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Elliptic basis for the Zernike system: Heun function solutions

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The differential equation that defines the Zernike system, originally proposed to classify wavefront aberrations of the wavefields in the disk of a circular pupil, had been shown to separate in three distinct coordinate systems obtained from polar coordinates on a half-sphere. Here we find and examine the separation in the generic elliptic coordinate system on the half-sphere and its projected disk, where the solutions, separated in Jacobi coordinates, contain Heun polynomials. *Published by AIP Publishing.*
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I. INTRODUCTION

The system that stems from the differential equation set forth by Zernike in 1934,[1] which led to the design of phase-contrast microscopy, was examined in Refs. 2–4 for its symmetry properties and the separability of its solutions. On the two-dimensional plane $\mathbf{r} = (x, y)$, this equation is

$$\widehat{Z}\Psi(\mathbf{r}) := \left(\nabla^2 - (\mathbf{r}\cdot\nabla)^2 - 2\,\mathbf{r}\cdot\nabla\right)\Psi(\mathbf{r}) = -E\,\Psi(\mathbf{r}), \tag{1}$$

restricted to the disk $\mathcal{D} := \{|\mathbf{r}| \le 1\}$ and to the space of square-integrable solutions $\Psi(\mathbf{r}) \in \mathcal{L}^2(\mathcal{D})$ with free but finite boundary conditions,

$$\Psi(\mathbf{r})\Big|_{|\mathbf{r}|^2=1} < \infty, \tag{2}$$

which are explicitly found in terms of separated polynomial solutions in the coordinates. These solutions are eigenfunctions of (1) with "energy" eigenvalues E that are quantized as

$$E_n = n(n+2), \qquad \text{for} \quad n \in \{0, 1, 2, \ldots\} =: \mathcal{Z}_0^+, \qquad \text{and} \quad (n+1)\text{-fold degenerate}, \tag{3}$$

as shown in detail in Ref. 3. Of course, these eigenvalues and degeneracies will be the same in any coordinate system. The spectrum (3) is also that of the two-dimensional harmonic oscillator, although the underlying dynamic is clearly different.[5]

In Refs. 2 and 3, we also found that both the classical and quantum Zernike systems possess the remarkable property of leading to a superintegrable cubic Higgs algebra.[6,7] In this article, we concentrate on finding further polynomial solutions separated in sets of non-orthogonal coordinates over the unit disk, which project from the standard elliptic coordinate systems of the sphere.

The key to find new coordinate systems where (1) separates is to project the disk vertically on a half-sphere

$$\mathcal{H}_+ := \{|\vec{\xi}\,| = 1,\ \xi_3 \ge 0\} \qquad \text{for } \vec{\xi} = (\xi_1, \xi_2, \xi_3), \qquad \xi_1 := x, \quad \xi_2 := y, \quad \xi_3 = \sqrt{1 - x^2 - y^2}, \tag{4}$$

whose orthogonal coordinate systems are known and where the constant value of one of the coordinates, $\xi_3 = 0$, matches the common boundary between the disk and the half-sphere. Polar coordinates on the two-sphere, with the pole on the $z := \xi_3 > 0$ axis, conform the solutions found by Zernike, involving Jacobi polynomials in the radius $|\mathbf{r}|$ and trigonometric angular functions, which were named solutions in coordinate System I. In Ref. 3, we set the pole of the coordinates on the $x = \xi_1 > 0$ and $y = \xi_2 > 0$ axes, whose solutions separated into Legendre and Gegenbauer polynomials, and were referred to as belonging to coordinate System II. The two sets of solutions were related in Ref. 4 through interbasis coefficients that turned out to be Hahn and Racah polynomials.

The generic class of orthogonal coordinate systems on the sphere is elliptic, with two pairs of antipodal foci. When a pair coincides, the coordinates become the polar coordinates of the previously studied Systems I and II on the half-sphere and on the disk. In Fig. 1, we show them together with the elliptic system where the foci are $\frac{1}{2}\pi$ apart; Systems I and II correspond to 0 and π. We may expect that the solutions separated in these coordinates will be of interest in the field of orthogonal function bases on two-dimensional compact manifolds.

In this paper, we examine separated polynomial solutions of the Zernike system projected from the generic orthogonal elliptic coordinate system on $\mathcal{H}_+$, and their projection on the unit disk $\mathcal{D}$, as non-orthogonal coordinates that will be also referred to as "elliptic." This coordinate system was used for the classical Zernike case in Ref. 2, but not further developed in the quantum/wave case;[3] this we do here.

