Canonical transforms, separation of variables, and similarity solutions for a class of parabolic differential equations

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Using the method of canonical transforms, we explicitly find the similarity or kinematical symmetry group, all "separating" coordinates and invariant boundaries for a class of differential equations of the form $[\alpha\,\partial^2/\partial q^2+\beta q\,\partial/\partial q+\gamma q^2+\delta q+\epsilon\,\partial/\partial q+\zeta]\;u(q,t)=-i(\partial/\partial t)\,u(q,t)$, or of the form $[\alpha'(\partial^2/\partial q^2+\mu/q^2)+\beta' q\,\partial/\partial q+\gamma' q^2]\;u(q,t)=-i(\partial/\partial t)\,u(q,t)$, for complex $\alpha,\beta,\ldots,\gamma'$. The first case allows a six-parameter WSL$(2,R)$ invariance group and the second allows a four-parameter $O(2)\otimes SL(2,R)$ group. Any such differential equation has an invariant scalar product form which, in the case of the heat equation, appears to be new. The proposed method allows us to work with the group, rather than the algebra, and reduces all computation to the use of 2×2 matrices.

I. INTRODUCTION

A. In a recent series of papers[1–3] we have dealt with realizations of Lie algebras in terms of second-order differential operators and their exponentiation to the group. In contradistinction with first-order differential realizations, which produce *geometric* transformations of the general form

$$f(q)\xrightarrow{h} f_h(q)=\mu(q,h)f[\bar{q}_h(q)], \tag{1.1}$$

where μ is a multiplier function, second-order differential operators, when exponentiated, will in general lead to an *integral transform*

$$f(q)\xrightarrow{g} f_g(q)=\int dq'\,K_g(q,q')\,f(q'), \tag{1.2}$$

where $K_g(q,q')$ is an integral kernel. The action (1.1) has been extensively treated[4] since the times of Lie,[5] while only recently[6,7] have forms (1.2) been subjected to intensive study. In Refs. 1 and 2, we have worked with the groups SL$(2,C)$ [the group of unimodular 2×2 complex matrices] and the associated mappings (1.2) as unitary transformations between Hilbert spaces, one of them being $\mathcal{L}^2(R)$ or $\mathcal{L}^2(R^+)$ (Lebesgue square-integrable functions on the real line R or on the positive half-line R^+), and the other one, a space of analytic functions over regions of the complex plane *à la* Bargmann.[8] When the mapping (1.2) belongs to the SL$(2,R)$ subgroup (of unimodular 2×2 real matrices), the "Bargmann" spaces collapse to ordinary $\mathcal{L}^2$ spaces. We have called these mappings *canonical transforms* since they arose from the study of complex canonical transformations in quantum mechanics. They include as particular cases the transforms of Fourier, Laplace, Weierstrass, Bargmann, Hankel, and Barut–Girardello.

B. If H is a second-order differential operator in a variable q, element of a Lie algebra (which in this paper will be sl$(2,R)$ or wsl$(2,R)$—semidirect sum of the Weyl and sl$(2,R)$ algebras), the solution of the parabolic differential equation

$$\theta H u(q,t)=-i\,\frac{\partial}{\partial t}\,U(q,t), \tag{1.3}$$

where θ is an in general complex constant, can be expressed as a *canonical transform* of the initial condition

$$u(q)\equiv u(q,0),$$

$$u(q,t)=\exp(it\theta H)\,u(q). \tag{1.4}$$

Now we can subject $u(q)$ to a general integral transform (1.2) to a $u_g(q)$, and the corresponding $u_g(q,t)$ will still be a solution of (1.3) and, in fact, a geometric transform of $u(q,t)$. This will be the group of symmetries, kinematical[9,10] or similarity[11,12] group of the differential equation (1.3). We can further look for the invariant lines (boundaries) under g_0 in the q–t plane, $v(q,t)$ and thus use the generator of the said transformation to separate Eq. (1.3) into two ordinary differential equations, one in v and one in t. The solution of (1.3) will then have the form of a general superposition of *separable* solutions,[13–17]

$$u_s(v(q,t),t)=\exp[iS(v,t)]\,V_s(v)\,T_s(t), \tag{1.5}$$

where $S(v,t)$ is a *multiplier* function (*not* expressible as a function in v plus a function in t).

C. Our claim in this article is that we can considerably simplify the process of finding these features for the class of differential equations (1.3) by starting with a given group and pair its realization in terms of a Lie algebra of second-order differential operators with the *matrix realization* of the group. Since we shall be dealing with real subgroups of WSL$(2,C)$, the algebra required is essentially that of 2×2 matrices. This can be used to replace the rather lengthy conventional methods for finding separable coordinates and similarity groups through the solution of partial and coupled differential equations and the exhaustive examination of multiparameter ranges.

The canonical transform method, as used here, has the following limitations: it applies only to differential equations where H in (1.3) is of the form

$$H=\alpha\,\frac{d^2}{dq^2}+\beta q\,\frac{d}{dq}+\gamma q^2+\delta q+\epsilon\,\frac{d}{dq}+\zeta, \tag{1.6a}$$

or of the form

$$H=\alpha'\left(\frac{d^2}{dq^2}+\frac{\mu}{q^2}\right)+\beta' q\,\frac{d}{dq}+\gamma' q^2, \tag{1.6b}$$

for complex $\alpha,\beta,\ldots,\gamma'$, i.e., it applies only to a particular class of parabolic, linear, second-order differential equations. Yet this class contains the physically interesting cases of the heat equation and the Schrödinger equations for the free particle or quadratic (attractive or repulsive) plus linear or inverse-quadratic potentials in one dimension. Through a simple point transformation, these can be related to the pseudo-Coulomb Schrödinger equation.[3] Our tabulated results are exhaustive within the group framework.[18]

D. The outline of the paper is the following. In Sec. II we assemble the mathematical tools: the algebra and group realizations in terms of second-order differential operators (1.6a) and their exponentiation to the six-parameter group, as acting on the space $\mathcal{L}^2(R)$ of functions and its adjoint action on the algebra; eigenfunctions and their eigenvalues for any operator in the algebra can thus be found in terms of their *orbit representatives*. In Sec. III we allow for the complexification of the group, and phrase the solution of (1.3) in terms of canonical transforms, reducing the problem of finding separating coordinates associated with a second operator in the algebra, to the manipulation of 2×2 matrices. We exemplify some of these developments for the heat equation as a complex canonical transform, pointing out the existence of a new quadratic—scalar product—invariant. Some of the group-integrated features of similarity methods are seen in Sec. IV. The free particle and heat equation are used as examples. In the latter, the set of bounded transformations constitute a semigroup. In Sec. V, differential equations with operators of the class (1.6b) are treated. Some connections, conclusions, and directions for further work are collected in Sec. VI.

II. THE GROUP WSL (2,R) AND ITS ORBIT STRUCTURE

A. The Heisenberg—Weyl algebra[19] w, of generators Q, P, and $\mathbb{1}$ is defined through the commutator brackets

$$[Q,P]=i\mathbb{1}, \quad [Q,\mathbb{1}]=0, \quad [P,\mathbb{1}]=0. \qquad (2.1)$$

On the Hilbert space $\mathcal{L}^2(R)$, it is known[20] that every representation of w is unitarily equivalent to the Schrödinger representation

$$Qf(q)=qf(q), \quad Pf(q)=-i\frac{d}{dq}f(q), \quad \mathbb{1}f(q)=f(q), \qquad (2.2)$$

which is densely defined and self-adjoint in $\mathcal{L}^2(R)$. The generator $\mathbb{1}$ is in the center of the algebra and thus denoted as the identity operator to start with.

B. We can exponentiate (2.2) to a unitary representation of the Weyl group w, where the elements[19] $\omega(x,y,z)\in W$ act on $f\in\mathcal{L}^2(R)$ as,

$$\begin{aligned}[\mathcal{T}_\omega(x,y,z)f](q) &\equiv \{\exp[i(xQ+yP+z\mathbb{1})f]\}(q) \\ &= \exp[i(xq+\tfrac{1}{2}xy+z)]f(q+y).\end{aligned} \qquad (2.3)$$

Defining for convenience $\xi=(x,y)$ as a two-component row vector, its transpose $\xi^T=\binom{x}{y}$ and $\mathbf{\Omega}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, the product law in W can be written, for $\omega(\xi,z)\equiv\omega(x,y,z)$ as,

$$\omega(\xi_1,z_1)\omega(\xi_2,z_2)=\omega(\xi_1+\xi_2,\, z_1+z_2+\tfrac{1}{2}\xi_1\mathbf{\Omega}\xi_2^T), \qquad (2.4)$$

so that the group indentity is $\omega(0,0)$ and $\omega(\xi,z)^{-1}=\omega(-\xi,-z)$.

C. Out of the enveloping algebra $\bar{w}$ of w, we want to produce other Lie algebras under the commutator bracket. The set of second-order expressions,

$$I_1=\tfrac{1}{4}(P^2-Q^2), \quad I_2=\tfrac{1}{4}(QP+PQ), \quad I_3=\tfrac{1}{4}(P^2+Q^2), \qquad (2.5)$$

are densely defined and self-adjoint on $\mathcal{L}^2(R)$, satisfying,

$$[I_1,I_2]=-iI_3, \quad [I_3,I_1]=iI_2, \quad [I_2,I_3]=iI_1, \qquad (2.6)$$

which we recognize as the $\mathrm{sl}(2,R)\cong\mathrm{su}(1,1)\cong\mathrm{so}(2,1)\cong\mathrm{sp}(2,R)$ algebra.[4] No other unitarily inequivalent, finite-dimensional algebra of finite-order expressions can be found in $\bar{w}$ besides (2.1), (2.5), and their composition.[21]

D. The algebra (2.5) can be exponentiated to the group $\mathrm{SL}(2,R)$ of real unimodular 2×2 matrices through its one-parameter subgroups,

$$\exp(i\alpha I_1): \begin{pmatrix}\cosh\frac{1}{2}\alpha & -\sinh\frac{1}{2}\alpha\\ -\sinh\frac{1}{2}\alpha & \cosh\frac{1}{2}\alpha\end{pmatrix}, \qquad (27a)$$

$$\exp(i\beta I_2): \begin{pmatrix}\exp(-\frac{1}{2}\beta) & 0\\ 0 & \exp(\frac{1}{2}\beta)\end{pmatrix}, \qquad (2.7b)$$

$$\exp(i\gamma I_3): \begin{pmatrix}\cos\frac{1}{2}\gamma & -\sin\frac{1}{2}\gamma\\ \sin\frac{1}{2}\gamma & \cos\frac{1}{2}\gamma\end{pmatrix}, \qquad (2.7c)$$

$$\exp(ic\tfrac{1}{2}Q^2): \begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}, \qquad (2.7d)$$

$$\exp(ib\tfrac{1}{2}P^2): \begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix}, \qquad (2.7e)$$

so that every $\mathbf{A}\equiv\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}(2,R)$ [with $ad-bc=1$ for unimodularity] can be decomposed in terms of two or more of the elements (2.7). Now, the representation of $\mathrm{sl}(2,R)$ on $\mathcal{L}^2(R)$ obtained from (2.2) can also be exponentiated to a unitary representation of $\mathrm{SL}(2,R)$ on the same space as[1,6,14]

$$\begin{aligned}[\mathcal{C}\begin{pmatrix}a&b\\c&d\end{pmatrix}f](q) &= \int_R dq'\,A(q,q')\,f(q') \\ &= (2\pi b)^{-1/2}\exp(-i\pi/4)\int_{-\infty}^{\infty}dq' \\ &\quad\times\exp[(i/2b)(aq'^2-2qq'+dq^2)]\,f(q').\end{aligned} \qquad (2.8a)$$

