The 2:1 anisotropic oscillator, separation of variables and symmetry group in Bargmann space

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(Received 3 March 1975)

We present a detailed analysis of the separation of variables for the time-dependent Schrödinger equation for the anisotropic oscillator with a 2:1 frequency ratio. This reduces essentially to the time-independent one, where the known separability in Cartesian and parabolic coordinates applies. The eigenvalue problem in parabolic coordinates is a multiparameter one which is solved in a simple manner by transforming the system to Bargmann's Hilbert space. There, the degeneracy space appears as a subspace of homogeneous polynomials which admit unique representations of a solvable symmetry algebra s_3 in terms of first order operators. These representations, as well as their conjugate representations, are then integrated to indecomposable finite-dimensional nonunitary representations of the corresponding group S_3. It is then shown that the two separable coordinate systems correspond to precisely the two orbits of the factor algebra $s_3/u(1)$ [$u(1)$ generated by the Hamiltonian] under the adjoint action of the group. We derive some special function identitites for the new polynomials which occur in parabolic coordinates. The action of S_3 induces a nonlinear canocical transformation in phase space which leaves the Hamiltonian invariant. We discuss the differences with previous works which present $su(2)$ as the algebra responsible for the degeneracy of the two-dimensional anisotropic oscillator.

1. INTRODUCTION

In this paper we will examine the quantum two-dimensional anisotropic harmonic oscillator with a 2 : 1 frequency ratio. This system, though particular, is interesting in two respects: First, the time-independent problem is known to separate in two coordinate systems, Cartesian and parabolic and, second, its energy levels exhibit a degeneracy pattern which has been attributed to a symmetry algebra. Both features will be shown to be related through the treatment of the problem in Bargmann space.

Winternitz *et al.* have shown[1] that there is a one-to-one correspondence between second-order differential operators which commute with quantum Hamiltonians H of the standard type [i. e., $-\Delta + V(x_1, x_2)$], and separable coordinate systems for the Schrödinger equation, that is, there exist functions $v_1(x_1,x_2)$, $v_2(x_1,x_2)$ such that the time-independent Schrödinger equation separates into two differential equations, one in v_1 and one in v_2. Reduced to a canonical form, these v's can be made to correspond to one of the four orthogonal coordinate systems in two-space: Cartesian, polar, parabolic, or elliptic. The 2 : 1 anisotropic oscillator, in particular, was shown to separate in Cartesian and parabolic coordinates with the corresponding "separation" operators S_C and S_P commuting with H. Section 2 recapitulates these developments in the light of the general procedure of separation of variables[2,3] involving the time, and shows that the time-dependent problem can be reduced to a study of the time-independent one. The wavefunctions of the system in parabolic coordinates are not known special functions.

In Sec. 3 we show that the introduction of Bargmann space[4] provides a very convenient tool for finding the eigenfunctions and spectra of the pair S_P and H. The parabolic basis eigenfunctions are seen to be given in terms of new orthogonal polynomials whose coefficients are given by a three-term recursion relation. The polynomials and their eigenvalues are tabulated up to the $n=15$ level. These coefficients also give the expansion of the parabolic basis eigenfunctions in terms of the Cartesian ones and vice versa as well as other special function identities.

In Sec. 4 we relate the polynomials and degeneracy pattern to the existence of a solvable symmetry group in Bargmann space. This group, or more accurately, its infinitesimal generators are found by looking for all first-order densely defined differential operators in Bargmann space which commute with the Hamiltonian. The representations we find are indecomposable non-unitary finite-dimensional representations of the symmetry group. We also discuss the relevance of the conjugate representations. Moreover, the orbit structure of the Lie algebra is analyzed and it is shown that the orbits relate to the two separable coordinate systems in the usual configuration space. We point out that this connection breaks down for any other rational frequency ratio.

The solvable symmetry group is a group of nonlinear canonical transformations of the coordinate-momentum space which are *geometrical* symmetries in Bargmann space. This is shown in Sec. 5. Finally, in Sec. 6 some conclusions are presented about the relations and differences between our and former work.[5-7] These question the necessity of unitary representations and of $su(2)$ in describing accidental degeneracy.

2. SEPARATION OF VARIABLES

Our first aim is to find all separable coordinate systems for the equation

$$U_{x_1x_1} + U_{x_2x_2} + iU_t - (4x_1^2 + x_2^2)U = 0 \tag{2.1}$$

where $U_z \equiv \partial U/\partial z$. The procedure that follows is quite analogous to Ref. 3 with only slight modifications due to the potential term. We will thus only give a rough sketch of the method used in deriving the result (2.5).

We look for all coordinate systems described by the change of variables

$$x_1 = X_1(v_1, v_2, v_3), \quad x_2 = X_2(v_1, v_2, v_3), \quad t = T(v_1, v_2, v_3), \tag{2.2}$$

such that a solution of (2.1) is of the form

$$U(v_1, v_2, v_3) = \exp[iS(v_1, v_2, v_3)]V_1(v_1)V_2(v_2)V_3(v_3), \tag{2.3}$$

and (2.1) reduces to three ordinary differential equations. The function $S(v_1, v_2, v_3)$ is called a multiplier and can be determined by the analysis. Moreover, the separation process[3] always allows us to choose $t = T(v_1, v_2, v_3) = v_3$ in (2.2).

For the purpose of finding the separable coordinate systems of (2.1) it is useful to consider its symmetry group. The Lie algebra of this group was determined in Ref. 8, where the integrated group $\mathcal{H}_2$ (the two-dimensional harmonic oscillator group) has the structure $\mathcal{H}_2 \approx \mathcal{R}_1 \otimes \mathcal{W}_2$, where $\mathcal{R}_1$ is the group of additive reals, $\mathcal{W}_2$ is the five-dimensional Weyl group in two-space and $\otimes$ denotes the semidirect product. The group action given in terms of the one-parameter subgroups ($\mathcal{R}_1$ is generated by R and $\mathcal{W}_2$ by B_i, P_i, $i=1,2$, and E with $[B_j, P_k] = i\delta_{jk}E\omega_j$), $\omega_1 = 2$, $\omega_2 = 1$, is

$$\exp(i\tau R)f(x_1, x_2, t) = f(x_1, x_2, t+\tau), \tag{2.4a}$$

$$\begin{aligned}\exp(i\boldsymbol{\beta}\cdot\mathbf{P})f(x_1, x_2, t) = \exp[&-i(4\beta_1 x_1 \sin 4t + 2\beta_2 x_2 \sin 2t)\\ &+ \beta_1^2 \sin 8t + \tfrac{1}{2}\beta_2^2 \sin 4t]\\ &\times f(x_1 - \beta_1 \cos 4t, x_2 - \beta_2 \cos 2t, t),\end{aligned} \tag{2.4b}$$

$$\begin{aligned}&\exp(i\boldsymbol{\alpha}\cdot\mathbf{B})f(x_1, x_2, t)\\ &\quad = \exp[i(4\alpha_1 x_1 \sin 4t + 2\alpha_2 x_2 \sin 2t) + \alpha_1^2 \sin 8t\\ &\qquad + \tfrac{1}{2}\alpha_2^2 \sin 4t] f(x_1 - \alpha_1 \sin 4t, x_2 - \alpha_2 \sin 2t, t),\end{aligned} \tag{2.4c}$$

$$\exp(\gamma E)f(x_1, x_2, t) = \exp(\gamma)f(x_1, x_2, t), \tag{2.4d}$$

where $f \in \mathcal{C}^\infty$, $\tau, \alpha_1, \ldots, \gamma \in \mathbb{R}$. Now, by a straightforward calculation following the procedure of Ref. 3, it can be shown that the only separable coordinates with a nontrivial multiplier S (i.e., *not* a sum of functions of the individual variables) are those given precisely by the change of variables induced by the transformations of the symmetry group (2.4). Indeed, such transformations give rise to separable solutions $V_1(v_1)$, $V_2(v_2)$ which satisfy the same ordinary differential equations with the usual separation in t, i.e., $T(v_3) = c \exp iEt$. Therefore, two separable coordinate systems which differ by a transformation of the type (2.4) are said to be equivalent. Hence, our problem reduces to the separation of the time-independent Schrödinger equation[1] and we find only *two* inequivalent separable coordinates:

(i) Cartesian

$$x_1 = v_1, \quad x_2 = v_2, \quad t = v_3, \quad x_1, x_2 \in \mathbb{R}; \tag{2.5a}$$

(ii) parabolic

$$x_1 = \tfrac{1}{2}(v_1^2 - v_2^2), \quad x_2 = v_1 v_2, \quad t = v_3. \tag{2.5b}$$

$v_1 \in \mathbb{R}$, $v_2 \in \mathbb{R}^+$; thus in what follows we consider the time-independent Schrödinger equation, viz.

$$H\psi = -\psi_{x_1x_1} - \psi_{x_2x_2} + (4x_1^2 + x_2^2)\psi = E\psi, \tag{2.6a}$$

obtained from (2.1) through

$$U(x_1, x_2, t) = \exp(iEt)\psi(x_1, x_2). \tag{2.6b}$$

Winternitz and collaborators[1] have characterized the two separable systems (2.5) of (2.6) by the two second-order symmetry operators

$$S_C = -\partial_{x_1x_1} + 4x_1^2, \tag{2.7a}$$

$$S_P = x_1\,\partial_{x_2x_2} - x_2\,\partial_{x_1x_2} - \tfrac{1}{2}\partial_{x_1} + x_1 x_2^2, \tag{2.7b}$$

corresponding to (i) and (ii) above, respectively. Indeed, it can be shown that (2.7) are the only second-order operators which commute with the Hamiltonian H. Now, the solutions of (2.1) in the Cartesian coordinate system (2.5a) are characterized by the equations

$$H\psi^C = E\psi^C, \tag{2.8a}$$

$$S_C\psi^C = \mu\psi^C, \tag{2.8b}$$

which give rise to the well-known eigenvalue problem for the one-dimensional harmonic oscillator with eigenfunctions normalized in the usual Hilbert space norm $\mathcal{L}^2(\mathbb{R}^2)$ given by

$$\begin{aligned}&\psi^C_{n_1n_2}(x_1, x_2)\\ &\quad = [2^{n_1+n_2-1/2}\pi n_1!\,n_2!]^{-1/2}\exp[-x^2 - \tfrac{1}{2}y^2]H_{n_1}(\sqrt{2}x_1)H_{n_2}(x_2),\end{aligned} \tag{2.9a}$$

with eigenvalues

$$E = 4n_1 + 2n_2 + 3 \equiv 2n + 3, \quad n_1, n_2, n = 0, 1, 2, \cdots, \quad \mu = 4n_1 + 2. \tag{2.9b}$$

Notice that the energy level labeled by n has degeneracy $[n/2]+1$, where $[r]$ is the integer part of r.

The solution of (2.1) in parabolic coordinates (2.5b) are

$$H\psi^P = E\psi^P, \tag{2.10a}$$

$$S_P\psi^P = \lambda\psi^P. \tag{2.10b}$$

These equations give rise to $\mathcal{L}^2(\mathbb{R}^2)$ solutions $\psi^P_{nl}(x_1, x_2)$ which are products of the form

$$\psi^P_{nl}(x_1, x_2) = \phi_{nl}(v_1)\phi_{nl}(iv_2), \tag{2.11}$$

where $\phi(v)$ is a solution of the equation

$$\phi_{vv} + (2\lambda + Ev^2 - v^6)\phi = 0. \tag{2.12}$$

We note that since the measure in parabolic coordinates is

$$d^2\mathbf{x} = dx_1\,dx_2 = (v_1^2 + v_2^2)\,dv_1\,dv_2, \tag{2.13}$$

and (2.12) depends on both λ and E, the eigenvalue problem is a (coupled) multiparameter one. However, we know the value of E from the Cartesian separation and we can use (2.10b) to derive a recursion relation for the overlap functions between the two bases. Then, since the degeneracy for each n is $[n/2]+1$, we look for the recursion relation to be cut off. Rather than implement this procedure here, in the next section we will analyze the system in Bargmann's Hilbert space where our problem reduces to a single Sturm–Liouville problem and the degeneracy of states appears simply as a subspace of homogeneous polynomials. It is further noticed that (2.12) is the equation for an anharmonic oscillator with a $-\omega^2 v^2 + v^6$ potential, and can be related to the confluent Heun equation.[9]

Equations (2.19) also exhibit an interesting discrete symmetry: It is easily seen from (2.6a) and (2.7b) that under the parity transformation $x_1 \to -x_1$, $H \to H$ while $S_P \to -S_P$; hence if $\psi^P_{nl}(x_1, x_2)$ is a solution of the eigenvalue problem (2.10) with eigenvalues E and λ then $\psi^P_{nl}(-x_1, x_2) = \phi_{nl}(v_2)\phi_{nl}(iv_1)$ is also a solution with eigenvalues E and $-\lambda$. In addition, Eqs. (2.10) are invariant under $x_2 \to -x_2$ $(v_1 \to -v_1)$ and the parity properties of the Cartesian basis (2.9a) are well known.

Note: Our Eq. (2.12) has been recently treated in an interesting work by Truong through the use of harmonic analysis on the Weyl group. [See T. T. Truong, a Weyl quantization of anharmonic oscillators, J. Math, Phys. **16**, 1034 (1975).]

3. SOLUTION IN BARGMANN SPACE

In this section we shall show that the treatment of the anisotropic oscillator in Bargmann's Hilbert space of analytic functions[4] allows a simple interpretation of the degeneracy pattern as well as a reduction for the parabolic coordinates to a simple Sturm—Liouville problem.[10] For an oscillator of frequency ω we define[11] out of the canonically conjugate operators $\hat{x}$ and $\hat{p}$ (with $[\hat{x}, \hat{p}] = i\mathbb{1}$),

$$\hat{x} = (2\omega)^{-1/2}(\hat{\eta} + i\hat{\xi}) \tag{3.1a}$$

$$\hat{p} = (2/\omega)^{-1/2}(i\hat{\eta} + \hat{\xi}), \tag{3.1b}$$

so that $\hat{\eta}$ and $\hat{\xi}$ also constitute a canonical pair ($[\hat{\eta}, \hat{\xi}] = i\mathbb{1}$). Under (3.1), the Hamiltonian becomes

$$H_\omega = \hat{p}^2 + \omega^2\hat{x}^2 = 2\omega(i\hat{\eta}\hat{\xi} + \tfrac{1}{2}). \tag{3.1c}$$