In Sec. II, we write Zernike's differential equation (1) in Jacobi elliptic coordinates (ρ_1, ρ_2),[8] where it separates into a simultaneous pair of differential equations that are of the Heun type.[9] In Sec. III, we find that the solutions of (1) are in fact products of Heun polynomials in ρ_1 and in ρ_2, with the spectrum and degeneracies (3). Section IV brings us back to the forms of the solutions on the Zernike disk and Sec. V presents some conclusions and directions of further inquiry. Although in Sec. III we seem to search only for polynomials solutions, the problem of assuring that there are no other $\mathcal{L}^2(\mathcal{D})$ solutions is rather subtle, so we reserve Appendix A for a fuller discussion on the existence *only* of Heun-type polynomials related to the solution set. Appendix B contains a list

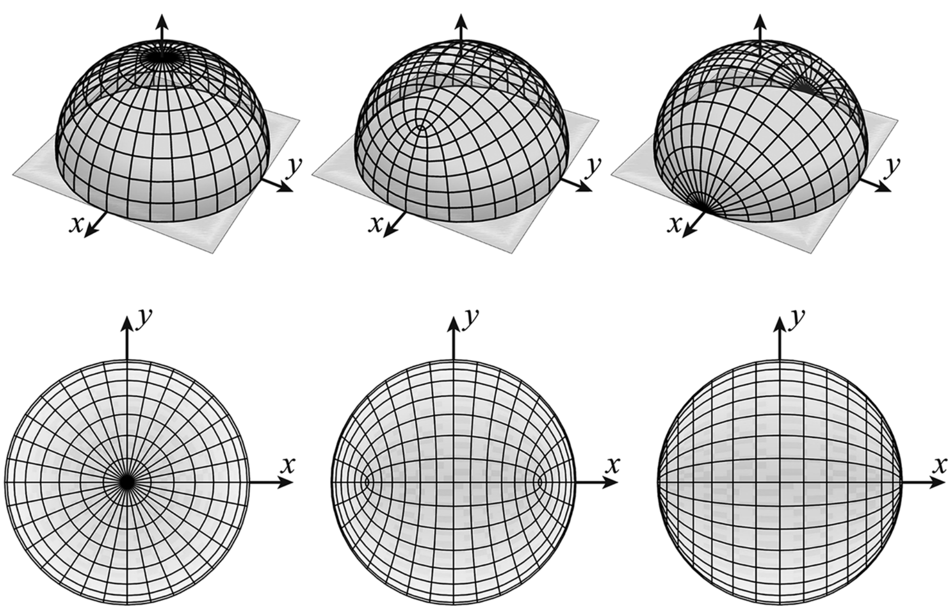

FIG. 1. *Left*: System I of polar coordinates on the upper half-sphere. *Middle*: Elliptic system of coordinates (for angles between foci $\frac{1}{2}\pi$ and eccentricity constant $k = 1/\sqrt{2}$). *Right*: System II of $\frac{1}{2}\pi$-rotated polar coordinates. The elliptic coordinate system interpolates between Systems I and II and provides the separating coordinates for solutions of the Zernike system on the disk.

of the lowest elliptic Zernike solutions that separate into Heun polynomials of coordinates on the disk.

II. SEPARATION IN JACOBI ELLIPTIC COORDINATES

Elliptic coordinate systems can be defined on the sphere where a pair of points and their antipodes are chosen as foci for ellipses whose points sum constant distances over the surface of the sphere: There are several parametrizations for generic coordinates on the sphere,[10] the trigonometric elliptic coordinates used in Ref. 2 and the class Jacobi coordinates[8] that we use here, which depend on two parameters that specify the orientation and eccentricity of the ellipses.

A. Coordinates on the half-sphere

Zernike's equation (1) presents the operator $\widehat{Z}$ as a Hamiltonian which is self-adjoint on the space $\mathcal{L}^2(\mathcal{D})$ of square-integrable functions on the disk,[23] whose surface element is dx dy. When projecting on $\mathcal{H}_+$ parameterized by $\vec{\xi}$, we replace this by the measure on the (half-) sphere,

$$d^2S = \frac{dx\,dy}{\sqrt{1-|\mathbf{r}|^2}} = \frac{d\xi_1\,d\xi_2}{\xi_3}. \tag{5}$$

The weight function,

$$w(\mathbf{r}) := 1\Big/\sqrt{(1-|\mathbf{r}|^2)} = 1/\xi_3, \tag{6}$$

is then used for a similarity transformation on operators and wave functions, leading us to re-write Zernike's equation in the form

$$\widehat{W}\,\Upsilon(\vec{\xi}\,) = \left(\Delta_{\text{LB}} + \frac{\xi_1^2+\xi_2^2}{4\xi_3^2} + 1\right)\Upsilon(\vec{\xi}\,) = -E\,\Upsilon(\vec{\xi}\,), \tag{7}$$