Notice that $\exp(i\pi/4)\mathcal{C}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ is the ordinary Fourier transform. When $|b|\to0$, the integration kernel in (2.8) appears indeterminate, but can be shown to be well defined and turn (2.8) into

$$[\mathcal{C}\begin{pmatrix}a&0\\c&a^{-1}\end{pmatrix}f](q)=a^{-1/2}\exp[(ic/2a)q^2]f(q/a). \qquad (2.8b)$$

Formulas (2.8) give a unitary representation of $\mathrm{SL}(2,R)$ on $\mathcal{L}^2(R)$. This is actually a true representation of $\overline{\mathrm{SL}(2,R)}$, the covering group of $\mathrm{SL}(2,R)$ with respect to the O(2) subgroup generated by I_3; for $\mathrm{SL}(2,R)$ it is a ray representation and the possible phase differences with a true representation have been discussed in Ref. 1.[22]

E. We can join the set of generators in w and $\mathrm{sl}(2,R)$ using the derivation property of the commutator bracket, and in the resulting algebra we find that w is an ideal. We thus define $\mathrm{wsl}(2,R)=w \oplus \mathrm{sl}(2,R)$, where $\oplus$ is the semidirect sum, as the collection of generators (2.1) and (2.5). Correspondingly, from W and $\mathrm{SL}(2,R)$ we build the semidirect product $\mathrm{WSL}(2,R)=\mathrm{W}\otimes \mathrm{SL}(2,R)$ of pairs $g=\{\mathbf{A},\omega\}$, and its unitary representation on $\mathcal{L}^2(R)$ is given by the composition of the constituent actions (2.3) and (2.8) as

$$[\mathcal{T}\{(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}),\ (xyz)\}f](q)\equiv[\mathcal{C}(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix})\mathcal{T}_\omega(x,y,z)f](q)$$

$$=\int_R dq'\, B_g(q,q')\,f(q'), \qquad (2.9a)$$

where the integral kernel $B_g(q,q')$ can be found[23] from (2.3) and (2.8); it will not be of interest by itself, indeed, the usefulness of the methods proposed in this article hinge upon our *not* needing the general form (2.9a), but only those transformations with $b=0$ where the integral transform collapses to a *geometric* transform,

$$\mathcal{G}(a,c;x,y,z)\equiv\mathcal{T}\{(\begin{smallmatrix} a & 0\\ c & a^{-1}\end{smallmatrix}),\ (x,y,z)\}, \qquad (2.9b)$$

which has the effect

$$[\mathcal{G}(a,c;x,y,z)f](q)$$

$$=a^{-1/2}\exp[i(cq^2/2a+xq/a+\tfrac{1}{2}xy+z)]\,f(q/a+y), \qquad (2.9c)$$

i.e., changes of scale (a), translations (y), multiplication by an exponential (x) and Gaussian (c), and an over-all phase (z). Notice that the composition of two geometric transforms is a geometric transform, and so is its inverse. Equation (2.9a) allows us, though, to write the $\mathrm{WSL}(2,R)$ product law for $g\{\mathbf{A},\ \omega(x,y,z)\}\equiv g\{\mathbf{A},\xi,z\}$ compactly as

$$g\{\mathbf{A}_1,\xi_1,z_1\}\,g\{\mathbf{A}_2,\xi_1,z_2\}$$

$$=g\{\mathbf{A}_1\mathbf{A}_2,\ \xi_1\mathbf{A}_2+\xi_2, z_1+z_2+\tfrac{1}{2}\xi_1\mathbf{A}_2\mathbf{\Omega}\xi_2^T\}, \qquad (2.10)$$

so that the group indentity is $g\{\mathbf{1},0,0\}$ and the inverse $g\{\mathbf{A},\xi,z\}^{-1}=g\{\mathbf{A}^{-1},-\xi\mathbf{A}^{-1},-z\}$, where we have used the fact that $\mathbf{A\Omega A}^T=\mathbf{\Omega}$ and $\xi\mathbf{\Omega}\xi^T=0$ for $\mathbf{A}\in\mathrm{SL}(2,R)$.

F. The action (2.9) of $\mathrm{WSL}(2,R)$ on $\mathcal{L}^2(R)$ induces its adjoint representation by automorphisms of the algebra,[4]

$$I_i \xrightarrow{g} I_i'=gI_ig^{-1}\equiv \mathrm{Ad}_g I_i=\sum_j M_{ij}I_j, \qquad (2.11)$$

for $I_i\in\mathrm{wsl}(2,R)$ denoting $I_4=Q$, $I_5=P$, and $I_6=\mathbb{1}$. Through (2.1), (2.3), (2.5)–(2.7), and (2.9) we obtain[2]

$$\mathbf{M}=\begin{pmatrix} \tfrac{1}{2}(a^2-b^2-c^2+d^2) & bd-ac & \tfrac{1}{2}(a^2-b^2+c^2-d^2) & \tfrac{1}{2}(cx-dy) & \tfrac{1}{2}(-ax+by) & \tfrac{1}{4}(x^2-y^2)\\ -ab+cd & ad+bc & -ab-cd & \tfrac{1}{2}(-dx-cy) & \tfrac{1}{2}(bx+ay) & -\tfrac{1}{2}xy\\ \tfrac{1}{2}(a^2+b^2-c^2-d^2) & -ac-bd & \tfrac{1}{2}(a^2+b^2+c^2+d^2) & \tfrac{1}{2}(cx+dy) & \tfrac{1}{2}(-ax-by) & \tfrac{1}{4}(x^2+y^2)\\ & & & d & -b & y\\ & 0 & & -c & a & -x\\ & & & 0 & 0 & 1 \end{pmatrix}. \qquad (2.12)$$

Since the group parameter z does not appear in (2.12), the latter is a faithful representation only of $\mathrm{wsl}(2,R)/\mathbb{1}$. An operator H built as a linear combination of the generators of the algebra,

$$H=\sum_j \theta_jI_j=\tfrac{1}{2}(\theta_1+\theta_3)P^2+\tfrac{1}{4}\theta_2(QP+PQ)+\tfrac{1}{2}(-\theta_1+\theta_3)Q^2$$

$$+\theta_4Q+\theta_5P+\theta_6\mathbb{1}, \qquad (2.13a)$$

will transform under the adjoint action of the group as

$$H\xrightarrow{g}H'=gHg^{-1}=\sum_i\sum_j\theta_iM_{ij}I_j=\sum_j\theta_j'I_j. \qquad (2.13b)$$

G. Two elements H and H' of the algebra are said to be on the same *orbit* under the group if there exists an element g in the group such that (2.13b) holds. Such elements H and H' generate one-parameter subgroups $g_0(\alpha)=\exp(i\alpha H)$ and $g_1(\beta)=\exp(i\beta H')$ which are conjugate through g, and thus $g_0(\alpha)$ and $g_1(\alpha)$ are in the same class in the group. Even if we perform an over-all change in scale $H''=\gamma H'$ [which is *not* a transformation (2.12)–(2.13) for $|\gamma|\neq 1$], the subgroup generated as $g_2(\alpha)=\exp(i\alpha H'')=g_1(\gamma\alpha)=gg_0(\gamma\alpha)g^{-1}$ will as *a whole* still be conjugate to the subgroup generated by H. Since the O(2) subgroup generated by $\mathbf{1}$ is a trivial phase, it will serve us to ignore it in our analysis, so that we will restrict our orbit analysis to the coset space[4] $\mathrm{WSL}(2,R)/\mathrm{O}(2)_{\mathbf{1}}$. In terms of the algebra $\mathrm{wsl}(2,R)/\mathbf{1}$, this means that operators differing by an additive term $\theta_6\mathbf{1}$ are considered equivalent. In choosing the orbit representatives, over-all factors will also be disregarded since they generate the same subgroup.

H. The orbit structure of $\mathrm{wsl}(2,R)/\mathbf{1}$ can now be analyzed,[14] noting that $\Theta\equiv\theta_3^2-\theta_1^2-\theta_2^2$ is an invariant under the transformation (2.13). As we are interested in operators equivalent up to over-all changes in scale γ (for which $\Theta''=\gamma^2\Theta'$) we consider three cases: (i) $\Theta>0$, (ii) $\Theta<0$, and (iii) $\Theta=0$. In each of these cases we can pick out an orbit *representative* operator H^ω, for each orbit ω. This is simplified by noting that we can choose the transformation to be a geometric transformation ($b=0$) and that (2.12) has a lower-left zero submatrix.

(i) $\Theta>0$ (*harmonic oscillator*):

$$H^h = 2I_3 = \tfrac{1}{2}(P^2 + Q^2), \tag{2.14a}$$

through

$$\begin{aligned} a_h &= [\theta_h/(\theta_1+\theta_3)]^{1/2}, \quad c_h = \theta_2[\theta_h(\theta_1+\theta_3)]^{-1/2}, \\ x_h &= 2\theta_h^{-2}[\theta_5(\theta_3-\theta_1) - \theta_4\theta_2], \\ y_h &= 2\theta_h^{-2}[\theta_5\theta_2 - \theta_4(\theta_3+\theta_1)], \end{aligned} \tag{2.14b}$$

where $\theta_h^2 = \Theta = \theta_3^2 - \theta_1^2 - \theta_2^2$ and the choice $\theta_h = 2$ leads to the form (2.14a). Clearly, the transformation (2.14b) is possible for all θ's except when $\theta_1 = -\theta_3$. This corresponds to the case when H has no P^2 (kinetic energy) term, which we can regard as unphysical. In this case, we can subject H to a Fourier transform, which is known and easy to implement, but is not a geometric transform.

(ii) $\Theta < 0$ (*repulsive oscillator*):

$$H^r = 2I_1 = \tfrac{1}{2}(P^2 - Q^2), \tag{2.15a}$$

through

$$\begin{aligned} a_r &= [\theta_r/(\theta_1+\theta_3)]^{1/2}, \quad c_r = \theta_2[\theta_r(\theta_1+\theta_3)]^{-1/2}, \\ x_r &= 2\theta_r^{-2}[\theta_5(\theta_1-\theta_3) + \theta_4\theta_2], \\ y_r &= 2\theta_r^{-2}[-\theta_5\theta_2 + \theta_4(\theta_3+\theta_1)], \end{aligned} \tag{2.15b}$$

where $\theta_r^2 = -\Theta = \theta_1^2 + \theta_2^2 - \theta_3^2$ and the choice $\theta_r = 2$ leads to (2.15a). Remarks as in (i) apply when $\theta_1 = -\theta_3$.