Upon introducing a scalar product over the complex plane $\mathbb{C}$

$$(f, g)_\omega = \omega^{1/2}\pi^{-1}\int_{\mathbb{C}} d^2\mu_\omega(\eta) f(\eta)^* g(\eta), \tag{3.2a}$$

where

$$d^2\mu_\omega(\eta) = \exp[-\omega|\eta|^2]\,d^2\eta, \quad d^2\eta \equiv d\,\mathrm{Re}\eta\, d\,\mathrm{Im}\eta, \tag{3.2b}$$

f and g are analytic functions in η over $\mathbb{C}$ of growth $(2, \omega/2)$, and completing with respect to the norm induced by (3.2) we obtain the Bargmann space $\mathcal{F}_\omega$. Bargmann has shown that the operators given by (3.1a, b, c) are self-adjoint in $\mathcal{F}_\omega$ defined with the domains

$$\mathcal{D}(\mathcal{O}) = \{f \in \mathcal{F}_\omega : \mathcal{O}f \in \mathcal{F}_\omega\}, \tag{3.3a}$$

where $\mathcal{O}$ is one of the operators (3.1). Thus in $\mathcal{F}_\omega$ we have the representation

$$\hat{\eta} f(\eta) = \eta f(\eta), \quad \hat{\xi} f(\eta) = -i\partial f(\eta)/\partial\eta, \tag{3.3b}$$

with the Hilbert space adjoint

$$\hat{\eta}^\dagger = i\hat{\xi}/\omega, \quad \hat{\xi}^\dagger = i\omega\hat{\eta}. \tag{3.3c}$$

The unitary mapping between $\mathcal{L}^2(\mathbb{R})$ and $\mathcal{F}_\omega$ is given by

$$\tilde{f}(\eta) = (\mathbf{A}_\omega f)(\eta) = \int_{\mathbb{R}} dx\, A(\eta, x) f(x) \tag{3.4a}$$

with the inverse

$$f(x) = (\mathbf{A}_\omega^{-1}\tilde{f})(x) = \int_{\mathbb{C}} d^2\mu(\eta) A(\eta, x)^* \tilde{f}(\eta) \tag{3.4b}$$

where

$$A_\omega(\eta, x) = \omega^{1/2}\pi^{-1/4}\exp[\omega(-\tfrac{1}{2}x^2 + \sqrt{2}x\eta - \tfrac{1}{2}\eta^2)], \tag{3.4c}$$

and $\tilde{f} \in \mathcal{F}_\omega$, $f \in \mathcal{L}^2(\mathbb{R})$, and the integrals are understood to be in the sense of limit in the mean.

We can now build the space $\mathcal{F} \equiv \mathcal{F}(\mathbb{C}^2)$ with the measure $d\mu_1^2 d\mu_2^2$ and the two-dimensional Hamiltonian which is the image under the unitary mapping $\mathbf{A} \equiv \mathbf{A}_2 \otimes \mathbf{A}_1$ of the Hamiltonian (2.6a) and the sum of two Hamiltonians (3.1c) with $\omega_1 = 2$, $\omega_2 = 1$. Hence in $\mathcal{F}$ our Hamiltonian is

$$\tilde{H} = 4\eta_1\partial_{\eta_1} + 2\eta_2\partial_{\eta_2} + 3. \tag{3.5}$$

Now the simple form of (3.5) allows us to immediately solve the two-dimensional eigenvalue problem

$$\tilde{H}\tilde{\psi}_n = E_n\tilde{\psi}_n, \quad \tilde{\psi}_n = \tilde{\psi}_n(\eta_1, \eta_2). \tag{3.6}$$

We find by the method of characteristics and the fact that $\tilde{H}\tilde{\psi}_n \in \mathcal{F}$ the general solution with (2.9b),

$$\tilde{\psi}_n(\eta_1, \eta_2) = \eta_2^n P_n(\eta_1/\eta_2^2), \tag{3.7}$$

where P_n is a polynomial of degree $[n/2]$. Hence the degeneracy of states in $\mathcal{F}$ makes its appearance by the simple fact that the solutions of (3.6) are homogeneous polynomials P_n of degree $[n/2]$. This polynomial subspace $\mathcal{P}_n$ maps under the Bargmann transform $\mathbf{A}$ given by (3.4) onto all $\mathcal{L}^2(\mathbb{R}^2)$ solutions of the Schrödinger equation (2.6a) with fixed energy E_n. In the next section we will find a group of transformations [*not* $SU(2)$] which maps the polynomial subspace $\mathcal{P}_n \subset \mathcal{F}$ defined by (3.7) into itself. We also emphasize here that the above analysis is quite general and applies to any anisotropic oscillator whose ratio of frequencies is rational, although there will be no connection with separable coordinate systems.

In the Cartesian basis described by the self-adjoint[4] operator

$$\tilde{S}_C = \mathbf{A}S_C\mathbf{A}^{-1} = 4\eta_1\partial_{\eta_1} + 2, \tag{3.8a}$$

along with (3.6) we obtain the orthonormal eigenfunctions

$$\tilde{\psi}^C_{n_1 n_2}(\eta_1, \eta_2) = (n_1!\,n_2!)^{-1/2}2^{n_1/2}\eta_1^{n_1}\eta_2^{n_2}. \tag{3.8b}$$

with E given by (2.9b). Note since $n = 2n_1 + n_2$, it follows that $\tilde{\psi}^C_{n_1 n_2} \in \mathcal{P}_n$. Moreover, under (3.4b) the eigenfunctions (3.8b) map onto the usual harmonic oscillator eigenfunctions (2.9a).

For the parabolic coordinates we find the operator

$$\tilde{S}_P = \mathbf{A}S_P\mathbf{A}^{-1} = \sqrt{2}\eta_1\partial_{\eta_2\eta_2} + \eta_2^2\partial_{\eta_1}/\sqrt{2}. \tag{3.9}$$

which is self-adjoint on the domain $\mathcal{D}(\hat{\eta}_1 \times \hat{\eta}_2^2)$ with the $\mathcal{D}$'s given by (3.3). From the operator S_P in (2.6b) one expects in general upon inserting (3.1) that $\tilde{S}_P$ be a third order operator in $\mathcal{F}$. It is a pleasant feature of the mapping that the third order terms cancel. The eigenvalue problem for (3.9) on $\mathcal{P}_n$, upon introducing in (3.7) the change of variables

$$z = \eta_2, \quad u = \eta_1/\eta_2^2, \tag{3.10a}$$

yields the differential equation

$$4\sqrt{2}u^3 P''_{nl}(u) + [-2\sqrt{2}(2n-3)u^2 + 1/\sqrt{2}]P'_{nl}(u)$$

$$+[\sqrt{2}n(n-1)u-\lambda_l]P_{nl}(u)=0, \qquad (3.10b)$$

where we have labelled the eigenvalue λ of $\tilde{S}_P$ through the index l in a fashion which will be described below. Expanding the polynomials $P_{nl}(u)$ as

$$P_{nl}(u)=\sum_{m=0}^{[n/2]} p_m^{nl}u^m, \qquad (3.11)$$

we find three-term recursion relation for the coefficients

$$\frac{1}{\sqrt{2}}(m+1)p_{m+1}^{nl}-\lambda_l p_m^{nl}+\sqrt{2}(2m-n-1)(2m-n-2)p_{m-1}^{nl}=0. \qquad (3.12)$$