$$\widehat{W} := w(\mathbf{r})^{-1/2}\widehat{Z}\,w(\mathbf{r})^{1/2}, \quad \Upsilon(\vec{\xi}\,) := w(\mathbf{r})^{-1/2}\,\Psi(\mathbf{r}), \tag{8}$$

where Δ_{LB} is the Laplace-Beltrami operator on the sphere,

$$\Delta_{\text{LB}} = \widehat{L}_1^2 + \widehat{L}_2^2 + \widehat{L}_3^2, \quad \widehat{L}_i = \xi_j\partial_{\xi_k} - \xi_k\partial_{\xi_j}, \quad i, j, k \text{ cyclic}, \tag{9}$$

and where the free boundary condition (2) translates to

$$\Upsilon(\vec{\xi}\,)/\sqrt{\xi_3}\Big|_{\xi_3=0} < \infty. \tag{10}$$

Written in the Schrödinger form $(-\frac{1}{2}\Delta_{\text{LB}} + V_{\text{W}})\Upsilon = \frac{1}{2}E\Upsilon$, Eq. (7) allows us to interpret the second summand as a potential,

$$V_{\text{W}}(\vec{\xi}\,) := -\frac{\xi_1^2+\xi_2^2}{8\xi_3^2} - \frac{1}{2} = -\frac{1}{8}\left(\frac{1}{\xi_3^2}+3\right) = -\frac{4-3|\mathbf{r}|^2}{8(1-|\mathbf{r}|^2)} = -\frac{1}{8(1-|\mathbf{r}|^2)} - \frac{3}{8}, \tag{11}$$

that describes a two-dimensional radial repulsive oscillator contained in the disk.

B. Jacobi elliptic coordinates

The Zernike operator (7) in the coordinates (ξ_1, ξ_2) does not separate, so we introduce the *Jacobi* coordinates (ρ_1, ρ_2), a class determined by two independent parameters contained in the multi-index $a \equiv \{a_1, a_2, a_3\}$ modulo a common scale, which is defined by

$$\xi_1^2 = \frac{(\rho_1-a_1)(\rho_2-a_1)}{(a_2-a_1)(a_3-a_1)}, \quad \xi_2^2 = \frac{(a_2-\rho_1)(\rho_2-a_2)}{(a_2-a_1)(a_3-a_2)}, \quad \xi_3^2 = \frac{(a_3-\rho_1)(a_3-\rho_2)}{(a_3-a_1)(a_3-a_2)}. \tag{12}$$

The range of the Jacobi coordinates (ρ_1, ρ_2) is determined by

$$a_1 \le \rho_1 \le a_2 \le \rho_2 \le a_3, \tag{13}$$

which is an $(a_2 - a_1) \times (a_3 - a_2)$ rectangle and where the boundary circle of $\mathcal{D}$ is the $\rho_2 = a_3$ edge, since there $\xi_3 = 0$.

The relation inverse to (12) requires solving simultaneous quadratic algebraic relations that can be shortened somewhat by defining $a_{i,j} := (a_i - a_j)$ to write

$$\rho_1(a;\xi_1^2,\xi_2^2) = \tfrac{1}{2}(S+T), \qquad \rho_2(a;\xi_1^2,\xi_2^2) = \tfrac{1}{2}(S-T), \tag{14}$$

$$S(a;\xi_1^2,\xi_2^2) := a_{3,1}\xi_1^2 + a_{3,2}\xi_2^2 + a_1 + a_2, \tag{15}$$

$$T(a;\xi_1^2,\xi_2^2) := \sqrt{(a_{3,1}\xi_1^2 + a_{3,2}\xi_2^2)^2 - 2a_{2,1}(a_{3,1}\xi_1^2 - a_{3,2}\xi_2^2) + a_{2,1}^2}, \tag{16}$$

noting that *four* distinct points on the sphere, $\{\pm\xi_1, \pm\xi_2\}$, correspond to a single point (ρ_1, ρ_2) on the Jacobi rectangular manifold. It follows that we shall have *four* distinct functions in the Jacobi manifold that will correspond to different quadrants in the half-sphere and disk manifolds, which will be characterized below by two parity indices.

Now, defining

$$r(a;\rho_i) := \sqrt{(\rho_i - a_1)(\rho_i - a_2)(\rho_i - a_3)}, \tag{17}$$

the Laplace-Beltrami operator (9) can be written as

$$\Delta^{\mathrm{J}}_{\mathrm{LB}}(a;\rho_1,\rho_2) = \frac{4}{\rho_2 - \rho_1}\Big(\widehat{D}^J(a;\rho_1) - \widehat{D}^J(a;\rho_2)\Big), \tag{18}$$