(iii) $\Theta = 0$ (*linear potential*):

$$H^l = I_1 + I_3 + Q = \tfrac{1}{2}P^2 + Q. \tag{2.16a}$$

Here we have several cases. As $\theta_1^2 + \theta_2^2 - \theta_3^2 = 0$ assume first θ_1, θ_2, and θ_3 are not all identically zero. Then through

$$\begin{aligned} a_l &= [2\theta_l/(\theta_1+\theta_3)]^{1/2}, \quad c_l = [(\theta_3-\theta_1)/2\theta_l]^{1/2}, \\ x_l&(\theta_3+\theta_1)^{1/2} - y_l(\theta_3-\theta_1)^{1/2} = 2\theta_5(\theta_3+\theta_1)^{-1/2} \end{aligned} \tag{2.16b}$$

we can bring H to the form H^l with θ_l a free parameter and $\theta_1' = \theta_l = \theta_3'$, while

$$\theta_4' = (2\theta_l)^{-1/2}[\theta_4(\theta_3+\theta_1)^{1/2} - \theta_5(\theta_3-\theta_1)^{1/2}]. \tag{2.16c}$$

The ratio $\rho \equiv \theta_4'/\theta_l$ can be varied by varying θ_l, and the choice (2.16a) corresponds to $\rho = 1$. We cannot make ρ vanish, however, unless to start with we have $\theta_4(\theta_3 + \theta_1)^{1/2} = \theta_5(\theta_3 - \theta_1)^{1/2}$. We distinguish this case:

(iii′) $\Theta = 0$, $\theta_4^2(\theta_3+\theta_1) = \theta_5^2(\theta_3-\theta_1)$ (*free particle*):

$$H^f = I_1 + I_3 = \tfrac{1}{2}P^2, \tag{2.17}$$

and we must add the remark following (i) in the case $\theta_1 = -\theta_3$. Now we examine the cases where $\theta_1 = \theta_2 = \theta_3 = 0$. We only have the lower-right submatrix (2.12), and we can always bring the operator to the form

(iii″) $\theta_1 = 0$, $\theta_2 = 0$, $\theta_3 = 0$ (*momentum*):

$$H^m = P, \tag{2.18}$$

through

$$a_m = \theta_5^{-1}, \quad c_m = \theta_4, \tag{2.19}$$

applying the Fourier transformation when $\theta_5 = 0$. The further case when $\theta_4 = 0 = \theta_5$, has $\mathbf{0}$ for its orbit representative in $\mathrm{wsl}(2,R)/\mathbb{1}$ and $\mathbb{1}$ in $\mathrm{wsl}(2,R)$.

To sum up: We have five orbits in $\mathrm{WSL}(2,R)/\mathrm{O}(2)_{\mathbb{1}}$ generated by H^ω ($\omega = h, r, l, f$ or m). We have found in each case the explicit transformation (2.12) leading a general operator (2.13a) to one of the five representatives, up to an over-all multiplicative constant and the (possible) addition of a multiple θ_6' of $\mathbb{1}$ given from (2.12)–(2.13) as

$$\begin{aligned} \theta_6' = \tfrac{1}{4}(x_\omega^2 - y_\omega^2)\theta_1 - \tfrac{1}{2}x_\omega y_\omega\theta_2 + \tfrac{1}{4}(x_\omega^2 + y_\omega^2)\theta_3 \\ + y_\omega\theta_4 - x_\omega\theta_5 + \theta_6 \end{aligned} \tag{2.20}$$

with x_ω, y_ω ($\omega = h, r, l, f$, or m) as in (2.14b), (2.15b), or (2.16b).

I. As the operators H as given by (2.13a) are self-adjoint in $\mathcal{L}^2(R)$, their eigenfunctions will constitute a complete orthonormal (possibly in the sense of Dirac) set of eigenvectors for the space, and since the transformations (2.9) are unitary, it suffices to give the results for the orbit representatives:

Harmonic Oscillator: These are well known[20] to be

$$\begin{aligned} \psi_\lambda^h(q) &= [2^n n!\sqrt{\pi}]^{-1/2}\exp(-\tfrac{1}{2}q^2)H_n(q), \\ \lambda &= n + \tfrac{1}{2}, \ n = 0, 1, 2, \cdots, \end{aligned} \tag{2.21}$$

where $H_n(q)$ are the Hermite polynomials. Orthonormality has the usual phrasing as $(\psi_\lambda^h, \psi_{\lambda'}^h) = \delta_{\lambda,\lambda'}$ (Kronecker delta) and completeness states $\psi(q) = \sum \psi_\lambda^h(q)(\psi_\lambda^{r-}, \psi)$in the norm for any $\psi \in \mathcal{L}^2(R)$.

Repulsive Oscillator: The basis and spectrum of $H^r = 2I_1$ can be found[14] in terms of that of $H^d \equiv -2I_2 = i(q\,d/dq + \tfrac{1}{2})$, which is on the same orbit: $H^r = g_{12}H^d g_{12}^{-1}$ with $g_{12} = \{(1/\sqrt{2})\left(\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right), (0)\}$ (this is the "square root" of the Fourier transform, as $g_{12}^2 = \{\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right), (0)\}$). The eigenfunctions of H^d are found from the theory of Mellin transforms to be, properly normalized,

$$\psi_\lambda^{d\pm}(q) = (2\pi)^{-1/2}q_\pm^{-i\lambda-1/2}, \quad \lambda \in R, \quad q_\pm \equiv \begin{cases} \pm q, & q \gtrless 0, \\ 0, & q \lessgtr 0, \end{cases} \tag{2.22a}$$

with a spectrum covering twice the real line. Using (2.9a) for g_{12}, we can find $\psi_\lambda^r = \mathcal{J}(g_{12})\psi_\lambda^d$ as

$$\begin{aligned} \psi_\lambda^{r\pm}(q) = 2^{-3/4}\pi^{-1}\exp[-(i/4)\pi(i\lambda + \tfrac{1}{2})] \\ \times\Gamma(-i\lambda + \tfrac{1}{2})D_{i\lambda-1/2}(\pm 2^{1/2}\exp(3i\pi/4)q), \quad \lambda \in R, \end{aligned} \tag{2.22b}$$

where $D_\nu(r)$ is the Parabolic Cylinder function.[24] Orthonormality means here $(\psi_\lambda^{r\pm}, \psi_{\lambda'}^{r\pm}) = \delta(\lambda - \lambda')$ (Dirac delta) and $(\psi_\lambda^{r\pm}, \psi_{\lambda'}^{r\mp}) = 0$. Completeness integrates twice over $\lambda \in R$, i.e., $\psi(q) = \int_R d\lambda\, \psi_\lambda^{r+}(q)(\psi_\lambda^{r+}, \psi) + \int_R d\lambda \psi_\lambda^{r-}(q)(\psi^{r-}, \psi)$, $\psi \in \mathcal{L}^2(R)$.

Linear potential: Again, the basis and spectrum of H^l is easier to analyze[14] for its Fourier transform $\tfrac{1}{2}Q^2 - P$ which gives rise to a first-order differential equation whose normalized solutions are $\tilde{\psi}_\lambda^l(q) = (2\pi)^{-1/2} \times \exp(-\lambda q + \tfrac{1}{6}q^3)$, for $\lambda \in R$. The inverse Fourier transform yields $\psi_\lambda^l(q)$ through Airy's integral[25]

$$\psi_\lambda^l(q)=2^{1/3}Ai(2^{1/3}[q-\lambda]), \quad \lambda\in R, \tag{2.23}$$

and the usual orthonormality and completeness statements are $(\psi_\lambda^l,\psi_{\lambda'}^l)=\delta(\lambda-\lambda')$ and $\psi(q)=\int_R d\lambda\, \psi_\lambda^l(q)(\psi_\lambda^l,\psi)$, $\psi\in \mathcal{L}^2(R)$.

Free particle: The basis and the spectrum of P is

$$\psi_\lambda^f(q)=(2\pi)^{-1/2}\exp(i\lambda q), \quad \lambda\in R. \tag{2.24}$$

This serves also as a convenient basis for $H^f=\frac{1}{2}P^2$ which in linearly, but not functionally independent of P. The spectrum of H^f is $\frac{1}{2}\lambda^2$, i.e., twice the half-line.

The eigenfunctions ψ_μ and eigenvalues μ of an operator H as given by (2.13a) can now be determined, knowing the ones for the orbit representatives H^ω, ψ_λ^ω, and λ ($\omega=h,r,l$ or $f-m$). We have

$$g_\omega H g_\omega^{-1}=\theta_\omega H^\omega+\theta_6'\mathbb{1} \tag{2.25a}$$

with g_ω a *geometric* transformation of the type (2.9b), with parameters given by (2.14b), (2.15b), or (2.16b) (save the cases when a Fourier transformation is needed) and the θ_ω determined correspondingly. Hence

$$\psi_\mu(q)=[\mathcal{G}(g_\omega^{-1})\psi_\lambda^\omega](q) \quad\text{and}\quad \mu=\theta_\omega\lambda+\theta_6'. \tag{2.25b}$$

Recall that geometric transforms are easily obtained as in (2.9c).

J. Example:

$$\begin{aligned}H&=2P^2+(QP+PQ)+\tfrac{1}{2}Q^2+Q+P+\zeta\mathbb{1}\\ &=3I_1+4I_2+5I_3+Q+P+\zeta\mathbb{1}.\end{aligned} \tag{2.26a}$$

We see that $\Theta=0$, so this case belongs to (iii). From (2.16b) we find $a_l=\frac{1}{2}\sqrt{\theta_l}$, $c_l=1/\sqrt{\theta_l}$ and $2x_l-y_l=\frac{1}{2}$. The transformation

$$\left\{\begin{pmatrix}\frac{1}{2}\sqrt{\theta_l} & 0\\ 1/\sqrt{\theta_l} & 2/\sqrt{\theta_l}\end{pmatrix}, (x_l, 2x_l-\tfrac{1}{2}, 0)\right\}$$

then maps H into $H'=\frac{1}{2}\theta_l P^2+1/\sqrt{\theta_l}\,Q+(x_l+\zeta-3/8)\mathbb{1}$ so we choose $\theta_l=1$ and $x_l=\frac{3}{8}-\zeta$. The spectrum of H is then $\mu=\lambda\in R$, while the basis functions are

$$\begin{aligned}\psi_\lambda(q)&\\ &=[\mathcal{G}(\tfrac{1}{2},1;\tfrac{3}{8}-\zeta,\tfrac{1}{4}-2\zeta,0)^{-1}\psi_\lambda^l](q)\\ &=[\mathcal{G}(2,-1;-\tfrac{1}{2},\zeta-\tfrac{1}{8},0)\psi_\lambda^l](q)\\ &=2^{-1/2}\exp i[-\tfrac{1}{4}(q^2+q+\zeta-1/8)]\psi_\lambda^l(\tfrac{1}{2}q+\zeta-\tfrac{1}{8})\\ &=2^{-1/6}\exp i[-\tfrac{1}{4}(q^2+q+\zeta-1/8)]\,\mathrm{Ai}(2^{1/3}\{\tfrac{1}{2}q+\zeta-\tfrac{1}{8}+\lambda\}).\end{aligned} \tag{2.26b}$$

III. COMPLEX CANONICAL TRANSFORMS AND TIME DEVELOPMENT OF A SYSTEM

A. We will now allow the group parameters of $g=\{\mathbf{A},\xi,z\}\in \mathrm{WSL}(2,R)$ ($\det\mathbf{A}=1$) to range over the complex field. The resulting set also forms a group which we denote by WSL$(2,C)$. The representation given by (2.3)–(2.8) and (2.9) does not follow for the whole new group: If f is assumed to be in $\mathcal{L}^2(R)$, $\mathcal{J}f$ will belong to $\mathcal{L}^2(R)$ only if the kernel $B_g(q,q')$ is bounded. This happens for the parameters in $\mathbf{A}$ only if $\mathrm{Im}(a/b)\geq 0$ so that the Gaussian factor will be decreasing and, when $a=0$, b must be real so that the kernel will be an oscillating exponential. For the $\omega(x,y,z)$ parameters it is only required that when $a=0$, x be real also. The product of two bounded operators is bounded and the group identity is bounded as well as all real elements in WSL$(2,R)$. Thus, (2.9a) represents properly a *subsemigroup* of WSL$(2,C)$ which we denote by HWSL$(2,C)$, following Refs. 1, 2, and 7 which deal with the SL$(2,C)$ part. As regards unitarity, those transformations in HWSL$(2,C)$ which are *not* in WSL$(2,R)$, are represented by integral nonunitary transformations from $\mathcal{L}^2(R)$ into $\mathcal{L}^2(R)$.