We remark that the coefficients p_m^{nl} have been chosen to be real and such that $p_0^{nl}>0$. Equation (3.12) allows us to solve the eigenvalue problem for λ_l when we require that $p_{[n/2]+1}^{nl}=0$. This problem is equivalent to diagonalizing a square tridiagonal matrix of dimension $[n/2]+1$. The resulting eigenvalues λ_l can be labelled by the index l running from $-\frac{1}{2}([n/2]+1)$ to $\frac{1}{2}([n/2]+1)$ in integer steps and such that $\lambda_{l_1}<\lambda_{l_2}$ iff $l_1<l_2$. The motivation for such a labeling stems from the parity properties discussed at the end of the last section. Clearly the inversion $x_1\to -x_1$ implies $\eta_1\to-\eta_1$, or equivalently $u\to -u$; and again if under $\tilde{S}_P$, λ_l is the eigenvalue of $\tilde{\psi}_{nl}^P(\eta_1,\eta_2)$, then $-\lambda_l$ is the eigenvalue of $\tilde{\psi}_{nl}^P(-\eta_1,\eta_2)$; and if $P_{nl}(u)$ satisfied (3.10b) with λ_l, $P_{nl}(-u)$ will satisfy the same equation with $-\lambda_l$. Our labeling convention for λ_l then implies that $-\lambda_l=\lambda_{-l}$ and $P_{n,-l}(u)=P_{nl}(-u)$. The eigenvalues λ_l appear thus in symmetrical pairs. When $[n/2]+1$ is even, the l's are half-intergers, while when $[n/2]+1$ is odd, the l's are integers and $\lambda_0=0$ is among the eigenvalues. We point out that although the label l resembles a "magnetic" quantum number suggesting an $su(2)$ symmetry algebra for the system, no such construction has been made.

The eigenvalues λ_l and the properly normalized coefficients p_m^{nl} for (3.11), (3.12) for the first 15 values of n have been computed and collected in Table I. We will refer to $P_{nl}(u)$ as *parabolic* polynomials. The entries of this table also give us the needed information about the expansion of the parabolic coordinate solutions in terms of the Cartesian basis, since from (3.7), (3.8b), and (3.11)

$$\begin{aligned}\tilde{\psi}_{nl}^P(\eta_1,\eta_2)&=\sum_{m=0}^{[n/2]} p_m^{nl}\eta_1^m\eta_2^{n-2m}\\ &=\sum_{m=0}^{[n/2]}[2^{-m}m!\,(n-2m)!]^{1/2}p_m^{nl}\tilde{\psi}_{m,n-2m}^C(\eta_1,\eta_2).\end{aligned} \qquad (3.13)$$

Choosing $\tilde{\psi}_{nl}^P(\eta_1,\eta_2)$ to be normalized in $\mathcal{J}$, we find

$$\sum_{m=0}^{[n/2]}[2^{-m}m!\,(n-2m)!]p_m^{nl}p_m^{nl'}=\delta_{l,l'}. \qquad (3.14)$$

The expansion inverse to (3.13) is easily obtained and reads

$$\tilde{\psi}_{m,n-2m}^C(\eta_1,\eta_2)=[2^{-m}m!\,(n-2m)!]^{1/2}\sum_{l=-[n/2]-1}^{[n/2]+1} p_m^{nl}\tilde{\psi}_{nl}^P(\eta_1,\eta_2) \qquad (3.15)$$

Again from the orthonormality properties we obtain

$$2^{-m}m!\,(n-2m)!\sum_{l=-[n/2]-1}^{[n/2]+1} p_{m'}^{nl}p_m^{nl}=\delta_{m,m'}. \qquad (3.16)$$

The solutions (3.10) of (3.11) are instrumental for the solutions (2.11), $\phi_{nl}(v)$, of (2.12) in the following manner: Transforming (3.10) to its standard form, we find the latter to be identical with (2.12) so that its solutions are

$$\phi_{nl}(v)\sim v^n\exp(-v^4/4)P_{nl}([2\sqrt{2}v^2]^{-1}), \qquad (3.17)$$

upon demanding the correct asymptotic properties for $\tilde{\psi}_{nl}^P$. It is emphasized that we have constructed polynomial solutions of the differential equation (2.12). The advantage of the Bargmann space treatment is now manifest: Through the unitary transform (3.4) we have reduced the coupled multiparameter eigenvalue problem (2.10) to the single Sturm—Liouville problem (3.10) whence upon transforming back and using (3.17), we have

$$\begin{aligned}\psi_{nl}^P(v_1,v_2)=c_{nl}(v_1v_2)^n\exp[-\tfrac{1}{4}(v_1^4+v_2^4)]P_{nl}([2\sqrt{2}v_1^2]^{-1})\\ \times P_{nl}(-[2\sqrt{2}v_2^2]^{-1}),\end{aligned} \qquad (3.18)$$

where c_{nl} is the normalization coefficient with respect to the measure (2.14) given by

$$c_{nl}=(p_{[n/2]}^{nl})^{-2}\pi^{-1/2}2^{n+\sigma(n)/4}\sum_{k=0}^{[n/4]}p_{2k}^{nl}(-1)^k\frac{(2k)!\,(n-4k)!}{k!\,([n/2]-2k)!} \qquad (3.19)$$

where $\sigma(n)\equiv(-1)^n$ and can be calculated with the use of the Table I.

Writing the transform (3.4a) explicitly, we find the integral identity

$$\begin{aligned}\int_{-\infty}^{\infty}dv_1\int_0^{\infty}dv_2(v_1^2+v_2^2)(v_1v_2)^n\exp[-\tfrac{1}{2}(v_1^4+v_2^4)+\sqrt{2}(v_1^2-v_2^2)\eta_1\\ +\sqrt{2}v_1v_2\eta_2]P_{nl}([2\sqrt{2}v_1^2]^{-1})P_{nl}(-[2\sqrt{2}v_2^2]^{-1})\\ =[c_{nl}\sqrt{2/\pi}]^{-1}\exp[\eta_1^2+\tfrac{1}{2}\eta_2^2]\eta_2^nP_{nl}(\eta_1/\eta_2^2).\end{aligned} \qquad (3.20a)$$

Equation (3.4b) yields

$$\begin{aligned}\int\int_{\mathbb{C}}d^2\eta_1d^2\eta_2\,\eta_2^nP_{nl}(\eta_1/\eta_2^2)\exp[-2|\eta_1|^2-|\eta_2|^2-\eta_1^{*2}\\ -\tfrac{1}{2}\eta_2^{*2}+\sqrt{2}(v_1^2-v_2^2)\eta_1^*+\sqrt{2}v_1v_2\eta_2^*]\\ =\tfrac{1}{2}\pi^{3/2}c_{nl}(v_1v_2)^nP_{nl}([2\sqrt{2}v_1^2]^{-1})P_{nl}(-[2\sqrt{2}v_2^2]^{-1}).\end{aligned} \qquad (3.20b)$$

Moreover, applying the unitary transform (3.4) to the expansions (3.13) and (3.15), we find the expansion formulae

$$\begin{aligned}\sum_{m=0}^{[n/2]}p_m^{nl}H_m(2^{-1/2}[v_1^2-v_2^2])H_{n-2m}(v_1v_2)\\ =\pi^{1/2}2^{n/2-1/4}c_{nl}(v_1v_2)^nP_{nl}([2\sqrt{2}v_1^2]^{-1})P_{nl}(-[2\sqrt{2}v_2^2]^{-1})\end{aligned} \qquad (3.21)$$

and

$$\begin{aligned}(v_1v_2)^n\sum_{l=-[n/2]-1}^{[n/2]+1}p_m^{nl}c_{nl}P_{nl}([2\sqrt{2}v_1^2]^{-1})P_{nl}(-[2\sqrt{2}v_2^2]^{-1})\\ =[\pi^{1/2}2^{n/2-m-1/4}m!\,(n-2m)!]^{-1}H_m(2^{-1/2}[v_1^2-v_2^2])H_{n-2m}(v_1v_2).\end{aligned} \qquad (3.22)$$

These formulas allowed us to calculate c_{nl} in (3.19) by evaluating (3.21) for even n at $\mathbf{x}=0$ and for odd n,

TABLE I. Eigenvalues and eigenvectors for parabolic polynomials. The eigenvectors $p_m^{n,\pm l}$ for fixed n and l are listed from top to bottom as m runs from 0 to $[n/2+1]$, respectively.

