be constructed.

III. SELF-RECIPROCAL FUNCTIONS UNDER THE FOURIER TRANSFORM

The results presented in the last section allow us to state that the eigenfunctions of H^h in (2.5a), namely the quantum harmonic oscillator wavefunctions

$$\Phi_n^h(x) = (\pi^{1/2} 2^n n!)^{-1/2} \exp(-x^2/2) H_n(x), \quad n = 0, 1, 2, \cdots \tag{3.1}$$

will be self-reciprocal under the Fourier transform (2.4a). This is a well-known fact[9] which will clearly illustrate our method. Since the spectrum of H^h is $\mu = n + \frac{1}{2}$, $n = 0, 1, 2, \cdots$, then

$$(\mathcal{F}\Phi_n^h)(x) = \exp(-i\pi n/2)\, \Phi_n^h(x). \tag{3.2}$$

Equation (3.2) allows us to split $\mathcal{C}_F^\infty$ [and its closure $\mathcal{L}^2(\mathbb{R})$] into four subspaces $\mathcal{S}_\lambda$, for $\lambda = 1, i, -1$, or $-i$ (recall that $\mathcal{F}^4 = 1$). Each of these subspaces is generated by the set Φ_n^h with $n \equiv (0, 1, 2$, or $3)$ mod4 respectively. Clearly, the intersection of two different $\mathcal{S}_\lambda$'s is empty, while the union of the four is $\mathcal{L}^2(\mathbb{R})$. The raising and lowering operators

$$2^{-1/2}\left[x - \frac{d}{dx}\right]\Phi_n^h(x) = (n+1)^{1/2}\Phi_{n+1}^h(x), \tag{3.3a}$$

$$2^{-1/2}\left[x + \frac{d}{dx}\right]\Phi_n^h(x) = n^{1/2}\Phi_{n-1}^h(x), \tag{3.3b}$$

are n-independent and will thus map the $\mathcal{S}_\lambda$ to $\mathcal{S}_{\lambda'}$ rotating the λ plane counterclockwise and clockwise by $\pi/2$.

IV. SELF-RECIPROCAL FUNCTIONS UNDER THE HANKEL TRANSFORM

Here we follow the general procedure as in the last section. It is well known that the normalized eigenfunctions of (2.5a) with (2.9) are

$$\Phi_n^{h(k)}(x) = \left(\frac{2n!}{\Gamma(n+2k)}\right)^{1/2} \exp(-x^2/2)\, x^{2k-1/2} L_n^{(2k-1)}(x^2),$$
$$n = 0, 1, 2, \cdots, \quad x \in \mathbb{R}^+. \tag{4.1}$$

The "radial" Schrödinger harmonic oscillator with centrifugal potential wavefunctions, for $k \geq 1$, (4.1) is the only set of eigenfunctions, while for $0 < k < 1$ we have more than one self-adjoint extension of (2.5a)—(2.9) in $\mathbb{R}^+$, one of which still has the eigenfunctions (4.1). The spectrum is $\mu = 2(n+k)$ with $n = 0, 1, 2 \cdots$ and thus

$$(\mathcal{H}_k \Phi_n^{h(k)})(x) = \exp(-in\pi)\, \Phi_n^{h(k)}(x), \tag{4.2}$$

exactly as in (3.2) and splitting the space $\mathcal{S}^{(k)}$ in $\mathcal{S}_\lambda^{(k)}$ ($\lambda = 1$ or -1) generated by $\Phi_n^{h(k)}$ with $n \equiv (0$ or $1)$ mod2, respectively, as before.