B. In Refs. 1 and 2, we constructed Hilbert spaces of analytic functions $\mathcal{J}_\mathbf{A}$ such that HSL$(2,C)$ is represented by *unitary* mappings between $\mathcal{L}^2(R)$ and $\mathcal{J}_\mathbf{A}$. The Hilbert spaces $\mathcal{J}_\mathbf{A}$ are characterized by a scalar product performed over the complex plane, as in Bargmann's case,[8] given by

$$(f,g)_\mathbf{A}=\int_\mathbb{C} d^2\mu(q)\,f(q)^*\,g(q), \tag{3.1a}$$

with the measure

$$\begin{aligned}d^2\mu(q)&=2(2\pi v)^{-1/2}\exp[(1/2v)(uq^2-2qq^*+u^*q^{*2})]\\ &\quad\times d\,\mathrm{Re}q\,d\,\mathrm{Im}q,\end{aligned} \tag{3.1b}$$

and where

$$u=a^*d-b^*c, \quad v=2\,\mathrm{Im}b^*a>0. \tag{3.1c}$$

Corresponding to the geometric transformations (2.9b), where $v=0$ the measure becomes singular and one can show that

$$\lim_{v\to 0}\int_\mathbb{C} d^2\mu(q)\,f(q)^*\,g(q)=\int_{Re^{i\psi}}dx\,\exp(-w|x|^2/2)f(x)^*g(x), \tag{3.1d}$$

where $w=2\,\mathrm{Im}c^*d$, and the integration contour is along a line in the complex $\mathbb{C}$-plane tilted with respect to the real axis by an angle $\psi=-\frac{1}{2}$ phase of u. Finally, for the general complex case, the transform inverse to (2.8) is given by

$$f(q)=\int_\mathbb{C} d^2\mu(q')A(q',q)^*\,[\mathcal{C}f](q'). \tag{3.1e}$$

With little extra labor we can build a similar scalar product and Hilbert spaces such that the transformations in HWSL$(2,C)$ will be unitary. The only application we will touch upon is the one provided by the heat equation, and so the construction of the general case beyond (3.1) is unnecessary here.

C. The action of WSL$(2,C)$ transformations on operators H of the form (2.13a) closely follows that seen in the last section, except for allowing all parameters to be complex. The orbit structure analyzed in II-H simplifies, in that the cases (i) and (ii) (attractive and repulsive oscillators) coalesce, if we allow for over-all complex factors. Indeed, the well-known Bargmann transformation,[8] $g_B\equiv\{(1/\sqrt{2})\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}, (0)\}$, bridges (i) and (ii), as $g_B I_3 g_B^{-1}=iI_2$ while $g_H\equiv\{\begin{pmatrix}\omega & 0\\ 0 & \omega^{-1}\end{pmatrix},(0)\}$, $\omega^2=-i$, performs $g_H H^f g_H^{-1}=-iH^f$ and takes us from the free particle Schrödinger equation to the heat equation.

D. The parabolic differential equations we want to analyze here are those of the general form

$$Hu(q,t)=-i(\partial/\partial t)u(q,t), \tag{3.2}$$

where H is an operator of the form (1.6a)–(2.13a).

Formally, the solution of (3.2) is given by the time-translated initial condition $u(q) \equiv u(q,0)$,

$$u(q,t) = \exp(t\,\partial/\partial t')\,u(q,t')\,|_{t'=0} = \exp(itH)\,u(q) \equiv [\mathcal{H}_t u](q). \tag{3.3}$$

The third term in (3.3) is a differential operator of infinite degree in q (termed also hyperdifferential operator[26]) densely defined in $\mathcal{L}^2(R)$, whose action on $u(q)$ is a time-dependent canonical transform $\mathcal{H}_t$ whose integral form is given by (2.9). Corresponding to the four orbits seen in the last section (excluding P), their four $\mathcal{H}_t^w$ time-evolution operators are represented by

$$\mathcal{H}_t^h = \exp(it\tfrac{1}{2}[P^2+Q^2]): \left\{\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix},\ (0,0,0)\right\} \tag{3.4a}$$

$$\mathcal{H}_t^r = \exp(it\tfrac{1}{2}[P^2-Q^2]): \left\{\begin{pmatrix}\cosh t & -\sinh t\\ -\sinh t & \cosh t\end{pmatrix},\ (0,0,0)\right\} \tag{3.4b}$$

$$\mathcal{H}_t^l = \exp(it[\tfrac{1}{2}P^2+Q]): \left\{\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix},\ (-t,\tfrac{1}{2}t^2,-(1/6)t^3)\right\} \tag{3.4c}$$

$$\mathcal{H}_t^f = \exp(it\tfrac{1}{2}P^2): \left\{\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix},\ (0,0,0)\right\}. \tag{3.4d}$$

All these expressions can be read off from (2.7), except for (3.4c), which requires some extra work in exponentiating.

For a general operator H [(2.13a)] we can find its geometric transformation g_ω ($\omega = h, r, l$ or f) relating it to its orbit representative. Its time-evolution transform will be

$$\mathcal{H}_t = \exp(itH) = \exp(it\theta_6')\exp(it\theta_\omega g_\omega^{-1}H^\omega g_\omega) = \exp(it\theta_6')g_\omega^{-1}\mathcal{H}^\omega_{\theta_\omega t}\,g_\omega, \tag{3.5a}$$

and its solutions

$$u(q,t) = \mathcal{H}_t u(q) = \exp(it\theta_6')\mathcal{G}(g_\omega^{-1})\mathcal{H}^\omega_{\theta_\omega t}\,\mathcal{G}(g_\omega)u(q). \tag{3.5b}$$

E. Simplest to consider, is the time evolution of the eigenfunctions $\psi_\lambda(q)$ of the operator H in (3.2), since

$$\psi_\lambda(q,t) = \mathcal{H}_t\psi_\lambda(q) = \exp(i\lambda t)\,\psi_\lambda(q). \tag{3.6}$$

These are the solutions of (3.2) *separable* in q and t: if we know the expansion coefficients, u_λ of an arbitrary function $u(q) = u(q,0)$ in terms of the ψ_λ-basis, the expansion coefficients of the $u(q,t)$ solution of (3.2) are $u_\lambda\exp(i\lambda t)$. But assume that the physically meaningful expansion for $u(q)$ is in terms of a $\psi_\lambda'(q)$-basis, eigenfunctions of an operator H' which may or may not be on the same orbit as H. Assume for definiteness that H and H' are the orbit representatives of the last section, with (3.4) their time-evolution transforms. Then, it is fundamental for our results that, at least in a region around $t=0$, we can write

$$\mathcal{H}_t = \mathcal{G}_t\mathcal{H}'_{t'}, \tag{3.7}$$

where $t' = t'(t)$. That is, the time-evolution transform $\mathcal{H}_t$ can be written as the time-evolution transform $\mathcal{H}'_{t'}$ for a rescaled time $t'(t)$, times a (time-dependent) *geometric* transform $\mathcal{G}_t$. Finding the group parameters of $\mathcal{G}_t$ and the function $t'(t)$ is an exercise in 2×2 matrix algebra.

F. Example: Let H be the harmonic oscillator Schrödinger Hamiltonian [so that $\mathcal{H}_t$ is $\mathcal{H}_t^h$ in (3.4a)]. We want to find the time evolution of plane waves [free particle eigenfunctions ψ_λ^f in (2.24), $\mathcal{H}'_{t'}$ being $\mathcal{H}^f_{t'}$] in that system. We write (3.7), where only the SL(2,R) parameters need to be considered, as

$$\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} = \begin{pmatrix}a_t & 0\\ c_t & a_t^{-1}\end{pmatrix}\begin{pmatrix}1 & -t'\\ 0 & 1\end{pmatrix}, \tag{3.8a}$$

and we find immediately,

$$t' = \tan t,\quad a_t = \cos t,\quad c_t = \sin t, \tag{3.8b}$$

so, from (3.6) and (2.9),

$$\begin{aligned}\psi_\lambda^f(q)\xrightarrow{\mathcal{H}_t^h}\psi_\lambda^f(q) &= \mathcal{G}_t\,\mathcal{H}^f_{t'}\,\psi_\lambda^f(q)\\ &= \exp(i\tfrac{1}{2}\lambda^2 t')\mathcal{G}_t\psi_\lambda^f(q)\\ &= \exp(i\tfrac{1}{2}\lambda^2 t')a_t^{-1/2}\exp(ic_t q^2/2a_t)\psi_\lambda^f(q/a_t)\\ &= (\cos t)^{-1/2}\exp[i\tan t(\tfrac{1}{2}\lambda^2+\tfrac{1}{2}q^2)]\psi_\lambda^f(q/\cos t)\\ &= \exp[i\tfrac{1}{2}\sin t\cos t(q/\cos t)^2]\psi_\lambda^f(q/\cos t)\\ &\quad\times(\cos t)^{-1/2}\exp(i\tfrac{1}{2}\lambda^2\tan t).\end{aligned} \tag{3.9c}$$

This result can be checked using the harmonic oscillator Green's function and performing the integration.

A few comments on (3.9c): Although the points $t = \pm\frac{1}{2}\pi$, $\pm(3/2)\pi$, ⋯ appear to be singular for some elements of the expression, since the transformation (3.7) is unitary in $\mathcal{L}^2(R)$, we are assured that any $\mathcal{L}^2(R)$ function expanded in the ψ_λ^f-basis will exhibit no singularities in its time development. Systems which classically are periodic or exhibit turning points will be in many-to-one correspondence with open systems. In Table I we give, for all pairs of orbit representatives, the geometric transformation which bridges them.

G. The next point to be remarked upon is that the final expression in (3.9c) is (from right to left) a product of a function in $t'(t)$ times a function in $v(q,t) = q/\cos t$ times a *multiplier* $\exp(i\frac{1}{4}v^2\sin 2t)$. If we follow the procedure of Kalnins, Miller, and Boyer[14,15] in finding coordinate systems $v(q,t) - t$ such that, in (3.2),

$$u_\lambda(q,t) = \exp[iS(v,t)]\,V_\lambda(v)\,T_\lambda(t) \tag{3.10}$$

separates into two ordinary differential equations in v and t, one of such systems will be the one found above. The presence of the exponent in $S(v,t)$ (specifically *not* a sum of a function in v plus a function in t), defines this case as *R-separable*, as opposed to ordinary separability, when $S(v,t)=0$. It is thus that, as detailed below, we obtain all "separating" coordinate systems for all parabolic equations (3.2). We follow the procedure of the example in subsection III.F to read them off Table I as

TABLE I. Expressions for the geometric transformations between pairs of time-development operators corresponding to the four orbit representatives $\mathcal{H}_t^{\omega}=\mathcal{G}_t(a,c;x,y,z)\mathcal{H}_{t'}^{\omega'}$. The entry "1" means $t=t'$ and $\mathcal{G}$ is the identity transformation. When x, y, z do not appear, they equal 0. The example in Sec. II. E corresponds to $\omega=h, \omega'=f$. The heat equation follows the f-system with the replacement $t\to 2it$.