with the two operators given by

$$\widehat{D}^J(a;\rho_i) := r(a;\rho_i)\frac{\partial}{\partial\rho_i} r(a;\rho_i)\frac{\partial}{\partial\rho_i} = r(a;\rho_i)^2\left(\frac{\partial^2}{\partial\rho_i^2} + \frac{1}{r(a;\rho_i)}\frac{\partial r(a;\rho_i)}{\partial\rho_i}\frac{\partial}{\partial\rho_i}\right) \tag{19}$$

for ρ_1 and ρ_2. With this, (7) becomes the differential equation for elliptic Zernike solutions $\Upsilon(a;\rho_1,\rho_2) \equiv \Upsilon(a;\vec{\xi}(\rho_1,\rho_2))$ in Jacobi coordinates,

$$\begin{aligned}\widehat{W}\Upsilon(a;\rho_1,\rho_2) &= \left(\Delta^{\mathrm{J}}_{\mathrm{LB}} + \frac{1}{4}\frac{(a_3-a_1)(a_3-a_2)}{(a_3-\rho_1)(a_3-\rho_2)} + \frac{3}{4}\right)\Upsilon(a;\rho_1,\rho_2)\\ &= -E\,\Upsilon(a;\rho_1,\rho_2),\end{aligned} \tag{20}$$

where the second term represents a Cartesian Kepler potential that is singular at the rectangle boundaries $\rho_1 = a_3$ and at $\rho_2 = a_3$, the latter being the edge of the disk $\mathcal{D}$.

The differential equation (20) has a manifest symmetry in the plane (ρ_1, ρ_2) that leads us to propose solutions that separate the these two coordinates as

$$\Upsilon(a;\rho_1,\rho_2) = P_1(a;\rho_1)\,P_2(a;\rho_2). \tag{21}$$

Equation (20) then separates into two differential equations of identical form, bound by a separation constant Λ, that we write as

$$\left(\widehat{D}^J(a;\rho_1) + \frac{(a_1-a_3)(a_2-a_3)}{16(\rho_1 - a_3)} - (E + \tfrac{3}{4})\frac{\rho_1}{4}\right)P_1(a;\rho_1) = \tfrac{1}{4}\Lambda\,P_1(a;\rho_1), \tag{22}$$

$$\left(\widehat{D}^J(a;\rho_2) + \frac{(a_1-a_3)(a_2-a_3)}{16(\rho_2 - a_3)} - (E + \tfrac{3}{4})\frac{\rho_2}{4}\right)P_2(a;\rho_2) = \tfrac{1}{4}\Lambda\,P_2(a;\rho_2). \tag{23}$$

We can thus set out to solve a common differential equation for a function $P(a;\rho)$ (suppressing the subindex 1 or 2) that will serve to build the Zernike separated solutions in Jacobi coordinates.

III. HEUN POLYNOMIAL ZERNIKE SOLUTIONS

The Jacobi coordinates separate the Zernike equation (1) in the manifestly symmetric form (22) and (23); each depends both on the "energy" E, whose values, known from previous papers[1,3] are $E_n = n(n+2)$, and on a separation constant Λ. We should stress at this point that the Zernike system is *two* dimensional, where the separation constant binds the differential equations in the *two* coordinates; as a physical system, it has *three* singularities in (ρ_1, ρ_2) at a_1, a_2, and a_3. Below we shall solve a *one*-dimensional system that will stand for $\rho_1 \in (a_1, a_2)$ or for $\rho_2 \in (a_2, a_3)$, each one having only *two* singularities at their endpoints.[13] The solutions to the one-dimensional problem below can involve polynomial as well as non-polynomial functions, but from the outset, we reserve for Appendix A the proof that when the analysis involves the *two* equations [(22) and (23)] and the two quantum numbers E and Λ, only Heun-type *polynomial* solutions exist that conform to the three said singularities.

A. Separation of solutions

Using the differential operator $\widehat{D}^J(a;\rho)$ in (19) and the factor $r(a;\rho)$ in (17), we can bring both these equations to the common form

$$\frac{\mathrm{d}^2P(a;\rho)}{\mathrm{d}\rho^2}+\frac{1}{r(a;\rho)}\frac{\mathrm{d}r(a;\rho)}{\mathrm{d}\rho}\frac{\mathrm{d}P(a;\rho)}{\mathrm{d}\rho} + \frac{1}{4r(a;\rho)^2}\left(\frac{\frac14(a_1-a_3)(a_2-a_3)}{\rho-a_3}-(E+\tfrac34)\,\rho-\Lambda\right)P(a;\rho)=0. \tag{24}$$