Level		Eigenvalue $\lambda_{\pm l}$	Eigenvectors $p_m^{n,\pm l}$
$n=2$	$l=1/2$	± 1.414214	5.000000×10^{-1} ± 1.000000
$n=3$	$l=1/2$	± 2.449490	2.886751×10^{-1} ± 1.000000
$n=4$	$l=1$	± 4.000000	1.250000×10^{-1} $\pm 7.071068\times10^{-1}$ 5.000000×10^{-1}
	$l=0$	0.000000	1.020621×10^{-1} 0.000000×10^{-4} -1.224745
$n=5$	$l=1$	± 5.656854	5.103104×10^{-2} $\pm 4.082483\times10^{-1}$ 6.123724×10^{-1}
	$l=0$	0.000000	5.590170×10^{-1} 0.000000×10^{-4} -1.118034
$n=6$	$l=3/2$	± 7.538754	1.863390×10^{-2} $\pm 1.986636\times10^{-1}$ 5.000000×10^{-1} $\pm 1.875925\times10^{-1}$
	$l=1/2$	± 1.779658	1.863390×10^{-2} $\pm 4.689812\times10^{-2}$ -5.000000×10^{-1} $\pm 7.946545\times10^{-1}$
$n=7$	$l=3/2$	± 9.579208	6.332529×10^{-3} $\pm 8.578707\times10^{-2}$ 3.151144×10^{-1} $\pm 2.791290\times10^{-1}$
	$l=1/2$	± 2.870326	7.688005×10^{-3} $\pm 3.120757\times10^{-2}$ -2.595565×10^{-1} $\pm 7.673029\times10^{-1}$
$n=8$	$l=2$	± 11.78082	2.008449×10^{-3} $\pm 3.346196\times10^{-2}$ 1.662750×10^{-1} $\pm 2.541743\times10^{-1}$ 6.102406×10^{-2}
	$l=1$	± 4.605675	2.518940×10^{-3} $\pm 1.640688\times10^{-2}$ -8.762825×10^{-2} $\pm 5.183905\times10^{-1}$ -3.183528×10^{-1}
	$l=0$	0.000000	2.010905×10^{-3} 0.000000×10^{-4} -1.126107×10^{-1} 0.000000×10^{-4} 6.756639×10^{-1}
$n=9$	$l=2$	± 14.12795	6.007569×10^{-4} $\pm 1.200309\times10^{-2}$ 7.665598×10^{-2} $\pm 1.744409\times10^{-1}$ -3.791902×10^{-1}
	$l=1$	± 6.356178	8.200552×10^{-4} $\pm 7.371471\times10^{-3}$ -2.591292×10^{-2} $\pm 2.840449\times10^{-1}$ -3.791902×10^{-1}
	$l=0$	0.000000	8.300199×10^{-4} 0.000000×10^{-3} -5.976143×10^{-2} 0.000000×10^{-4} 5.976143×10^{-1}
$n=10$	$l=5/2$	± 16.61347	1.704169×10^{-4} $\pm 4.003944\times10^{-3}$ 3.169879×10^{-2} $\pm 9.877367\times10^{-2}$ 1.046898×10^{-1} $\pm 1.782333\times10^{-2}$
	$l=3/2$	± 8.353982	2.447240×10^{-4} $\pm 2.891216\times10^{-3}$ -4.946480×10^{-3} $\pm 1.274183\times10^{-1}$ -3.021387×10^{-1} $\pm 1.022969\times10^{-1}$
	$l=1/2$	± 2.050659	2.210312×10^{-4} $\pm 6.410059\times10^{-4}$ -1.896333×10^{-2} $\pm 4.226254\times10^{-2}$ 2.538089×10^{-1} $\pm 3.500729\times10^{-1}$
$n=11$	$l=5/2$	± 19.22960	4.608940×10^{-5} $\pm 1.253390\times10^{-3}$ 1.197299×10^{-2} $\pm 4.837152\times10^{-2}$ 7.743018×10^{-2} $\pm 3.416695\times10^{-2}$
	$l=3/2$	± 10.48809	6.979429×10^{-5} $\pm 1.035217\times10^{-3}$ -0.000000×10^{-5} $\pm 4.969040\times10^{-2}$ -1.842569×10^{-1} $\pm 1.490712\times10^{-1}$
	$l=1/2$	± 3.197242	7.436786×10^{-5} $\pm 3.362605\times10^{-4}$ -7.420250×10^{-3} $\pm 2.732426\times10^{-2}$ 1.249380×10^{-1} $\pm 3.315776\times10^{-1}$
$n=12$	$l=3$	± 21.97026	1.193308×10^{-5} $\pm 3.707685\times10^{-4}$ 4.184841×10^{-3} $\pm 2.109578\times10^{-2}$ 4.668927×10^{-2} $\pm 3.698379\times10^{-2}$ 4.761253×10^{-2}
	$l=2$	± 12.78129	1.890156×10^{-5} $\pm 3.416546\times10^{-4}$ 5.927794×10^{-4} $\pm 1.692769\times10^{-2}$ -9.309182×10^{-2} $\pm 1.334035\times10^{-1}$ -2.952144×10^{-2}
	$l=1$	± 5.093752	2.029600×10^{-5} $\pm 1.462053\times10^{-4}$ -2.152465×10^{-3} $\pm 1.394086\times10^{-2}$ 3.416276×10^{-2} $\pm 2.179504\times10^{-1}$ 1.210221×10^{-1}
	$l=0$	0.000000	1.626302×10^{-5} 0.000000×10^{-5} -2.146719×10^{-3} 0.000000×10^{-5} -6.010814×10^{-2} 0.000000×10^{-5} -2.404325×10^{-1}

(continued)

TABLE I. (*continued*)