$\omega \backslash \omega'$	h	r	l	f
h	1	$\tanh t'=\tan t$ $a_t=(\cos 2t)^{1/2}$ $c_t=\sin 2t(\cos 2t)^{-1/2}$	$t'=\tan t$ $a_t=\cos t$ $c_t=\sin t$ $x=t', y=\frac{1}{2}t'^2$ $z=\frac{1}{6}t'^3$	$t'=\tan t$ $a_t=\cos t$ $c_t=\sin t$
r	$\tan t'=\tanh t$ $a_t=(\cosh 2t)^{1/2}$ $c_t=\sinh 2t(\cosh 2t)^{-1/2}$	1	$t'=\tanh t$ $a_t=\cosh t$ $c_t=\sinh t$ $x=t', y=\frac{1}{2}t'^2$ $z=\frac{1}{6}t'^3$	$t'=\tanh t$ $a_t=\cosh t$ $c_t=\sinh t$
l	$\tan t'=t$ $a_t=(1+t^2)^{1/2}$ $c_t=t(1+t^2)^{-1/2}$ $x_t=-t(1+\frac{1}{2}t^2)(1+t^2)^{-1/2}$ $y_t=-\frac{1}{2}t^2(1+t^2)^{-1/2}$ $z_t=-\frac{1}{6}t^3$	$\tanh t'=t$ $a_t=(1-t^2)^{1/2}$ $c_t=t(1-t^2)^{-1/2}$ $x_t=-t(1-\frac{1}{2}t^2)(1-t^2)^{-1/2}$ $y_t=-\frac{1}{2}t^2(1-t^2)^{-1/2}$ $z_t=-\frac{1}{6}t^3$	1	$t'=t$ $a_t=1,\ c_t=0$ $x_t=-t$ $y_t=-\frac{1}{2}t^2$ $z_t=-\frac{1}{6}t^3$
f	$\tan t'=t$ $a_t=(1+t^2)^{1/2}$ $c_t=-t(1+t^2)^{-1/2}$	$\tanh t'=t$ $a_t=(1-t^2)^{1/2}$ $c_t=t(1-t^2)^{-1/2}$	$t'=t$ $a_t=1,\ c_t=0$ $x_t=t,\ y_t=\frac{1}{2}t^2$ $z_t=\frac{1}{6}t^3$	1

follows: From (2.9c) and (3.7),

$$\mathcal{H}_t^{\omega}\psi_\lambda^{\omega'}(q)$$

$$=\mathcal{G}_t\mathcal{H}_{t'}^{\omega'}\psi_\lambda^{\omega'}(q)=\exp(i\lambda t')\mathcal{G}_t\psi_\lambda^{\omega'}(q)$$

$$=(a_t)^{-1/2}\exp\{i[c_t/2a_t)q^2+(x_t/a_t)q+z_t+\tfrac{1}{2}x_ty_t+\lambda t']\}$$

$$\times\psi_\lambda^{\omega'}(q/a_t+y_t)$$

$$=(a_t)^{-1/2}\exp\{i[\tfrac{1}{2}c_ta_tv^2+(v-\tfrac{1}{2}y_t)(x_t-c_ta_ty_t)$$

$$+z_t+\lambda t']\}\psi_\lambda^{\omega'}(v),$$

where

$$v(q,t)=q/a_t+y_t, \tag{3.11b}$$

and all other parameters in $\mathcal{G}_t$, a_t, b_t, $\ldots$, z_t, and t' depend on t only. Thus $\mathcal{H}_t^{\omega}\psi_\lambda^{\omega'}(q)$ is a separable function in the sense (3.10) in v and t, where the multiplier $S(v,t)$ can be read off (3.11a) and is

$$S(v,t)=\tfrac{1}{2}c_ta_tv^2+(x_t-c_ta_ty_t)v, \tag{3.11c}$$

where as stated, a_t, c_t, x_t, y_t depend on t.

The differential equation (3.2) for H^{ω} generating $\mathcal{H}_t^{\omega}$ will separate in two differential equations, one of the form of an eigenvalue equation for $H^{\omega'}$ in the variable $v(q,t)$ and the other, a first-order differential equation in t. This can be seen by writing (3.7) for $t\to 0$, as $\partial t'/\partial t|_{t=0}=1$; it yields

$$H^{\omega}=G+H^{\omega'}, \tag{3.12}$$

where G generates $\mathcal{G}_t$ and is thus a first-order differential operator in q. The part in the separable function which depends only on v is $\psi_\lambda^{\omega'}$, which was chosen as an eigenfunction of $H^{\omega'}$ to start with. We have used $H^{\omega'}$ to separate the variables for H^{ω} in (3.2).

We can now see *a posteriori* why the factorization (3.7) works for all orbit representatives: They all have the form $\frac{1}{2}P^2+V(Q)$ so that G will only be a function of Q. A disentanglement of the Baker–Campbell–Hausdorff type to produce (3.7) out of (3.12) will introduce the P and $PQ+QP$ parts which generate the translations and scale transformation. In Table II we have collected the separating coordinates and multipliers for all pairs of orbit representatives. The results can be compared with the literature.[27] In order to describe the general form of the separating coordinates and to determine the H' to which they correspond, defining equivalence between coordinate pairs, we must present first the material of the next section. The general case, however, can be formulated as follows.

H. We are given arbitrary H and H', and we can determine the (geometric) transformation relating them to their orbit representatives. We are thus able to know their time-evolution transforms $\mathcal{H}_t$ and $\mathcal{H}'_t$ through (3.5). We can write $\mathcal{H}_t=\{\left(\begin{smallmatrix}h_a & h_b\\ h_c & h_d\end{smallmatrix}\right),(h_x,h_y,h_z)\}$ with $h_i=h_i(t)$, $(i=a,b,\ldots,z)$, where, it should be noticed, the $h_i(t)$ are linear combinations of trigonometric, hyperbolic, or power functions of t when H lies in the h, r, or l–f orbits. A similar construction is done for $\mathcal{H}'_{t'}$ with $h'_i(t')$, and the product with a general $\mathcal{G}_t$ is made as in (3.7). Comparison of the ratio of the 1–1 and 1–2 matrix elements gives

TABLE II. Expressions for the coordinate systems $(v(q,t),t)$ which separate the equation $H^{\omega}\psi = -i\partial_t\psi$ into two ordinary differential equations in v and t, such that $\psi(q,t) = e^{iS(v,t)}V(v)T(t)$. The separation operator is $H^{\omega'}$. The heat equation follows the f-case with $t \to 2it$.

$\omega \backslash \omega'$	h	r	l	f
h	$v=q$ $S=0$	$v=q(\cos 2t)^{-1/2}$ $S=\frac{1}{2}v^2\sin 2t$	$v=q/\cos t+\frac{1}{2}\tan^2 t$ $S=\frac{1}{4}v^2\sin 2t$ $+v\tan t(1-\frac{1}{2}\sin^2 t)$	$v=q/\cos t$ $S=\frac{1}{4}v^2\sin 2t$
r	$v=q(\cosh 2t)^{-1/2}$ $S=\frac{1}{2}v^2\sinh 2t$	$v=q$ $S=0$	$v=q/\cosh t+\frac{1}{2}\tanh^2 t$ $S=-\frac{1}{4}v^2\sinh 2t$ $+v\tanh t(1+\frac{1}{2}\sinh^2 t)$	$v=q/\cosh t$ $S=-\frac{1}{4}v^2\sinh 2t$
l	$v=q(1+t^2)^{-1/2}$ $S=\frac{1}{2}v^2t-vt(1+t^2)^{1/2}$	$v=q(1-t^2)^{-1/2}$ $S=\frac{1}{2}v^2t-vt(1-t^2)^{-1/2}$	$v=q$ $S=0$	$v=q-\frac{1}{2}t^2$ $S=-vt$
f	$v=q(1+t^2)^{-1/2}$ $S=-\frac{1}{2}v^2t$	$v=q(1-t^2)^{-1/2}$ $S=\frac{1}{2}v^2t$	$v=q+\frac{1}{2}t^2$ $S=vt$	$v=q$ $S=0$

$$h(t) \equiv h_a(t)/h_b(t) = h'_a(t')/h'_b(t') \equiv h'(t'), \qquad (3.13a)$$

whereby all h'_i's are known as functions of t. This is valid whenever h_b and h'_b are different from zero (this is *not* the case when H or H' is θI_2, for example). The parameters in the geometric transformation are then found as

$$a_t = h_a/h'_a = h_b/h'_b,$$
$$c_t = h_c/h'_a - h'_c/h_a = h_d/h'_b - h'_d/h_b, \qquad (3.13b)$$
$$(x_t, y_t) = (h_x - h'_x, h_y - h'_y)\begin{pmatrix} h'_d & -h'_b \\ -h'_c & h'_a \end{pmatrix},$$

and the separating variables and multipliers are found as in (3.11b)—(3.11c).

I. These developments also apply to complex transforms. Of particular interest is the heat equation,

$$\frac{\partial^2}{\partial q^2}u(q,t) = \frac{\partial}{\partial t}u(q,t), \qquad (3.14a)$$

i.e., in the form (3.2), $H^H = 2iH^f$. In the form (2.25a) this corresponds to $\theta_f = 2$, $\theta'_6 = 0$ and a scale transformation with $a^2 = i$ (subsection III.C). Better still, we can set $\theta_f = 2i$ and Eqs. (3.4d)—(3.5a) then state that the time-evolution transform is,

$$\mathcal{H}^H_t = \mathcal{H}^f_{2it} = \exp\left(t\frac{\partial^2}{\partial q^2}\right) : \left\{\begin{pmatrix} 1 & -2it \\ 0 & 1 \end{pmatrix}, (0)\right\}. \qquad (3.14b)$$

The separable solutions, coordinates, and multipliers for the heat equation, with respect to each of the orbit representatives we have considered, can thus be read off the bottom row of Table II, replacing $t \to 2it$. We have thus the separable solutions in terms of oscillator, parabolic cylinder, Airy, and exponential functions.[28]

J. In comparing with the literature,[29] we notice that one of the better-known separating coordinate systems, that giving rise to the heat polynomials[30] $v_n(q,t) \equiv (-t)^{n/2}H_n(\frac{1}{2}q[-t]^{-1/2})$, solutions of (3.14a), is apparently missing. We proceed to show that it is related to the entry in the h-orbit.

The Hermite differential equation can be written as

$$DH_n(q) \equiv \left(-\frac{1}{2}\frac{d^2}{dq^2} + q\frac{d}{dq} + \frac{1}{2}\right)H_n(q)$$
$$= (I_1 + 2iI_2 + I_3)H_n(q) = (n + \tfrac{1}{2})H_n(q), \qquad (3.15a)$$

so that $\Theta = 4 > 0$ and we can write $g_h D g_h^{-1} = \theta_h I_3 = \frac{1}{2}\theta_h H^h$ finding g_h to be a geometric SL(2,C) transformation given by (2.14) with $\theta_h = 2$, $a_h = 1$, $c_h = i$. This is a complex canonical transform, so that the eigenfunctions of D, $H_n(q)$, will be orthogonal with respect to the measure given by (3.1d) which is $e^{-q^2}dq$ and the integration performed over the real line[31] as in (3.1d). The time-development operator is

$$\mathcal{D}_{t'} = g_h^{-1}\mathcal{H}^h_t \cdot g_h : \left\{\begin{pmatrix} \exp(-it') & -\sin t' \\ 0 & \exp(it') \end{pmatrix}, (0)\right\} \qquad (3.15b)$$

and the decomposition $\mathcal{H}^H_t = \mathcal{G}_t\mathcal{D}_{t'}$ is possible with $a = \exp(it') = (1-4t)^{1/2}$, $c_t = 0$. This yields

$$\mathcal{H}^H_t H_n(q) = \exp[i(n+1/2)t']\mathcal{G}_t H_n(q)$$
$$= (1-4t)^{n/2}H_n(q[1-4t]^{-1/2}) = 2^n v_n(q, t - \tfrac{1}{4}), \qquad (3.15c)$$

which is a polynomial in q and $t - \frac{1}{4}$.