We also note that

$$\frac{1}{r(a;\rho)}\frac{\partial r(a;\rho)}{\partial\rho}=\frac12\left(\frac{1}{\rho-a_1}+\frac{1}{\rho-a_2}+\frac{1}{\rho-a_3}\right), \tag{25}$$

so (24) becomes

$$\frac{\mathrm{d}^2P(a;\rho)}{\mathrm{d}\rho^2}+\frac12\left(\frac{1}{\rho-a_1}+\frac{1}{\rho-a_2}+\frac{1}{\rho-a_3}\right)\frac{\mathrm{d}P(a;\rho)}{\mathrm{d}\rho} +\frac{1}{4(\rho-a_1)(\rho-a_2)(\rho-a_3)}\left(\frac{\frac14(a_1-a_3)(a_2-a_3)}{\rho-a_3}-(E+\tfrac34)\,\rho-\Lambda\right)P(a;\rho)=0, \tag{26}$$

within the range $\rho\in(a_1,a_2)$ for $\rho=\rho_1$ in (22), or the range $\rho\in(a_2,a_3)$ for $\rho=\rho_2$ in (23), where the condition of square-integrability (10) is now supplemented by

$$P(a;\rho)\Big/(\rho-a_3)^{1/4}\Big|_{\rho=a_3}<\infty. \tag{27}$$

The differential equation (26) has two regular singular points at $\rho=a_1,a_2$, one at infinity and a regular inverse-quadratic singularity at $\rho=a_3$. To single out the singularities at a_1,a_2 and ensure the boundary condition (27) for polynomial solutions, we define the new function $Q(a;\rho)$ through

$$Q(a;\rho):=(\rho-a_1)^{-\frac12\alpha_1}(\rho-a_2)^{-\frac12\alpha_2}(\rho-a_3)^{-\frac14}\,P(a;\rho). \tag{28}$$

The exponents of the singularities at a_1, a_2 are the same (see Appendix A), and it is required that $\alpha_i(\alpha_i-1)=0$, i.e., they are *parity numbers* $\alpha_i\in\{0,1\}$, and $\alpha_i^2=\alpha_i$. This brings (26) to the form

$$\frac{\mathrm{d}^2Q(a;\rho)}{\mathrm{d}\rho^2}+\left(\frac{\alpha_1+\frac12}{\rho-a_1}+\frac{\alpha_2+\frac12}{\rho-a_2}+\frac{1}{\rho-a_3}\right)\frac{\mathrm{d}Q(a;\rho)}{\mathrm{d}\rho} -\frac{1}{4(\rho-a_1)(\rho-a_2)(\rho-a_3)}\Big(\big(E-(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+2)\big)\rho +(2\alpha_2+\tfrac14)a_1+(2\alpha_1+\tfrac14)a_2+\big((\alpha_1+\alpha_2)^2+\tfrac14\big)a_3+\Lambda\Big)Q(a;\rho)=0, \tag{29}$$

where we note the invariance under the exchange $(a_1,\alpha_1)\leftrightarrow(a_2,\alpha_2)$. The differential equation (26) is Fuchsian with four regular singularities at $\rho=a_1$, a_2, a_3, and infinity. Such equations are said to be of *Heun type*[11] (see Appendix A); their Erdélyi standard form[12] places these four singularities at $\tau=0,1,t,\infty$ and contains seven parameters.

B. The Frobenius series expansion

In the following, we shall find the particular Heun polynomials that satisfy the differential equation (29). We consider solutions expanded in a Frobenius power series around the middle singularity a_2 so that it serves for both equations as ρ_1-a_2 and ρ_2-a_2. This is

$$Q(a;\rho)=\sum_{m=0}^{\infty}C_m\left(\frac{\rho-a_2}{a_3-a_1}\right)^m, \tag{30}$$

whereupon we obtain from (29) the following three-term recursion relation for the series coefficients C_m:

$$U_m C_{m+1} + (V_m - \lambda) C_m + W_m C_{m-1} = 0, \qquad C_{-1} = 0, \quad C_0 = 1. \tag{31}$$

To obtain *polynomial* solutions of degree N in the Frobenius series (30), we must have $W_{N+1} = 0$. This implies that the "energy" E in the Zernike system is quantized as previously announced, depending on the *principal* quantum number $n \in \{0, 1, 2, \ldots\}$ and on the parities (α_1, α_2),

$$E_n = n(n+2), \quad n := 2N + \alpha_1 + \alpha_2 = \begin{cases} 2N & \text{for } (0,0), \\ 2N+1 & \text{for } (0,1) \text{ and } (1,0), \\ 2N+2 & \text{for } (1,1). \end{cases} \tag{32}$$