The n-independent differential operators which raise and lower the index n in (4.1) will similarly rotate the λ plane. From the raising and lowering operators for the upper index of the Laguerre polynomials, we find

$$\left(x + \frac{2k - 1/2}{x} - \frac{d}{dx}\right)\Phi_n^{h(k)}(x) = 2(n+2k)^{1/2}\, \Phi_n^{h(k+2)}(x), \tag{4.3a}$$

$$\left(x + \frac{2k - 3/2}{x} + \frac{d}{dx}\right)\Phi_n^{h(k)}(x) = 2(n+2k+1)^{1/2}\Phi_n^{h(k-2)}(x), \tag{4.3b}$$

which will thus map $\mathcal{S}_\lambda^{(k)}$ onto $\mathcal{S}_\lambda^{(k\pm2)}$, i.e., self-reciprocal functions of the Hankel transform of index k to index $k \pm 2$.

The characterization of the self-reciprocal functions under the Hankel transform with either eigenvalue λ thus seems almost trivial and certainly simpler than that presented in Refs. 10 and 11.

V. SELF-RECIPROCAL FUNCTIONS UNDER THE BILATERAL LAPLACE TRANSFORM

The bilateral Laplace transform can be realized through the hyperdifferential operator (1.2) with exponent (2.5b), the quantum repulsive oscillator Hamiltonian, and $\tau = -i\pi$, on a suitable function space. The eigenfunctions of (2.5b) are the repulsive oscillator quantum eigenfunctions

$$\Phi_\mu^{r\pm}(x) = C_\mu D_{i\mu-1/2}[\pm\sqrt{2}\exp(3i\pi/4)x], \quad \mu \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{5.1a}$$

$$C_\mu = 2^{-3/4}\pi^{-1}\Gamma(-i\mu + \tfrac{1}{2})\exp[-\tfrac{1}{4}i\pi(i\mu + \tfrac{1}{2})], \tag{5.1b}$$

where D_ν is the parabolic cylinder function. [See Ref. 16, Eq. (2.22) for their computation; the method closely follows that of Miller *et al.*, Ref. 17.] The spectrum of H^r covers twice the real line, so $\mu \in \mathbb{R}$ and Φ_μ^{r+} and Φ_μ^{r-} are mutually orthogonal and a generalized basis for $\mathcal{L}^2(\mathbb{R})$. Our statement is now that (5.1) are self-reciprocal under the Laplace transform and that, due to (2.4b),

$$(\mathcal{L}\Phi_\mu^{r\pm})(x) = i\sqrt{2\pi}\, \exp(\pi\mu/2)\, \Phi_\mu^{r\pm}(x). \tag{5.2}$$

The statements (5.1)—(5.2) can be made more transparent with the use of the technique of Ref. 17, of transforming them to simpler operators on the same orbit under the group as that generated by (2.4e). For this it is sufficient to notice that $\mathcal{T}_L$ in (2.4b) [as well as the Bargmann transform $\mathcal{T}_B$ in (2.4c), next section] is on the same orbit as the dilatation transformation (2.4e), that is

$$\mathcal{T}_{M_0}\mathcal{T}_L\mathcal{T}_{M_0}^{-1} = \mathcal{T}_{G(-i\pi/2,0)}, \quad M_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \tag{5.3}$$

(where M_0 represents the square root of the inverse Fourier transform, as $M_0^2 = F^{-1}$). Equation (5.3) can be verified by simply multiplying the corresponding 2×2 matrices. Now, the generalized eigenfunctions of H^d in (2.6) are

$$\Phi_\mu^{d\pm}(x) = (2\pi)^{-1/2} x_\pm^{i\mu-1/2}, \quad x_\pm \equiv \begin{cases}\pm x, & x \gtrless 0,\\ 0, & x \lessgtr 0,\end{cases} \tag{5.4}$$

with eigenvalue $\mu \in \mathbb{R}$. As (5.3) holds, we have that

$$\Phi_\mu^{r\pm}(x) = (\mathcal{T}_{M_0}^{-1}\Phi_\mu^{d\pm})(x). \tag{5.5}$$

The generalized orthonormality of the set (5.4) and its completeness for $\mathcal{L}^2(\mathbb{R})$ is known from the theory of Mellin transforms,[3] and from here the same statement follows for their unitary transforms (5.1) whose direct verification is far less straightforward than for (5.4). The action of transform (5.3) is that of dilatation by a factor $e^\beta = \exp(-i\pi/2)$. On the basis (5.4) this is clearly seen to be

$$(\mathcal{T}_{G(-i\pi/2,0)}\Phi_\mu^{d\pm})(x) = \exp(-i\pi/4)\, \Phi_\mu^{d\pm}[\exp(-i\pi/2)x]$$
$$= \exp(\pi\mu/2)\, \Phi_\mu^{d\pm}(x). \tag{5.6}$$

From here and (2.4b), Eq. (5.2) follows.