The separating coordinates are $v = q(1-4t)^{-1/2}$ and t equivalent under time translation to $\frac{1}{2}q(-t)^{-1/2}$, t and the multiplier $S(v,t)$ is zero. From (3.15c) we see that if the temperature distribution of a conducting rod at $t = 0$ is $H_n(q) = 2^n v_n(q, -\frac{1}{4})$, it will evolve in time as $2^n v_n(q, t-\frac{1}{4})$ and at $t = \frac{1}{4}$, $2^n v_n(q,0) = (2q)^n$. It should be observed that the $v_n(q, t-\frac{1}{4})$ are not elements of $\mathcal{L}^2(R)$ [nor is D self-adjoint in $\mathcal{L}^2(R)$]. However, as remarked above, D *is* self-adjoint if we take the measure $e^{-q^2}dq$, and there its eigenvectors are orthogonal and complete. Were we looking for the separating operator which produces the heat polynomials themselves, as $v_n(q,0) = q^n$, the operator would have been $H' \sim iI_2$. For this operator, however, we have $h'_b = 0$ and the decomposition (3.7) fails.

It should be observed that, since $H^H = -i\,\partial^2/\partial q^2$ is not Hermitian in $\mathcal{L}^2(R)$, the time development operator for the solutions of the heat equation (3.14b) is not unitary and does not preserve the orthogonality of two functions $f(q,t)$, $g(q,t)$ in $\mathcal{L}^2(R)$. However, if we use the formalism of complex canonical transforms, $\mathcal{H}_t^H$ is made a unitary mapping between $\mathcal{L}^2(R) \equiv \mathcal{J}_0$ and spaces $\mathcal{J}_t$ where the scalar product is, from (3.14b) and (3.1),

$$(f(\cdot,t),\ g(\cdot,t))_t \equiv \int_{\mathbf{C}} d\,\mathrm{Re}q\, d\,\mathrm{Im}q\,(2\pi t)^{-1/2} \exp[-(\mathrm{Im}q)^2/t]\, f(q,t)^*\, g(q,t), \quad t \geqslant 0. \tag{3.16}$$

Thus we can state that the quantity (3.16) is a *quadratic invariant* of the heat equation under time translations. This invariant is distinct from the total heat (a linear invariant), and is apparently new. Indeed, any differential equation (3.2) of the type we are studying will have its corresponding quadratic invariant.

IV. INVARIANCE GROUP AND INVARIANT BOUNDARIES

A. Lie theory has been used to solve partial differential equations through exploring their invariance under infinitesimal transformations, reducing thus by one the number of variables and then determining the subgroup—which leaves invariant a particular set of boundary conditions.[12] These methods apply to linear or nonlinear equations of any order. By contrast, our procedure is designed for linear parabolic equations of the type (1.6)—(3.2) and solves the problem through the use of matrix algebra in a global rather than infinitesimal manner.

The *invariance* of (3.2) under a transformation $g \in \mathrm{WSL}(2,C)$ can be stated as follows: when $u(q,t)$ is a solution of (3.2), then $v(q,t) \equiv \mathcal{J}_g^{(t)} U(q,t)$, where $\mathcal{J}_g^{(t)}$ is a two-variable representation of a canonical transform, is also a solution of (3.2). Notice that we have not said "if": *Any* such function will be a solution and the full invariance group of the equation will be the group WSL(2,C of six (complex) parameters. We will show below that, moreover, $v(q,t)$ will have the form

$$v(q,t) = \mathcal{J}_g^{(t)} u(q,t) = \mu_g(q,t)\, u(\overline{q}_g(q,t),\ \overline{t}_g(t)), \tag{4.1}$$

where μ_g, $\overline{q}_g$, and $\overline{t}_g$ are determinable functions of q and t. We should impose the additional conditions, however, that if q and t are real, then $\overline{q}_g$ and $\overline{t}_g$ should be also real and that if u is either square-integrable or real (the latter case in the heat equation, for example), then so should (4.1) be. This will reduce the acceptable symmetry group to a real subgroup of $\mathrm{WSL}(2,C)$.

B. In order to prove (4.1) and find the functions involved, use (2.9), (3.2)—(3.3), and (3.11): if $u(q,t)$ is the time development of the initial conditions $u(q) \equiv u(q,0)$ then $v(q,t) = \mathcal{J}_g^{(t)} u(q,t)$ is the time development of $v(q) = (\mathcal{J}_g u)(q)$:

$$\begin{aligned} v(q,t)) &= (\mathcal{J}_g^{(t)} u)(q,t) = (\mathcal{H}_t v)(q) \\ &= (\mathcal{H}_t \mathcal{J}_g u)(q) = (\mathcal{G}_{\overline{g}(g,t)}\ \mathcal{H}_{\overline{t}_g(t)}\, u)(q) \\ &= (\mathcal{G}_{\overline{g}} u)(q, \overline{t}_g) \\ &= \overline{a}^{-1/2} \exp\{i[(\overline{c}/2\overline{a})q^2 + (\overline{x}/\overline{a})q + \tfrac{1}{2}\overline{x}\overline{y} + \overline{z}]\} \\ &\quad \times u(\overline{q}_g(q,t),\ \overline{t}_g(t)), \end{aligned} \tag{4.2a}$$

where $\overline{a} = \overline{a}(t), \ldots, \overline{z} = \overline{z}(t)$ and

$$\overline{q}_g(q,t) = (q/\overline{a}) + \overline{y}, \quad h(\overline{t}_g) = [dh(t) + b]/[a + ch(t)] \tag{4.2b}$$

with the function $h(t)$ defined as in (3.13a). The key step in (4.2a) has been that of writing $\mathcal{H}_t \mathcal{J}_g = \mathcal{G}_{\overline{g}} \mathcal{H}_{\overline{t}}$, i.e., time development × canonical transform = geometric transform × time development in $\overline{t}_g(t)$. The last member of (4.2a) and (4.2b) were obtained from (3.11a)—(3.11b).

C. As a first illustration of (4.2) consider the case of the free particle, closely related to the heat equation, where the results are known[12,14]:

$$\begin{aligned} \mathcal{H}_t^f \mathcal{J}_g &= \left\{\begin{pmatrix} 1 & -t \\ 0 & 0 \end{pmatrix}, (0)\right\} \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (xyz)\right\} \\ &= \left\{\begin{pmatrix} a - ct & b - dt \\ c & d \end{pmatrix}, \quad (x, y, z)\right\} \\ &= \mathcal{G}_{\overline{g}} \mathcal{H}_{\overline{t}}^f = \left\{\begin{pmatrix} \overline{a} & 0 \\ \overline{c} & \overline{a}^{-1} \end{pmatrix} (\overline{x}\overline{y}\overline{z})\right\} \left\{\begin{pmatrix} 1 & -\overline{t} \\ 0 & 1 \end{pmatrix}, (0)\right\} \\ &= \left\{\begin{pmatrix} \overline{a} & -\overline{a}\overline{t} \\ \overline{c} & -\overline{c}\overline{t} + \overline{a}^{-1} \end{pmatrix}, (\overline{x}, \overline{y} - \overline{x}\overline{t}, \overline{z})\right\}. \end{aligned} \tag{4.3a}$$

Equation (4.3a) contains six independent simultaneous equations which yield

$$\overline{a} = a - ct, \quad \overline{c} = c, \quad \overline{x} = x, \quad \overline{y} = y + x\overline{t}, \quad \overline{z} = z \tag{4.3b}$$

and from (4.2b)

$$\begin{aligned} \overline{q} &\equiv \overline{q}_g(q,t) = (q + xdt - xb)/(a - ct) + y, \\ \overline{t} &\equiv \overline{t}_g(t) = (dt - b)/(a - ct). \end{aligned} \tag{4.3c}$$

Hence, if $u(q,t)$ is a solution of the free-particle Schrödinger equation, then so is

$$\begin{aligned} \mathcal{V}(q,t) &= \mathcal{G}_{\overline{g}} u(q, \overline{t}) \\ &= (a - ct)^{-1/2} \\ &\quad \exp(i\{(a - ct)^{-1}[cq^2 + xq + \tfrac{1}{2}x^2(dt - b)] \\ &\quad + \tfrac{1}{2}xy + z\}) \\ &\quad u((q + xdt - xb)/(a - ct) \\ &\quad + y, (dt - b)/(a - ct)). \end{aligned} \tag{4.4}$$

The physical meaning of each of the one-parameter subgroups in (4.4) can be readily ascertained when we put all others to their identity values. Thus y can be seen to represent coordinate translations $(q \to q + y)$, $-b$ time translations $(t \to t - b)$, $a = d^{-1}$ space—time scale transformations $(q \to q/a,\ t \to t/a^2)$, z phase multiplication $(u \to \exp(iz)u)$, x Galilean transformations $(q \to q + xt,\ u \to \exp(ixq)u)$, c conformal transformations $(q \to q/(1 - ct),\ t \to t/(1 - ct),\ u \to (1 - ct)^{-1/2} \exp\{i[cq^2/(1 - ct)]\}u)$. The last two are not "inspectionally" obvious symmetries of the equation.

If we further require that, under the transformation $\mathcal{J}$, q and t remain real and u remains in $\mathcal{L}^2(R)$, the values of the parameters $a, b, \ldots, z$ must be real. Thus the symmetry group of the free-particle Schrödinger equation is the six-parameter $\mathrm{WSL}(2,R)$ group.

D. The results for the heat equation can be read off (4.4) when we replace $t \to 2it$. It is convenient to define $\beta \equiv \tfrac{1}{2}ib$, $\gamma \equiv -2ic$, $\xi \equiv -2ix$, $\zeta \equiv -iz$. Here we require q, t, and u to be real. In terms of the new variables, we can see that the symmetry group of the heat equation is

TABLE III. Action of the general group transformation $g = \{\mathrm{A},\omega\} \in \mathrm{WSL}(2,C)$ on a function $u(q,t)$, solution of $H^{\omega}u - i\partial_t u$ for $\omega = h, r, l$ or f, as given by Eq. (4.2).