Replacing the recursion relation (31) in (29) yields the coefficients U_m, V_m, and W_m that depend on N, the parities (α_1, α_2), and the *eccentricity* parameter k of the elliptic coordinates

$$k^2 := (a_2 - a_1) \big/ (a_3 - a_1), \tag{33}$$

which on $\mathcal{H}_+$ relates to the angle ϕ between their two foci as $k = \sin \frac{1}{2}\phi$. The replacement yields their expressions

$$\begin{aligned} U^{(\alpha_2)}_{N,m} &= k^2 (1-k^2)(m+1)(m+\alpha_2+\tfrac{1}{2}), \\ V^{(\alpha_1,\alpha_2)}_{N,m} &= \tfrac{1}{4}(1-k^2)(\alpha_1+\alpha_2+2m)^2 - \tfrac{1}{4}k^2(\alpha_2+2m+\tfrac{1}{2})^2, \\ W^{(\alpha_1,\alpha_2)}_{N,m} &= \tfrac{1}{4}E_n - \tfrac{1}{4}(\alpha_1+\alpha_2+2m)(\alpha_1+\alpha_2+2m-2). \end{aligned} \tag{34}$$

The coefficients $\{C_m\}_{m=0}^N$ in (31) can then obtained from a tridiagonal matrix eigen-equation $\mathbf{M}\,\mathbf{C}_\mu = \lambda_\mu \mathbf{C}_\mu$ as $N+1$ eigenvectors of dimension $N+1$ with eigenvalues λ_μ numbered by $\mu \in \{0, 1, \ldots, N\}$,

$$\begin{pmatrix} V^{(\alpha_1,\alpha_2)}_{N;0} & U^{(\alpha_2)}_{N;0} & 0 & \cdots & 0 \\ W^{(\alpha_1,\alpha_2)}_{N;1} & V^{(\alpha_1,\alpha_2)}_{N;1} & U^{(\alpha_2)}_{N;1} & \cdots & 0 \\ 0 & W^{(\alpha_1,\alpha_2)}_{N;2} & V^{(\alpha_1,\alpha_2)}_{N;2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & V^{(\alpha_1,\alpha_2)}_{N;N-1} & U^{(\alpha_2)}_{N;N-1} \\ 0 & 0 & \cdots & W^{(\alpha_1,\alpha_2)}_{N;N} & V^{(\alpha_1,\alpha_2)}_{N;N} \end{pmatrix} \begin{pmatrix} C^{N;\alpha_1,\alpha_2}_{\mu,0} \\ C^{N;\alpha_1,\alpha_2}_{\mu,1} \\ \vdots \\ C^{N;\alpha_1,\alpha_2}_{\mu,N-1} \\ C^{N;\alpha_1,\alpha_2}_{\mu,N} \end{pmatrix} = \lambda_\mu \mathbf{C}^{N;\alpha_1,\alpha_2}_\mu. \tag{35}$$

It has been proven that the $N+1$ eigenvalues λ_μ of these tridiagonal matrices are real and distinct.[14] With the coefficients $C^{N;\alpha_1,\alpha_2}_{\mu,m}$ obtained from (35), we determine the polynomials $Q^{N;\alpha_1,\alpha_2}_\mu(a;\rho)$ in (30), now endowed with the indices N, μ and the parities α_1, α_2, as

$$Q^{(\alpha_1,\alpha_2)}_{N,\mu}(a;\rho) := \sum_{m=0}^{N} C^{N;\alpha_1,\alpha_2}_{\mu,m} \Big(\frac{\rho - a_2}{a_3 - a_1}\Big)^m, \tag{36}$$

up to a common factor, which we set choosing $C^{N;\alpha_1,\alpha_2}_{\mu,m=0} = 1$.

C. Four polynomial forms

The solutions $P(a;\rho)$ of the original differential equation (26), expressed through the polynomials $Q(a;\rho)$ of degree N in (28) with the series expansion (30), can be now labeled as Heun functions $Hp^{(\alpha_1,\alpha_2)}_{N,\mu}(a;\rho)$. There are four cases determined by the parities α_1, α_2 that we list as follows:

$$Hp^{(0,0)}_{n=2N,\mu}(a;\rho) = (\rho - a_3)^{\frac{1}{4}} Q^{(0,0)}_{N,\mu}(a;\rho), \tag{37}$$

$$Hp^{(0,1)}_{n=2N+1,\mu}(a;\rho) = (\rho - a_2)^{\frac{1}{2}} (\rho - a_3)^{\frac{1}{4}} Q^{(0,1)}_{N,\mu}(a;\rho), \tag{38}$$

$$Hp^{(1,0)}_{n=2N+1,\mu}(a;\rho) = (\rho - a_1)^{\frac{1}{2}} (\rho - a_3)^{\frac{1}{4}} Q^{(1,0)}_{N,\mu}(a;\rho), \tag{39}$$

$$Hp^{(1,1)}_{n=2N+2,\mu}(a;\rho) = (\rho - a_1)^{\frac{1}{2}} (\rho - a_2)^{\frac{1}{2}} (\rho - a_3)^{\frac{1}{4}} Q^{(1,1)}_{N,\mu}(a;\rho). \tag{40}$$