One point which merits brief consideration is the following fact: $(\Phi_\mu^{r\sigma}, \Phi_\mu^{r\sigma'}) = \delta(\mu-\mu')\delta_{\sigma\sigma'}$ $(\sigma=\pm)$, under the ordinary scalar product on the real line. This implies, through the parseval identity for the Laplace transform, that for the scalar product

$$(\bar{f},\bar{g})_L = \frac{1}{2\pi}\int_{-i\infty}^{i\infty} dx\, \bar{f}(x)^*\bar{g}(x) \tag{5.7a}$$

the same generalized orthogonality relation holds,

$$(\Phi_\mu^{r\sigma}, \Phi_{\mu'}^{r\sigma'})_L = \exp(\pi\mu)\delta(\mu-\mu')\,\delta_{\sigma\sigma'}. \tag{5.7b}$$

Completeness does not hold, however, as the transform is only isometric.

VI. SELF-RECIPROCAL FUNCTIONS UNDER THE BARGMANN TRANSFORM

The Bargmann transform[4] is closely related to the bilateral Laplace transform, as it has the same generating operator (1.2), namely in (2.5b). The value of the parameter τ is here $i\pi/4$. Indeed, we can point to the fact that $B^2 = L^{-1}$ through (2.4b) and (2.4c). This can be verified easily through integration. The self-reciprocal functions (5.1) of the Laplace transform will thus also be self-reciprocal under the Bargmann transform. With the proper constants from (2.4c) we have that

$$(\mathcal{B}\Phi_\mu^{r\pm})(x) = (2\pi)^{1/4}\exp(-\pi\mu/4)\Phi_\mu^{r\pm}(x). \tag{6.1}$$

This result can also be derived by noting that $\mathcal{T}_{M_0}\mathcal{T}_B\mathcal{T}_{M_0}^{-1} = \mathcal{T}_{G(i\pi/4,0)}$ where M_0 is given by (5.3), and repeating the argument of the last section. Finally, direct verification through integration[18] is also possible. As $B^{-2}=L$ this proves the result for Laplace transforms as well.

Now, use of the Parseval identity for Bargmann transforms and the unitarity of the transform, informs us that the set $\Phi_\mu^{r\pm}$ is a complete, orthonormal generalized basis for the Bargmann space $\mathcal{F}_B$ of analytic functions. Recall that the better-known bases for Bargmann's Hilbert space are the denumerable ("harmonic oscillator") monomials, i.e., powers of x, and the overcomplete coherent-state basis.[4,12] The repulsive oscillator basis can now be added to the list. The orthogonality relation (5.7) amounts to the same statement for the $\mathcal{F}_L$ space. On similar grounds, at the end of Sec. IX we will show that the Airy functions can be used to construct another such generalized basis.

VII. SELF-RECIPROCAL FUNCTIONS UNDER THE WEIERSTRASS-GAUSS TRANSFORM

For the Weiserstrass–Gauss transform we can use the eigenfunctions of (2.5c) and (2.5d) (recalling that the spectrum of this operator covers twice the positive half-axis), and choose

$$\Phi_\mu^f(x) = (2\pi)^{-1/2}\exp(i\mu x),\quad x\in\mathbb{R},\quad \mu\in\mathbb{R} \tag{7.1}$$

which has eigenvalues $\frac{1}{2}\mu^2$ as an appropriate basis. The exponentiation (1.2) with $\tau=2it$ then yields

$$(\mathcal{W}_t\Phi_\mu^f)(x) = \exp(-\mu^2 t)\,\Phi_\mu^f(x) \tag{7.2}$$

which is a well-known property of the heat equation Green's function.[5] Thus the functions (7.1) also provide a generalized orthogonal basis for the corresponding spaces $\mathcal{F}_{W_t}$.