ω	time transformation	geometrical transformation
h	$\tan\bar{t} = \dfrac{d\tan t - b}{a - c\tan t}$	$\bar{a} = (a\cos t - c\sin t)/\cos\bar{t} = (d\sin t - b\cos t)/\sin\bar{t}$ $\bar{c} = (c\cos t + a\sin t - \bar{a}^{-1}\sin\bar{t})/\cos\bar{t}$ $(\bar{x},\bar{y}) = (x,y)\begin{pmatrix}\cos\bar{t} & \sin\bar{t}\\ -\sin\bar{t} & \cos\bar{t}\end{pmatrix}, \ \bar{Z} = Z$
r	$\tanh\bar{t} = \dfrac{d\tanh t - b}{a - c\tanh t}$	$\bar{a} = (a\cosh t - c\,\mathrm{cinh}\,t)/\cosh\bar{t} = (d\sinh t - b\cosh t)/\sinh\bar{t}$ $\bar{c} = (c\cosh t - a\sinh t + \bar{a}^{-1}\sinh\bar{t})/\cosh t$ $(\bar{x},\bar{y}) = (x,y)\begin{pmatrix}\cosh\bar{t} & \sinh\bar{t}\\ \sinh\bar{t} & \cosh\bar{t}\end{pmatrix}, \ \bar{Z} = Z$
l	$\bar{t} = \dfrac{dt - b}{a - ct}$	$\bar{a} = a - ct, \ \bar{c} = c, \ \bar{Z} = Z$ $(\bar{x},\bar{y}) = \{(x,y) + (-t, \tfrac{1}{2}t^2)\begin{pmatrix}a & b\\ c & d\end{pmatrix} + (\bar{t}, \tfrac{1}{2}\bar{t}^2)\}\begin{pmatrix}1 & \bar{t}\\ 0 & 1\end{pmatrix}$
f	$\bar{t} = \dfrac{dt - b}{a - ct}$	$\bar{a} = a - ct, \ \bar{c} = c, \ \bar{Z} = Z$ $(\bar{x},\bar{y}) = (x,y)\begin{pmatrix}1 & \bar{t}\\ 0 & 1\end{pmatrix}$

given by the subgroup of $\mathrm{WSL}(2,C)$ represented by the matrices

$$\left\{\begin{pmatrix} a & -2i\beta \\ \tfrac{1}{2}i\gamma & d\end{pmatrix}, \ (\tfrac{1}{2}i\xi, y, i\zeta)\right\}, \quad ad - \beta\gamma = 1 \tag{4.5a}$$

with $\alpha, \beta, \ldots, \zeta$ real.[32]

The operators which represent the canonical transformation (4.5) in (2.8) will be *bounded* when

$$a \geq 0, \quad \beta \geq 0, \quad \gamma \geq 0, \quad d \geq 0, \quad \xi = 0 \ \text{ when } \gamma = 0. \tag{4.5b}$$

The transformations (4.5a) with the restrictions (4.5b) form a *semigroup*, the $\mathrm{SL}(2,R)$ part of which is identical with the $\mathrm{HSL}(2,R)$ semigroup introduced in Ref. 7. It lies on the same orbit—through complex transformations—as the semigroup of real transformations in $\mathrm{SL}(2,R)$ with nonnegative matrix elements.[33] It is here augmented by the Weyl group and can be seen to be a subsemigroup of (4.5a) which preserves the positivity of the time displacements.[34]

E. The treatment of the four quantum Hamiltonians chosen in the last section as orbit representatives, follows the procedure of Eqs. (4.2a,b). We give in Table III the expressions for the time and geometric transformations as done in (4.2). It should be noted, though, that the physical transformations represented by the parameters $a, b, \ldots, y$ differ from case to case.

F. In solving a differential equation, we usually have to contend with *boundary conditions* $u_0(q,t)$ on the boundaries $\beta(q,t) = \text{const}$. Similarity methods choose the transformation $\mathcal{J}_g^{(t)}$ to leave these boundaries invariant: $\beta(q,t) = \beta(\bar{q},\bar{t})$. We will now show that the separating coordinates $(v(q,t),\ t'(t))$ of subsection III.G provide such boundaries in the form $v(q,t) = \text{const}$. Consider an example: Assume the transformation $\mathcal{J}_g^{(t)}$ in (4.3) is of the particular kind $\mathcal{J}_{g(\alpha)}^{(t)} = \mathcal{H}_{\alpha}^{h(t)}$ as in (3.4a). Then (4.3) tells us that $\bar{q} = q/(\cos\alpha - t\sin\alpha)$ and $\bar{t} = (t\cos\alpha + \sin\alpha)/(\cos\alpha - t\sin\alpha)$. Taking the lead from the entries f—h of Tables I and II, we can verify that $v \equiv q(1+t^2)^{-1/2} = \bar{q}(1+\bar{t}^2)^{-1/2} = \text{const}$, while for $t = \tan t'$ and $\bar{t} = \tan\bar{t}'$, $\bar{t}' = t' + \alpha$ simply. Hence, the family of hyperbolae $q^2 = v^2(1+t^2)$ for any $v \in R$ remains invariant under $\mathcal{J}_{g(\alpha)}^{(t)}$.

G. The general proof of this fact hinges on writing $\mathcal{J}_{g(\alpha)}^{(t)} = \mathcal{H}_{\alpha}^{\omega'(t)}$ for some generating operator $H^{\omega'}$. If now we are looking at the solution $u(q,t) = \mathcal{H}_t^{\omega}u(q)$, we should write $\mathcal{H}_t^{\omega} = \mathcal{G}_t\mathcal{H}_{t'}^{\omega'}$ and look for the corresponding separation of variables $\{v(q,t), t'(t)\}$ as done in (3.11). The action of $\mathcal{H}_{\alpha}^{\omega(t)}$ will thus be $t' \to t' + \alpha$, and leave $v(q,t)$ as a family of invariant lines on the q—t plane.

H. As for the inverse problem, if we know $\{v(q,t), t'(t)\}$ to be system of coordinates where the operator H^{ω} is separated by a second operator $H^{\omega'}$ [see Eqs. (3.7), (3.11), and (3.12)] with a multiplier $S(v,t)$, then $\bar{v} \equiv v(\bar{q},\bar{t})$, $\bar{t}' \equiv t'(\bar{t})$ as given by (4.2b) will be the separating coordinates of H^{ω} by the operator $gH^{\omega'}g^{-1}$ with multiplier $S(\bar{v},\bar{t})$. In order to see this, let $\mathcal{J}_g^{(t)}$ [the two-variable representation (4.1)—(4.2) of a transformation g associated with the time development $\mathcal{H}_t^{\omega}$] act on (3.11). The result of this action will still be a solution of (3.2) for H^{ω}:

$$\begin{aligned}\mathcal{J}_g^{(t)}\mathcal{H}_t^{\omega}\psi_{\lambda}^{\omega'}(q) &\equiv \mathcal{H}_t^{\omega}\mathcal{J}_g\psi_{\lambda}^{\omega'}(q)\\ &= \exp\{i[S(\bar{v},\bar{t}) - \tfrac{1}{2}\bar{y}(\bar{x} - \bar{c}\bar{a}\bar{y}) + \bar{z} + \lambda\bar{t}']\}\bar{a}^{-1/2}\psi_{\lambda}^{\omega'}(\bar{v}),\end{aligned} \tag{4.6}$$

where all barred variables depend on $\bar{q}$ and $\bar{t}$, while $\mathcal{J}_g\psi_{\lambda}^{\omega'}(q)$ is an eigenfunction of $gH^{\omega'}g^{-1}$.

I. As an illustration, we can apply this relation to the separable solutions of the heat equation seen in subsections III.I and III.J. When the separating operator is H^h (see entries f—h in Table II with $t \to 2it$), then $v = q(1-4t^2)^{-1/2}$. Hence, when we use gH^hg^{-1} to separate, the corresponding variables are

$$\bar{v} = \frac{q + t(\gamma y - d\xi) + (ay - \beta\xi)}{[t^2(\gamma^2 - 4d^2) + 2t(a\gamma - 4d\beta) + (a^2 - 4\beta^2)]^{1/2}}, \quad \bar{t} = \frac{dt + \beta}{a + \gamma t}, \tag{4.7}$$

with $a, \beta, \ldots, y$ as in (4.5). Now, the Hermite separating operator (3.15) is related to H^h through $D = gH^hg^{-1}$ [$g = g_h^{-1}$ as defined below Eq. (3.15a)]; hence, for $g = \{\left(\begin{smallmatrix}1 & 0\\ -i & 1\end{smallmatrix}\right), (0)\}$, $a = 1 = d$, $\gamma = -2$, $\xi = 0 = y$ the separating variables (4.7) become precisely those of (3.15c), namely $\bar{v} = q(1-4t)^{-1/2}$. Conversely, proposing a form for $\bar{v}$, we can find the group element which takes the separating operator to one of the four orbit representatives. We must compare the proposed form with (4.7) and the corresponding expressions for the $r, l,$ and f orbits, solving (nonlinear) algebraic equations for the parameters of g. If these equations are incompatible, the separating operator does not lie on the proposed orbit. If two operators are related through a similarity transformation in the symmetry group of the differential equation of a third, the variables they separate in the latter can be called *equivalent* in a general sense.[35]

Hence, while in Sec. III we found the separating variables for any given operator in the algebra; here we have solved the converse problem.

V. EQUATIONS CONTAINING TERMS IN q^{-2}

A. A class of operators containing terms in q^{-2} is amenable to a treatment parallel to the previous sections. The analysis is in fact simpler, and much of the groundwork has been done in Refs. 2 and 3, so only the general outline and conclusions will be presented. The operators we are refering to are

$$J_1=\frac{1}{4}\left(-\frac{d^2}{dq^2}+\frac{\mu}{q^2}-q^2\right), \tag{5.1a}$$

$$J_2=-\frac{i}{2}\left(q\frac{d}{dq}+\frac{1}{2}\right), \tag{5.1b}$$

$$J_3=\frac{1}{4}\left(-\frac{d^2}{dq^2}+\frac{\mu}{q^2}+q^2\right), \tag{5.1c}$$

which, together with $\mathbb{1}$ close onto an $o(2)\oplus sl(2,R)$ algebra as (2.6), the commuting o(2) is the one generated by $\mathbb{1}$. The operators (5.1) can be seen as the radial part of (2.5) for n-dimensional vectors $\mathbf{Q}$ and $\mathbf{P}$ in the space of angular momentum L, with $\mu=(\frac{1}{2}n+L-1)^2-\frac{1}{4}$ and subjected to a similarity transformation with the factor $|q|^{(n-1)/2}$ in order to cancel the term $[(n-1)/q]d/dq$ in $\mathbf{P}^2$. The operators (5.1) are densely defined and have self-adjoint extensions [36] for the ranges of μ specified below, in $\mathcal{L}^2(R^+)$. There is no underlying Weyl algebra here. [2]

Define now k through

$$\mu=(2k-1)^2-\tfrac{1}{4},\quad 2k=1\pm(\mu+\tfrac{1}{4})^{1/2} \tag{5.2}$$

so that the Casimir invariant for the algebra (5.1) can be seen to be $k(1-k)$.

Exponentiating the algebra (5.1) to an $O(2)\otimes SL(2,R)$ group, we associate a realization through 2×2 matrices as in (2.7). As the O(2) part generated by $\mathbb{1}$ corresponds to over-all phase transformations, it is rather trivial and we shall work henceforth with the $SL(2,R)$ part only. The action of the $SL(2,R)$ group on $f\in\mathcal{L}^2(R^+)$ is [2,3,6,7]

$$[\mathcal{C}\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)f](q)=b^{-1}\exp(i\pi k)\int_0^\infty dq'(qq')^{1/2}\exp[(i/2b)(aq'^2+dq^2)]\times J_{2k-1}(qq'/b)f(q') \tag{5.3a}$$

and, when $b=0$, we have the geometric transformation

$$[\mathcal{C}\left(\begin{smallmatrix}a&0\\c&a^{-1}\end{smallmatrix}\right)f](q)=|a|^{-1/2}\exp[(ica/2|a|^2)q^2]f(|a|^{-1}q), \tag{5.3b}$$

which, save for the absolute values, is identical with (2.8b). The transformations for complex group parameters and the definitions of Hilbert spaces into which these transformations are unitary was detailed in Ref. 2.