These functions are of Heun *type* related, except for normalization under the $\mathcal{L}^2(\mathcal{D})$ norm, to the Heun polynomials of degree N (of which there are eight classes[11]). The number of functions $d_N = N + 1$ for each value of the *principal* quantum number n are the following:

$$\begin{aligned} n = 2N \quad &\rightarrow \quad (\alpha_1, \alpha_2) = (0,0) &&\rightarrow \quad d_N = \tfrac{1}{2}n + 1, \\ n = 2N+1 \quad &\rightarrow \quad (\alpha_1, \alpha_2) = (0,1),\ (1,0) &&\rightarrow \quad d_N = \tfrac{1}{2}(n+1), \\ n = 2N+2 \quad &\rightarrow \quad (\alpha_1, \alpha_2) = (1,1) &&\rightarrow \quad d_N = \tfrac{1}{2}n. \end{aligned} \tag{41}$$

Hence, when n is even, the sum of the first and last cases is $n + 1$, and when n is odd we have twice the middle case; thus we obtain the $n + 1$ solutions for all integer values of the principal quantum number n.

Putting together the two separate one-variable solutions, $Hp_{n,\mu}^{(\alpha_1,\alpha_2)}(a;\rho_1)$ and $Hp_{n,\mu}^{(\alpha_1,\alpha_2)}(a;\rho_2)$, the four cases in (41) yield the Zernike elliptic solutions on the half-sphere $\mathcal{H}_+$,

$$n = 2N: \ \Upsilon_{N,\mu}^{(0,0)}(a;\vec{\xi}) := c_{N,\mu}^{(0,0)}\, Hp_{2N,\mu}^{(0,0)}(a;\rho_1)\, Hp_{2N,\mu}^{(0,0)}(a;\rho_2), \tag{42}$$

$$n = 2N+1: \ \Upsilon_{N,\mu}^{(0,1)}(a;\vec{\xi}) := c_{N,\mu}^{(0,1)}\, Hp_{2N+1,\mu}^{(0,1)}(a;\rho_1)\, Hp_{2N+1,\mu}^{(0,1)}(a;\rho_2), \tag{43}$$

$$n = 2N+1: \ \Upsilon_{N,\mu}^{(1,0)}(a;\vec{\xi}) := c_{N,\mu}^{(1,0)}\, Hp_{2N+1,\mu}^{(1,0)}(a;\rho_1)\, Hp_{2N+1,\mu}^{(1,0)}(a;\rho_2), \tag{44}$$

$$n = 2N+2: \ \Upsilon_{N,\mu}^{(1,1)}(a;\vec{\xi}) := c_{N,\mu}^{(1,1)}\, Hp_{2N+2,\mu}^{(1,1)}(a;\rho_1)\, Hp_{2N+2,\mu}^{(1,1)}(a;\rho_2), \tag{45}$$

where $\rho_i = \rho_i(\vec{\xi})$ are the expressions (14)–(16) of the Jacobi coordinates ρ_i in terms of the coordinates $(\xi_1, \xi_2) = (x, y)$ on the half-sphere and disk and the $c_{N,\mu}^{(\alpha_1,\alpha_2)}$ are normalization constants that absorb the common denominators $(a_2 - a_1)(a_3 - a_1)$ in (12). The parity indices of $\Upsilon_{N,\mu}^{(\alpha_1,\alpha_2)}$ characterize the sign under inversions of the disk with $(1-2\alpha_i) = (-1)^{\alpha_i} \in \{1,-1\}$,

$$\begin{aligned} \Pi_{(\xi_1)} : \Upsilon_{N,\mu}^{(\alpha_1,\alpha_2)}(a;\xi_1,\xi_2) &= \Upsilon_{N,\mu}^{(\alpha_1,\alpha_2)}(a;-\xi_1,\xi_2) \\ &= (1-2\alpha_1)\Upsilon_{N,\mu}^{(\alpha_1,\alpha_2)}(a;\xi_1,\xi_2), \end{aligned} \tag{46}$$

$$\begin{aligned} \Pi_{(\xi_2)} : \Upsilon_{N,\mu}^{(\alpha_1,\alpha_2)}(a;\xi_1,\xi_2) &= \Upsilon_{N,\mu}^{(\alpha_1,\alpha_2)}(a;\xi_1,-\xi_2) \\ &= (1-2\alpha_2)\Upsilon_{N,\mu}^{(\alpha_1,\alpha_2)}(a;\xi_1,\xi_2). \end{aligned} \tag{47}$$