A "radial" Weierstrass–Gauss transform for the matrix W_t in (2.4) with a modified Bessel function in the kernel (2.7) would have its self-reciprocal functions given by the eigenfunctions of $-\frac{1}{2}\Delta^{(k)}$ in (2.9), viz.,

$$\Phi_\mu^{f(k)}(x) = (\mu x)^{1/2}J_{2k-1}(\mu x),\quad x\in\mathbb{R}^+,\quad \mu\in\mathbb{R}^+ \tag{7.3}$$

with eigenvalue $\frac{1}{2}\mu^2$. An equation identical to (7.2) for these transforms follows. These functions will be used below.

VIII. SELF-RECIPROCAL FUNCTIONS UNDER THE BARUT-GIRARDELLO TRANSFORM

The Barut–Girardello transform[6] has the integral kernel (2.7) which stems from (2.8b) generated by (2.5b) with Δ given by (2.9). The generator is the quantum repulsive oscillator Hamiltonian with a centrifugal potential. Its eigenfunctions are

$$\Phi_\mu^{r(k)}(x) = C'_\mu x^{-1/2}M_{i\mu/2,k-1/2}(-ix^2),\quad x\in\mathbb{R}^+,\quad \mu\in\mathbb{R}, \tag{8.1a}$$

$$C'_\mu = 2^{(i\mu-1)/2}\pi^{-1/2}\Gamma(k+i\mu/2)\exp[i\pi(-i\tfrac{1}{4}\mu+k)]/\Gamma(2k), \tag{8.1b}$$

with eigenvalue μ, where $M_{\alpha\beta}$ is the Whittaker function. These functions can be found through the technique of Ref. 17, parallel to that of (5.5) through the use of the appropriate kernel (2.7). Since $\tau=i\pi/4$, as stated before, it follows that

$$(\mathcal{G}_k\Phi_\mu^{r(k)})(x) = \exp(-\pi\mu/4)\,\Phi_\mu^{r(k)}(x). \tag{8.2}$$

Remarks similar to those made in Sec. VI can be made to point out that (8.1) constitutes a new generalized basis for the Barut–Girardello space[8] $\mathcal{F}_B^{(k)}$.

IX. SELF-REPRODUCING FUNCTIONS UNDER CANONICAL TRANSFORMS

A useful generalization to the concept of self-reciprocal functions (1.1) to the class of canonical transforms $\mathcal{T}_M$, $M=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ given by (2.1) or (2.7) is to ask for functions $\phi_\mu^\omega(x)$ such that

$$[\mathcal{T}_M\phi_\mu^\omega](x) = C_\mu^M\exp i(\alpha_M x^2+\beta_M x)\,\phi_\mu^\omega(\gamma_M x+\delta_M), \tag{9.1}$$

where C_μ^M, α_M, ..., δ_M are constants. We can call such functions "self-reproducing" under $\mathcal{T}_M$. This has been used in Ref. 19 in order to find the irreducible representation matrix elements of SL(2,R) for all subgroup reduction chains as well as for the nonsubgroup Airy function basis.[20] In Ref. 16 we were able to state some general results on separation of variables for a class of two-variable parabolic differential equations through exploring relations of the type (9.1), where $\mathcal{T}_{M(t)}$ represented the time evolution of a system governed by such an equation. It also allows us, via the unitarity of the transform to find new generalized function bases for spaces of analytic functions á la Bargmann, as we have done with the parabolic cylinder functions in Sec. VI.