Level		Eigenvalue	Eigenvectors $p_m^{n,\pm l}$
$n=13$	$l=3$	± 24.82997	2.968235×10^{-6}
			$\pm 1.042292\times10^{-4}$
			1.366953×10^{-3}
			$\pm 8.356661\times10^{-3}$
			2.415046×10^{-2}
			$\pm 2.921621\times10^{-2}$
			9.984215×10^{-3}
	$l=2$	± 15.21229	4.902887×10^{-6}
			$\pm 1.054779\times10^{-4}$
			3.697447×10^{-4}
			$\pm 5.083554\times10^{-3}$
			-4.065197×10^{-2}
			$\pm 8.950889\times10^{-2}$
			-4.992728×10^{-2}
	$l=1$	± 6.932454	5.644911×10^{-6}
			$\pm 5.534254\times10^{-5}$
			-6.093177×10^{-4}
			$\pm 6.049697\times10^{-3}$
			7.107675×10^{-3}
			$\pm 1.155716\times10^{-1}$
			1.414589×10^{-1}
	$l=0$	0.000000	5.582380×10^{-6}
			0.000000×10^{-6}
			-8.708512×10^{-4}
			0.000000×10^{-5}
			3.135064×10^{-2}
			0.000000×10^{-4}
			-2.090043×10^{-1}
$n=14$	$l=7/2$	± 27.80398	7.114271×10^{-7}
			$\pm 2.797386\times10^{-5}$
			4.204971×10^{-4}
			$\pm 3.049722\times10^{-3}$
			1.105699×10^{-2}
			$\pm 1.864007\times10^{-2}$
			1.158708×10^{-2}
			$\pm 1.178723\times10^{-3}$
	$l=5/2$	± 17.77729	1.220946×10^{-6}
			$\pm 3.069567\times10^{-5}$
			1.636458×10^{-4}
			$\pm 1.329819\times10^{-3}$
			-1.572227×10^{-2}
			$\pm 4.926638\times10^{-2}$
			-4.921069×10^{-2}
			$\pm 7.829590\times10^{-3}$
	$l=3/2$	± 9.040977	1.455154×10^{-6}
			$\pm 1.860541\times10^{-5}$
			-1.458948×10^{-4}
			$\pm 2.259073\times10^{-3}$
			-6.557908×10^{-4}
			$\pm 4.892627\times10^{-2}$
			1.108188×10^{-1}
			$\pm 3.466912\times10^{-2}$
	$l=1/2$	± 2.273209	1.273209×10^{-6}
			$\pm 4.093117\times10^{-6}$
			-2.251448×10^{-4}
			$\pm 6.014597\times10^{-4}$
			9.648122×10^{-3}
			$\pm 1.967606\times10^{-2}$
			-8.593878×10^{-2}
			$\pm 1.069288\times10^{-4}$
$n=15$	$l=7/2$	± 30.88805	1.647271×10^{-7}
			$\pm 7.195660\times10^{-6}$
			1.225688×10^{-4}
			$\pm 1.036347\times10^{-3}$
			4.576218×10^{-3}
			$\pm 1.013316\times10^{-2}$
			9.706253×10^{-3}
			$\pm 2.666413\times10^{-3}$
	$l=5/2$	± 20.46843	2.929299×10^{-7}
			$\pm 8.479361\times10^{-6}$
			6.120959×10^{-5}
			$\pm 2.912478\times10^{-4}$
			-5.474195×10^{-3}
			$\pm 2.330407\times10^{-2}$
			-3.579068×10^{-2}
			$\pm 1.483719\times10^{-2}$
	$l=3/2$	± 11.26606	3.638824×10^{-7}
			$\pm 5.797598\times10^{-6}$
			-3.022984×10^{-5}
			$\pm 7.634970\times10^{-4}$
			-1.378486×10^{-3}
			$\pm 1.759614\times10^{-2}$
			6.602422×10^{-2}
			$\pm 4.972759\times10^{-2}$
	$l=1/2$	± 3.470994	3.701417×10^{-7}
			$\pm 1.816924\times10^{-6}$
			-7.327036×10^{-5}
			$\pm 3.088482\times10^{-4}$
			3.650857×10^{-3}
			$\pm 1.247904\times10^{-2}$
			-4.090263×10^{-2}
			$\pm 9.999164\times10^{-2}$

$\partial^2/\partial v_1 \partial v_2$ of both sides at $\mathbf{x}=0$. We add that from (3.21) and (3.22) many $\mathcal{L}^2(\mathbb{R}^2)$ expansions can be derived for the parabolic polynomials P_{nl}. To conclude this section we give explicitly the parabolic polynomials for the first few n values:

$$n=0:\quad P_{00}(u)=1, \tag{3.23a}$$

$$n=1:\quad P_{10}(u)=1, \tag{3.23b}$$

$$n=2:\quad P_{2,\pm1/2}(u)=\pm u+\tfrac{1}{2}, \tag{3.23c}$$

$$n=3:\quad P_{3,\pm1/2}(u)=\pm u+1/2\sqrt{3}, \tag{3.23d}$$

$$n=4:\quad P_{4,\pm1}(u)=\frac{1}{\sqrt{106}}\,(10u^2\pm4\sqrt{2}u+1),$$

$$P_{4,0}=\frac{\sqrt{6}}{2}\,(-u^2+\tfrac{1}{12}), \tag{3.23e}$$

$$n=5:\quad P_{5,\pm1}(u)=\frac{1}{\sqrt{394}}\,(12u^2\pm8u+1),$$

$$P_{5,0}=\frac{\sqrt{5}}{2}\,(-u^2+\tfrac{1}{20}). \tag{3.23f}$$

When written in terms of the variables z, u as (3.10a) we can perform the z integral and obtain orthogonality for n as well as a weight function in u which depends on n. This weight function appears in terms of parabolic cylinder functions.

4. A SYMMETRY GROUP IN BARGMANN SPACE

Here we will show how the information of the previous section can be obtained by studying the group of geometrical symmetry transformations in Bargmann space. We look for all *first order* differential operators of the form

$$A=\sum_{i=1}^{2} a_i(\eta_1,\eta_2)\partial_{\eta_i}+b(\eta_1,\eta_2) \tag{4.1}$$

which commute with the Hamiltonian (3.5), i.e., $[A,\tilde{H}]=0$. If we further demand that all our symmetry operators A be densely defined in $\mathcal{J}$ then the functions a_i and b must be analytic functions of η_1 and η_2, we find

$$A_1=\eta_1\partial_{\eta_1},\quad A_2=\eta_2\partial_{\eta_2},\quad A_3=\eta_2^2\partial_{\eta_1},\quad A_0=\mathbf{1}, \tag{4.2}$$

with domains $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$ given by (3.3) and $\mathcal{D}(A_3)=\mathcal{D}(\eta_1\times\eta_2^2)$. When we disregard the central element A_0 we find the three-dimensional solvable algebra[12] s_3 with Lie brackets

$$[A_1,A_2]=0,\quad [A_1,A_3]=-A_3,\quad [A_2,A_3]=2A_3. \tag{4.3}$$

It is easily seen that

$$\tilde{H}=4A_1+2A_2+3A_0, \tag{4.4}$$

and so we have the structure $s_3\approx u(1)\oplus s_2$, where s_2 is the two-dimensional algebra spanned by, say, A_1 and A_3, and $u(1)$ is spanned by $\tilde{H}$. Now on the space $\mathcal{P}_n$ of homogeneous polynomials the representation (4.2) of s_3 acts on the normalized Cartesian basis in Bargmann space, calling $\tilde{\phi}^n_m\equiv\tilde{\psi}^c_{m,n-2m}$, as

$$A_1\tilde{\phi}^n_m=m\tilde{\phi}^n_m, \tag{4.5a}$$

$$A_2\tilde{\phi}^n_m=(n-2m)\tilde{\phi}^n_m, \tag{4.5b}$$

$$A_3\tilde{\phi}^n_m=[2m(n+2-2m)(n+1-2m)]^{1/2}\tilde{\phi}^n_{m+1}. \tag{4.5c}$$