B. The adjoint action of $SL(2,R)$ on the algebra is found exactly as in Sec. II. It is represented as in (2.11)–(2.13):

$$K=\sum_j\eta_jJ_j\xrightarrow{g}K'=\sum_j\sum_k\eta_jN_{jk}J_k=\sum_k\eta_k'J_k$$

where $\|N_{jk}\|$ is the 3×3 upper-left submatrix of (2.12).

The orbit structure of $SL(2,R)$ is well known: there are three orbits corresponding to the sign of the invariant $\Theta=\eta_3^2-\eta_1^2-\eta_2^2$. The orbit representatives are chosen to be $2J_3$ $(\Theta>0)$, $2J_1$ $(\Theta<0)$, and J_1+J_3 $(\Theta=0)$, corresponding respectively to the Schrödinger Hamiltonians for harmonic oscillator plus centrifugal force, repulsive oscillator plus centrigural force, and pure centrifugal force. The relative strength of the oscillator and centrifugal parts can be varied through dilatation transformations in $SL(2,R)$ and the transformations leading a general operator K to one of the orbit representatives are calculated through the use of (2.14b), (2.15b), and (2.16b) excluding the expressions for x, y, θ_4, and θ_5.

For completeness, we list the eigenfunctions and spectrum of the orbit representatives [3]:

Harmonic Oscillator $+\mu/q^2$, $K^h=2J_3$, *spectrum* $\lambda=2(n+k)$, $n=0,1,2,\cdots$:

$$\phi_\lambda^h(q)=[2n!/\Gamma(n+2k)]^{1/2}\exp(-q^2/2)\,q^{2k-1/2}L_n^{(2k-1)}(q^2). \tag{5.4}$$

Repulsive Oscillator $+\mu/q^2$, $K^r=2J_1$, *spectrum* $\lambda\in R$:

$$\phi_\lambda^r(q)=(2\pi q)^{-1/2}\exp(i\pi k)\exp(\pi\lambda/4)\,2^{i\lambda/2}\times[\Gamma(k+\tfrac{1}{2}i\lambda)/\Gamma(2k)]M_{i\lambda/2,k-1/2}(-iq^2), \tag{5.5}$$

where $M_{\mu\nu}$ is the Whittaker function. [24]

Pure Centrifugal, μ/q^2, $K^f=J_1+J_3$, *spectrum* $\frac{1}{2}\lambda^2$, $\lambda\in R^+$:

$$\phi_\lambda^f(q)=(\lambda q)^{1/2}J_{2k-1}(\lambda q). \tag{5.6}$$

These functions are orthogonal and complete for $\mathcal{L}^2(R^+)$. It should be noted that the ϕ_λ^ω are, up to a phase, functions of $|q|$ and in fact $\phi_\lambda^\omega(e^{i\pi}q)=\exp[i\pi(2k-1/2)]\phi_\lambda^\omega(q)$. The operators (5.1) are invariant under $q\to -q$. Thus, the analysis of the eigenfunctions for $q\in R$ and harmonic analysis for functions in $\mathcal{L}^2(R)$ makes use of (5.4)–(5.6) with a few extra facts [3,36]:

(i) For $\mu\geq\frac{3}{4}$ (repulsive centrifugal force), the operators (5.1) have unique self-adjoint extensions in $\mathcal{L}^2(R^+)$ so that $k=\frac{1}{2}(1+[\mu+\frac{1}{4}]^{1/2})\geq 1$ and $\phi_\lambda^\omega(0)=0$.

(ii) For $\frac{3}{4}>\mu>0$ (repulsive), we have two square-integrable solutions and $\phi_\lambda^\omega(q)\sim q^{2k-1/2}$ at $q\to 0$, one for $k_1=\frac{1}{2}(1+[\mu+\frac{1}{4}]^{1/2})$, $\frac{3}{4}<k_1<1$, where the solutions are regular at the origin and one for $k_2=\frac{1}{2}(1-[\mu+\frac{1}{4}]^{1/2})$, $0<k_2<\frac{1}{4}$, where the solutions are irregular, but still square-integrable. We thus have to impose an extra boundary condition at $q=0$. (For example, if we have an infinite potential wall for $q\leq 0$, only the first solutions are acceptable). In $\mathcal{L}^2(R)$, the two families of solutions must be considered.

(iii) At $\mu=0$ the centrifugal "barrier" has disappeared, $k_1=\frac{3}{4}$ and $k_2=\frac{1}{4}$ represent the odd and even solutions, which become zero and constant as $q\to 0$. Their union gives back the spectrum and eigenvalues of the corresponding operators (2.5) on the whole of R.

(iv) For $-\frac{1}{4}<\mu<0$ (attractive centrifugal force), $\frac{1}{2}<k_1<\frac{3}{4}$ and $\frac{1}{4}<k_2<\frac{1}{2}$. Both solutions are regular at the origin. At $\mu=-\frac{1}{4}$ they coalesce.

(v) The centrifugal part cannot be more attractive than $\mu = -\frac{1}{4}$; otherwise, the k's become $\frac{1}{2} \pm i\nu$ (ν real): the spectrum of K^h is no longer lower-bound and the functions belong to the principal series rather than the lower-bound "discrete" representations of SL$(2,R)$.

From these observations, eigenfunctions of any other operator K in sl$(2,C)$ can be constructed as in II. J as a geometric transform of the eigenfunctions of their orbit representatives.

C. When we come to analyze differential equations of the type

$$Ku(q,t) = -i\frac{\partial}{\partial t}u(q,t) \tag{5.7}$$

with K in the algebra o$(2,C) \oplus$ sl$(2,C)$ generated by (5.1) the time-evolution transforms associated with K can be constructed out of the basis (3.4a, c, d) (the linear potential does not appear here). Copying Sec. II. E we can describe the time evolution of a function, solution of (5.7), expanded in terms of eigenfunctions of an operator K'. In particular the example II. E applies (replacing ψ by ϕ) for $K = 2J_3$ and $K' = J_1 + J_3$ with no change at all. Here we have three instead of the four cases of former sections and Tables I, II, and III on separating coordinates and multipliers apply here when we take out the l-rows and columns. The geometrical action of a is replaced by $|a|$.

Following the results of Sec. IV, we can see that the full invariance group of the class of differential equations (5.7) is the four-parameter group O(2)$\otimes$ SL$(2,R)$ when the appropiate reality and square-integrability conditions are imposed. The illustration in subsection IV. C is valid for the Schrödinger equation with a μ/q^2 potential when we eliminate the variables x and y, and its invariant boundaries are found as in the ensuing discussion.

VI. CONCLUSION

A. First, we would like to compare our approach with that of the "kinematical" invariance groups of Niederer and Boyer. We have dealt with representations of WSL$(2,C)$ on spaces of functions $u(q)$ on the real line q. The time development of a system (3.2) is a particular one-dimensional subgroup of such transformations: $u(q,t) = \mathcal{H}_t u(q)$. Then, we found that the action of WSL$(2,C)$ on the space of functions of two variables could be written as $u(q,t) \xrightarrow{g} v(q,t) = \mathcal{J}_g^{(t)} u(q,t)$ as in (4.1)–(4.2). Clearly $\mathcal{J}_g^{(t)} \equiv \mathcal{H}_t \mathcal{J}_g \mathcal{H}_t^{-1}$. If these transformations are generated as $\mathcal{J}_{g(\alpha)} = \exp(i\alpha F)$ and $\mathcal{J}_{g(\alpha)}^{(t)} = \exp(i\alpha F^{(t)})$, then also $F^{(t)} = \mathcal{H}_t F \mathcal{H}_t^{-1}$, so that F and $F^{(t)}$ are the Schrödinger and Heisenberg pictures of the same operators,[37] while $u(q,t)$ and $u(q)$ are the corresponding wavefunctions. We have

$$[H + i\partial/\partial t,\ F^{(t)}] = 0. \tag{6.1}$$

B. It should be noticed that $F^{(t)}$ generates geometric transformations in q—t space, i. e., $v(q,t)$ is a multiplier function times the function u of the transformed arguments $\bar{q}$ and $\bar{t}$. Thus $F^{(t)}$ can also be realized as a *first-order* differential operator in q and t. Indeed, if now, whenever H appears as a summand in $F^{(t)}$ we replace it by $-i\partial/\partial t$ in such a way that the resulting operator $F^{[t]}$ contain no second-order derivative terms in q and $F^{(t)} - F^{[t]} = f(t)(H + i\partial/\partial t)$, where $f(t)$ is a function only of t which appears among the matrix elements in the representation of $\mathcal{H}_t$ through (2.12). We will have $-[i\partial_t, F^{[t]}] = G^{[t]}$, where $G^{[t]}$ is in the algebra and has similarly H replaced by $-i\partial/\partial t$ and no second-order derivative terms. Now, it is still true that $[H, F^{[t]}] = G^{(t)}$ since H commutes with the H part in $F^{(t)}$. Hence, for some function $g(t)$ which we can find in (2.12),

$$[H + i\partial/\partial t,\ F^{[t]}] = G^{(t)} - G^{[t]} = g(t)(H + i\partial/\partial t), \tag{6.2}$$

acting on the space of differentiable functions of q and t.

Equation (6.2) can be recognized as the starting point for Niederer[9] who proposed definite forms for H (free particle and harmonic oscillator), and Boyer,[10] who left H in the general form $\frac{1}{2}P^2 + V(Q)$ and then determined the possible two-variable first-order differential operators $F^{[t]}$ satisfying (6.2). It was then found that only potentials of the form studied here allowed such a kinematical invariance group.[38,39] A wider class of time-dependent operators, not necessarily polynomials in P and Q have been considered by Anderson, Wulfman, *et al.*[40]

C. Boyer[10] pursued the study of (6.2) for n-dimensional systems and found the symmetry algebra (and group) to be subgroups of $W_n \circledS$ (SO$(n) \otimes$ SL$(2,R)$), called the Schrödinger group. Our method appears applicable to quadratic operators of the type

$$\sum\sum \alpha_{ij}P_iP_j + \sum\sum \beta_{ij}(P_iQ_j + Q_jP_i) + \sum\sum \gamma_{ij}Q_iQ_j + \sum \delta_i Q_i + \sum \epsilon_i P_i + \eta. \tag{6.3}$$

The symmetry algebra will be generated by the operators appearing in the summands and the generated group will be WSp$(2n,R)$, complexified. This group contains the Schrödinger group but cannot appear out of the starting equation (6.2) since the transformations in WSp$(2n,R)$ which are not in the Schrödinger group are not geometric transformations in q—t space and hence are not representable as first-order differential operators in these variables satisfying (6.2).

D. Our analysis should reduce the examination of the symmetry group of quadratic Hamiltonians of the type (6.3) to the complete orbit analysis of WSp$(2n,R)$ or of different real forms of its complex algebra.[41] Presence of "centrifugal force barriers," radial or plane, would cut down the full symmetry and some of the more interesting cases up to three dimensions have been analyzed through separation of variables in the conventional way.[15,16,18] Further, one need not restrict oneself to $\mathcal{L}^2(R^n)$ spaces of functions, but use any differentiable group coset manifold[17] and look for finite- or infinite-dimensional subalgebras in the enveloping algebra[42] of the group. Eventually, one would also like to extend the application of the global group method through matrix algebra (on an extended space, if possible), to other types of differential equations.

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