The basic step is to write

$$\mathcal{T}_M = \mathcal{T}_{G(\beta,c)}\,\mathcal{H}_t^\omega, \tag{9.2}$$

where $\mathcal{T}_{G(\beta,c)}$ is the geometric transform (2.4e) and

TABLE I. Self-reciprocating functions under canonical transforms of the linear (2.1) ($\mathbb{I}=\mathbb{R}$) and radial (2.7) ($\mathbb{I}=\mathbb{R}^+$) types. The table headings refer to the constants appearing in Eq. (9.1).

Function ϕ_μ^ω	C_μ^M	α_M	β_M	γ_M	δ_M
Harmonic oscillator (3.1) (Hermite × Gaussian) $\mu=n+\frac{1}{2}$, $n=0, 1, 2, \cdots$ Radial [k] Harm. Oscill. (4.1) (Laguerre × Gaussian) $\mu=2(n+k)$, $n=0, 1, \cdots$	$(a^2+b^2)^{-1/4} \times \exp\left(-i\mu\tan^{-1}\frac{b}{a}\right)$	$\frac{ac+bd}{2(a^2+b^2)}$	0	$\frac{1}{(a^2+b^2)^{1/2}}$	0
Repulsive oscillator (5.1) (Parabolic cylinder) $\mu\in\mathbb{R}$ (twice) Radial [k] Rep. Oscill. (8.1) (Whittaker function)	$(a^2-b^2)^{-1/4} \times \exp\left(-i\mu\tanh^{-1}\frac{b}{a}\right)$	$\frac{ac-bd}{2(a^2-b^2)}$	0	$\frac{1}{(a^2-b^2)^{1/2}}$	0
Schrödinger free particle (7.1) (imaginary exponential) $\mu\in\mathbb{R}$ Radial [k] centrifugal pot. (7.3) (Bessel function) $\mu\in\mathbb{R}^+$	$a^{-1/2} \times \exp\left(-\frac{i}{2}\mu^2\frac{b}{a}\right)$	$\frac{c}{2a}$	0	$\frac{1}{a}$	0
Linear potential (9.4) (Airy function) $\mu\in\mathbb{R}$	$a^{-1/2}\exp i\left(-\mu\frac{b}{a}-\frac{5b^3}{12a^3}\right)$	$\frac{c}{2a}$	$-\frac{b}{a^2}$	$\frac{1}{a}$	$\frac{b^2}{2a^2}$

$\mathcal{H}_t^\omega=\exp(itH^\omega)$, H^ω being any of the operators considered in (2.5) or any other operator in the $\mathrm{SL}(2,C)$ orbit of one of these. By writing the corresponding 2×2 matrices for the transforms in (9.2), we can easily find β, c, and t in terms of the matrix elements of M through a set of coupled algebraic equations. Thus, when ϕ_μ^ω is an eigenfunction of H^ω, the action of $\mathcal{H}_t^\omega$ is to multiply ϕ_μ^ω by $\exp(i\mu t)$ and that of $\mathcal{T}_{G(\beta,c)}$ is given by (2.4e), yielding the form (9.1) for the transform function.

A simple example will illustrate the procedure for the harmonic oscillator functions $\Phi_n^h(x)$ in (3.1),

$$\left[\mathcal{T}_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)}\Phi_n^h\ (x)\right]=\left[\mathcal{T}_{\left(\begin{smallmatrix}\alpha&0\\\gamma&\alpha^{-1}\end{smallmatrix}\right)}\mathcal{T}_{\left(\begin{smallmatrix}\cos t&-\sin t\\\sin t&\cos t\end{smallmatrix}\right)}\Phi_n^h\right](x), \tag{9.3a}$$

$$\alpha=(a^2+b^2)^{1/2},\quad \gamma=(ac+bd)/\alpha,\quad \tan t=-b/a. \tag{9.3b}$$