This action can be integrated to a representation of the solvable Lie group S_3 as

$$\begin{aligned}T(g(\alpha\beta\gamma))f(\eta_1,\eta_2)&\equiv\exp(\alpha A_1+\beta A_2+\gamma A_3)f(\eta_1,\eta_2)\\&=\exp(\alpha A_1)\exp(\beta A_2)\exp(\delta A_3)f(\eta_1,\eta_2)\\&=f(e^{\alpha}\eta_1+e^{2\beta}\delta\eta_2^2,e^{\beta}\eta_2)\\&=e^{n\beta}\eta_2^nP_n(e^{\alpha-2\beta}\eta_1/\eta_2^2+\delta)\end{aligned} \tag{4.6a}$$

where $f\in\mathcal{P}_n$, $\alpha,\beta,\gamma\in\mathbb{C}$, and

$$\delta=\gamma(e^{\alpha-2\beta}-1)/(\alpha-2\beta). \tag{4.6b}$$

The transformations (4.6) form the group of geometrical symmetry transformations in Bargmann space. The group composition law is $g(\alpha_1,\beta_1,\gamma_1)g(\alpha_2,\beta_2,\gamma_2)=g(\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma_3)$ where γ_3 is related to δ_3 through (4.6b) and $\delta_3=\delta_2+e^{\alpha_2-2\beta_2}\delta_1$. This yields the representation matrices

$$\begin{aligned}D^n_{m'm}(\alpha\beta\gamma)&\equiv(\tilde{\phi}^n_{m'},T(g(\alpha\beta\gamma))\tilde{\phi}^n_m)\\&=e^{m'\alpha}e^{(n-2m)\beta}(\sqrt{2}\delta)^{m-m'}\frac{1}{(m-m')!}\left(\frac{m!\,(n-2m')!}{m'!\,(n-2m)!}\right)^{1/2},\end{aligned} \tag{4.7}$$

where $0\le m',\ m\le[n/2]$ and the matrix is upper-triangular, δ being given by (4.6b). Now since A_1 and A_2 are self-adjoint on $\mathcal{D}$ given by (3.3), by choosing α,β pure imaginary, the representation of the Abelian subgroup generated by them defines a unitary representation on $\mathcal{J}$. Of course A_3 is not Hermitean (symmetric) in $\mathcal{J}$, so its integrated group representation is not unitary. Moreover, $\exp(\delta A_3)$ is an unbounded operator in $\mathcal{J}$, since functions of growth $(2,1)$ in η_1 and $(2,\frac{1}{2})$ in η_2 are mapped onto functions of growth $(2,1)$ in η_1 and $(4,\delta)$ in η_2. However, it can be seen easily from (4.6a) that $T(g)$ maps $\mathcal{P}_n$ into itself and thus is densely defined on $\mathcal{J}$. Using the binomial theorem and (4.6a), it follows that all polynomials $\mathcal{P}_j$ of degree $j\le[n/2]$ form an invariant subspace under s_3. The complement $\bar{\mathcal{P}}_j$ of $\mathcal{P}_j$ in $\mathcal{P}_n$ is *not* invariant under (4.6a). Thus the representation (4.6a) of s_3 on $\mathcal{P}_n$ is indecomposable and nonunitary with dimension $[n/2]+1$. This is consistent with a a theorem of Lie which states[12] that all finite-dimensional representations of a solvable Lie group over $\mathbb{C}$ are indecomposable. From the point of view of the Lie algebra (4.3) this means that we have only a lowering operator given by A_3. It can be seen how another representation of the same algebra s_3 contains a raising operator: Indeed, consider the operators defined by

$$A_1'\equiv-A_1,\quad A_2'\equiv-A_2,\quad A_3'\equiv A_3^\dagger=2\eta_1\partial_{\eta_2\eta_2}, \tag{4.8}$$

where $\mathcal{D}(A_3^\dagger)=\mathcal{D}(A_3)$. The primed operators (4.8) form a representation of s_3 conjugate to that of (4.2). In fact, we easily find

$$A_3'\tilde{\phi}^n_m=[2(m+1)(n-2m)(n-2m-1)]^{1/2}\tilde{\phi}^n_{m+1}. \tag{4.9}$$

Since A_3' is a second-order operator, its exponentiation will be represented through an integral kernel in Bargmann space. There is a striking analogy between this exponentiation and the development in time of the solutions of the heat equation.[13] Using this analogy, the general element of the conjugate representation of S_3 can thus be found as

$$\begin{aligned}T(g'(\alpha\beta\gamma))f(\eta_1,\eta_2)&\equiv\exp(\alpha A_1'+\beta A_2'+\gamma A_3')f(\eta_1,\eta_2)\\&=\exp(\delta'A_3')\exp(-\beta A_2)\exp(\alpha A_1)f(\eta_1,\eta_2)\\&=\int\int_{\mathbb{C}}d^2\mu_1(\eta_1')\,d^2\mu_2(\eta_2')\\&\quad\times K_{g'(\alpha\beta\gamma)}(\eta_1,\eta_2;\eta_1',\eta_2')f(\eta_1',\eta_2'),\end{aligned} \tag{4.10a}$$

where $\delta'=\gamma(e^{2\beta-\alpha}-1)/(2\beta-\alpha)$ and the integral kernel is

$$K_{g'(\alpha\beta\gamma)}=\exp(2e^{-\alpha}\eta_1\eta_1'^*+e^{-\beta}\eta_2\eta_2'^*+2\delta'e^{-2\beta}\eta_1\eta_2'^{*2}). \tag{4.10b}$$

Finally, it is straightforward to see that the matrix elements of the representation (4.10) are the adjoints of the matrix elements (4.7) of the group S_3. However, it must be noted that if we try to embed the two representations (4.2) and (4.8) of the algebra s_3 into a higher-dimensional Lie algebra, we are led to an algebra of infinite dimension.

Now from the relations (4.5c) and (4.9) one can derive the recursion relation (3.12). Forming the inner product of $(\tilde{S}_p-\lambda)$ from (4.9b) between the Cartesian and parabolic bases, this yields

$$\begin{aligned}\lambda(\tilde{\psi}^P_{nl},\tilde{\phi}^n_m)&=[m(n+2-2m)(n+1-2m)]^{1/2}(\tilde{\psi}^P_{nl},\tilde{\phi}^n_m)\\&\quad+[(m+1)(n-2m)(n-2m-1)]^{1/2}(\tilde{\psi}^P_{nl},\tilde{\phi}^n_{m+1}).\end{aligned} \tag{4.11a}$$

Then upon defining

$$(\tilde{\psi}^P_{nl},\tilde{\phi}^n_m)=[(n-2m)!\,(n-2m+1)m!\,]^{1/2}2^{-m/2}p^{nl}_m, \tag{4.11b}$$

we regain (3.12). We thus could have made these calculations using the harmonic oscillator raising and lowering operator formalism in ordinary configuration space; however, the analysis of the differential equations that was made previously requires specific Lie algebra models.