Now, the right-factor matrix is generated by (2.5a), (see Sec. III), hence

$$\begin{aligned}\left[\mathcal{T}_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)}\Phi_n^h\right](x)&=\int_{-\infty}^{\infty}dx'A_M(x,x')\Phi_n^h(x')\\&=\exp[i(n+1/2)t]\left[\mathcal{T}_{\left(\begin{smallmatrix}\alpha&0\\\gamma&\alpha^{-1}\end{smallmatrix}\right)}\Phi_n^h\right](x)\\&=\alpha^{-1/2}\exp[i(n+1/2)t]\exp i\left(\frac{\gamma}{2\alpha}x^2\right)\Phi_n^h\left(\frac{x}{\alpha}\right).\end{aligned} \tag{9.3c}$$

Equation (9.3) can of course be verified directly using integral tables. The fact that Φ_n^h appears in the integrand and in the right-hand side of this equation, has thus been given a group-theoretical interpretation. The list of self-reproducing functions can then be drawn from the eigenfunctions of the operators (2.5) or the "radial" ones with (2.9). There is one further extension which for economy we have not mentioned at all in this article, but which appears in full detail in Ref. 16: The extension of the $\mathrm{sl}(2,R)$ algebra through a semidirect sum with a Heisenberg–Weyl algebra w with generators $x, -id/dx$, and 1 to an algebra $w\oplus\mathrm{sl}(2,R)$. When exponentiated to the group $W\otimes\mathrm{SL}(2,R)$, this brings in one new interesting orbit generated by $H^l=\frac{1}{2}P^2+Q$, i.e., the quantum free-fall or linear potential Schrödinger Hamiltonian. Its generalized eigenfunctions can be found to be

$$\Phi_\mu^l(x)=2^{1/3}Ai(2^{1/3}[x-\mu]),\quad x,\mu\in\mathbb{R}. \tag{9.4}$$

Now, in following Ref. 16 to deal with $W\otimes\mathrm{SL}(2,R)$ we can add (9.4) to the list of self-reproducing functions. In Table I, we summarize the results for the harmonic oscillator (3.1), repulsive oscillator (5.1), free (7.1), and linear potential (9.4), as well as the radial harmonic oscillator (4.1), radial repulsive oscillator (8.1), and pure centrifugal (7.3) Schrödinger eigenfunctions. It should be noted that the choice of these eigenfunctions (rather than the most general eigenfunction of a linear combination of these Schrödinger Hamiltonians) is no restriction at all, since assume we wish to ascertain the self-reproducing formula for a function $\mathcal{T}_{M_1}\phi_\mu^\omega$ where ϕ_μ^ω is one of the functions above. Now $\mathcal{T}_{M_1}\phi_\mu^\omega$ has the form in the right-hand side of (9.1). Write (9.2) as

$$\mathcal{T}_M\mathcal{T}_{M_1}=\mathcal{T}_{M_2}=\mathcal{T}_{M_1}\mathcal{T}_{G(\beta',c')}\mathcal{H}_{t'}^\omega, \tag{9.5}$$

where it is as easy to find t' and $G(\beta',c')$ in terms of M as it was before. Thus, the most general self-reproducing functions under canonical transforms are given by $\mathcal{T}_{M_1}\phi_\mu^\omega$ where ϕ_μ^ω appear in the table and have the structure (9.1). Table I can be used for all values such that the entries are nonsingular.[21] In particular, the table gives the results on self-reciprocal functions found in Secs. III–VIII as can be checked by replacing the proper matrix elements (2.4) and (2.6) into the six first entries of the table. As far as the last entry on Airy functions is concerned, it is of interest to point out that the Bargmann transform (2.4c) of (9.4), namely

$$\begin{aligned}[\mathcal{B}\Phi_\mu^l](x)=&2^{1/2}\pi^{1/4}\exp[\tfrac{1}{2}x^2-\sqrt{2}x+\tfrac{5}{12}-\mu]\\&\times Ai(2^{5/6}x-2^{-2/3}-2^{1/3}\mu),\end{aligned} \tag{9.6}$$

constitutes a generalized orthonormal complete set of functions for Bargmann's Hilbert space $\mathcal{F}_B$. The argument follows that of the $\Phi_\mu^{r\pm}$ basis in Sec. VI.

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