Another important consequence of the symmetry algebra is the correspondence between the separable coordinate systems (2.5) and the orbits of the factor algebra $s_2 \approx s_3/u(1)$ under the adjoint action of S_3. An easy calculation shows that we have essentially two orbits:

$$\text{(i) } \alpha A_1 \text{ and (ii) } A_3. \tag{4.12}$$

As discussed previously, A_1 is a self-adjoint operator on $\mathcal{J}$; in fact, from (3.8a) we have $A_1 = \frac{1}{4}\tilde{S}_C - \frac{1}{2}$. Thus orbit (i) describes the Cartesian basis. In constrast, the operator A_3 is not Hermitian in $\mathcal{J}$. However, by considering the Hermitian part of A_3, i.e., $\frac{1}{2}(A_3 + A_3^*)$, we find the self-adjoint operator $\tilde{S}_p = 2^{-1/2}(A_3 + A_3^*)$. Thus orbit (ii) describes the parabolic basis and we have found the correspondence between the two orbits (4.12) of the symmetry algebra and the two separable coordinate systems (2.5). It can also be remarked that the preceeding description of the symmetry algebra also carries over to the case of any anisotropic two-dimensional oscillator whose frequency ratio is $k:1$ (k integer). What does *not* carry over is the connection with separation of variables, and the reason clearly is that for any other ratio of frequencies the operator $A_3^\dagger$ is higher than second order, giving rise to a higher than second order operator for not only the analog of $\tilde{S}_P$ but also of S_P.

5. CANONICAL TRANSFORMATIONS INDUCED BY THE SYMMETRY GROUP ACTION

In this section we want to show explicitly that the S_3 group action induces a canonical transformation in the Bargmann phase space and point out some of its characteristics. Consider the action of $\exp(\alpha A_1)$ and $\exp(\beta A_2)$; these produce dilatations of the canonical operators $\hat{\eta}_i, \hat{\zeta}_i$, i.e., $\hat{\eta}_1 \to e^{\alpha}\hat{\eta}_1$, $\hat{\zeta}_1 \to e^{-\alpha}\hat{\zeta}_1$ under the first one and $\hat{\eta}_2 \to e^{\beta}\hat{\eta}_2$, $\hat{\zeta}_2 \to e^{-\beta}\hat{\eta}_2$ under the second one. It is clear that they preserve the canonical commutation relations $[\hat{\eta}_j, \hat{\zeta}_k] = i\delta_{jk}\mathbf{1}$ and the form of the Hamiltonian (3.5). The adjoint group action of $\exp(\gamma A_3)$ gives

$$\hat{\eta}_1 \to \hat{\eta}_1' = \exp(\gamma A_3)\hat{\eta}_1 \exp(-\gamma A_3) = \hat{\eta}_1 + \gamma\hat{\eta}_2^2, \tag{5.1a}$$

$$\hat{\eta}_2 \to \hat{\eta}_2' = \hat{\eta}_2, \quad \hat{\zeta}_1 \to \hat{\zeta}_1' = \hat{\zeta}_1, \tag{5.1b}$$

$$\hat{\zeta}_2 \to \hat{\zeta}_2' = \hat{\zeta}_2 - 2\gamma\hat{\eta}_2\hat{\zeta}_1, \tag{5.1c}$$

which can be similarly checked to preserve the canonical commutation relations and the form of the Hamiltonian. Thus S_3 can be said to induce a *canonical transformation*[11,14,15] in the η-space which, moreover, is a nonlinear *point* transformation, as $\hat{\eta}_i'$ is only a function of the $\hat{\eta}_i$ and the group element. The translation to ordinary description of phase space can be made through (3.1) and seen to mix the configuration and momentum components. The transformation (5.1) is not in general a unitary transformation since as was seen in (4.6), $\exp(\gamma A_3)$ is not unitary.

The action of the conjugate representation of S_3 can be obtained through adjunction from (5.1) and similar considerations apply as a canonical transformation. It is not a point transformation, however.

By looking at the transformations (5.1) it is seen that two new operators appear, namely

$$A_4 = \eta_2^2, \quad A_5 = \eta_2\partial_{\eta_1}. \tag{5.2}$$

The generators of s_3 together with (5.2) and the five-dimensional Heisenberg–Weyl algebra w_2 form a solvable dynamical noninvariance algebra of dimension ten, with the structure $s_{10} \approx s_5 \oplus w_2$ where s_5 is a five-dimensional solvable algebra with basis $A_1, \ldots, A_5$ and w_2 is an ideal in s_{10}. Similarly, one can construct the conjugate representation from the Hilbert space adjoint operators. The algebra s_{10} can be integrated to the corresponding group on a dense invariant subspace of $\mathcal{J}$. This group is a Lie subgroup of the pseudogroup of all canonical transformations.[14]

6. DISCUSSION ON SYMMETRY GROUPS AND ACCIDENTAL DEGENERACY

The degeneracy pattern for the anisotropic oscillator has usually been attributed[5–7,15] to the group $SU(2)$. We feel, however, that the role of this $SU(2)$ is still not well understood since in contradistinction to the isotropic oscillator case, the formal Lie algebra $su(2)$ for the anisotropic oscillator cannot be written in terms of finite-order differential operators in Hilbert space. The generators of $su(2)$ are written in terms of shift operators which are well defined over the finite-dimensional subspaces; however, their extension to a dense subspace of Hilbert space seems to have been overlooked. Moreover, in order to obtain a unitary irreducible representation[7,15] of the group $SU(2)$ on one of the finite-dimensional subspaces, a new norm must be introduced. This is the meaning of the factors containing the number operator and modulo numbers: One has to rescale the basis functions so that they form a properly normalized $SU(2)$ basis, for they do not do so in the ordinary norm. As a consequence, the representations are nonunitary in the usual Hilbert space norm.

Second, when we follow the procedure of Refs. 5 and 7 for n-dimensional anisotropic oscillators ($n > 2$), the group $SU(n)$ does not in general give a full account of the degeneracy of the system, that is, representations are in general reducible, in fact, completely reducible. This occurs already in the $n = 3$ case and constitutes the major failure for $SU(n)$ as the symmetry group explaining the accidental degeneracy.

Thirdly, the choice of the group $SU(2)$ [$U(2)$ including the action of the Hamiltonian] is not unique. In Ref. 7 this choice was dictated in order to find the quantum counterpart of a classical canonical transformation which maps the general anisotropic oscillator onto the isotropic oscillator whose geometrical symmetry group in Bargmann space *is* $U(2)$. It is of interest to study the former system on its own, since the two quantum problems are not unitarily equivalent.

The generators of the solvable group S_3 on the other hand, are all the first order symmetry operators in Bargmann's description of phase space. They are thus the generators of all the *geometrical* symmetry transformations in Bargmann space, and in this sense they are unique. While the representations are reducible, they cannot be decomposed into irreducible parts, i.e., they are indecomposable. We can find no fundamental reason why, when explaining accidental degeneracy through a symmetry group, one should exclude non-unitary indecomposable representations. Clearly, completely reducible representations should be excluded. It is thus of interest to consider the n-dimensional generalization of the geometrical symmetry group discussed above.

To sum up, the connection between accidental degeneracy and symmetry groups seems to be still an open question. In this context one should understand the role played by the infinite-dimensional Lie algebras of symmetry transformations and its corresponding Lie pseudogroup. Perhaps more immediate is the possibility of finding, for all systems with discrete spectra exhibiting accidental degeneracy, a Hilbert space *á la* Bargmann such that its group of geometrical symmetry transformations explains the accidental degeneracy. Work in this direction is currently in progress.

ACKNOWLEDGMENTS

The results of Table I were obtained in collaboration with Dr. Victor Guerra (CIMAS). We would like to thank Professor Marcos Moshinsky for his interest in our work.

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