The normalization constants $c_{N,\mu}^{(\alpha_1,\alpha_2)}$ in (42)–(45) should be such that the Zernike solutions on the disk $\Psi(\mathbf{r}) \in \mathcal{L}^2(\mathcal{D})$, on the half-sphere $\Upsilon(\vec{\xi}) \in \mathcal{L}^2(\mathcal{H}_+)$, and in the Jacobi coordinates $(a;\rho_1,\rho_2)$, be unity. The inner products in these spaces are

$$(\Psi_1, \Psi_2)_{\mathcal{D}} = \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, \Psi_1(\mathbf{r})^*\, \Psi_2(\mathbf{r}) = \int_{\mathcal{H}_+} \frac{\mathrm{d}\xi_1\, \mathrm{d}\xi_2}{\xi_3} \Upsilon_1(\xi_1,\xi_2)^*\, \Upsilon_2(\xi_1,\xi_2) \tag{48}$$

$$= \int_{a_1}^{a_2} \mathrm{d}\rho_1 \int_{a_2}^{a_3} \mathrm{d}\rho_2\, \frac{\rho_2 - \rho_1}{4r(a;\rho_1)r(a;\rho_2)} \Upsilon_1(a;\rho_1,\rho_2)^*\, \Upsilon_2(a;\rho_1,\rho_2). \tag{49}$$

The Zernike solutions in Jacobi coordinates, $\Upsilon_{N,\mu}^{(\alpha_1,\alpha_2)}(a;\rho_1,\rho_2)$, being eigenfunctions of the self-adjoint Hamiltonian $\widehat{W}$ in (20), are thus orthogonal in their "energy" eigenvalues E and, due to (46) and (47) in their parities (α_1, α_2), hence in N and in the principal quantum number n, so

$$(\Upsilon_{N,\mu}^{(\alpha_1,\alpha_2)}(a;\cdot,\cdot), \Upsilon_{N',\mu}^{(\alpha_1',\alpha_2')}(a;\cdot,\cdot))_{\mathcal{D}} = 0 \ \text{ when } \begin{cases} (\alpha_1,\alpha_2) \neq (\alpha_1',\alpha_2'), \text{ or} \\ N \neq N' \text{ i.e., } n \neq n'. \end{cases} \tag{50}$$

Setting out from the separation of functions by Jacobi coordinates in (21), we find that there are two additional orthogonality relations which are related with the resolution of the degeneracy in the "energy" spectrum through the label μ that in (35) enumerates the eigenvalues of the separation constant $\{\Lambda_\mu^N\}_{\mu=0}^N$ in (22) and (23). Dividing the operator (19) by $r(a;\rho_1,\cdot)$, applying it to $\Upsilon_{N,\mu'}^{(\alpha_1,\alpha_2)}(a;\rho_1,\cdot)$

in (50), multiplying this by another $\Upsilon^{(\alpha_1,\alpha_2)}_{N,\mu}(a;\rho_1,\cdot)^*$, and subtracting a similar term exchanging $\mu \leftrightarrow \mu'$, as is usual when proving orthogonality of eigenfunctions, one obtains that

$$\begin{aligned}(\Lambda_\mu - \Lambda_{\mu'})\int_{a_1}^{a_2}\frac{\mathrm{d}\rho_1}{4r(a;\rho_1)}\Upsilon_{N,\mu}(\rho_1,\cdot)^*\Upsilon_{N,\mu'}(\rho_1,\cdot)\\ = r(a;\rho_1)\Big(\Upsilon_{N,\mu'}\frac{\mathrm{d}\Upsilon_{N,\mu}}{\mathrm{d}\rho_1} - \Upsilon_{N,\mu}\frac{\mathrm{d}\Upsilon_{N,\mu'}}{\mathrm{d}\rho_1}\Big)\Big|_{a_1}^{a_2} = 0, \quad \text{when } \Lambda_\mu \neq \Lambda_{\mu'},\end{aligned} \tag{51}$$

and one similarly proves orthogonality for the factor in ρ_2. We thus have the second orthogonality relation for Zernike solutions with the same "energy" E as resolved by the label $\mu \in \{0, 1, \ldots, N\}$.

IV. ELLIPTIC ZERNIKE SOLUTIONS ON THE DISK

To organize the solutions (42)–(45) into the pyramid-shaped pattern common to the original System I solutions (with n growing down and angular momentum across) and System II (with $n = n_1 + n_2$ down and $n_1 - n_2$ across),[3] we should first consider the solutions on $\mathcal{H}_+$, $\Upsilon^{(\alpha_1,\alpha_2)}_{N,\mu}$, as primarily classified by their "energy" $E_n = n(n + 2)$, i.e., through their principal quantum number $n = 2N + \alpha_1 + \alpha_2$, into the $d_n := n + 1$ multiplets guaranteed by (41).

The solutions $\Upsilon^{(\alpha_1,\alpha_2)}_{N,\mu}$ are labeled by $N \in \{0, 1, \ldots\}$, but further classified by $\mu \in \{0, 1, \ldots, N\}$, whose meaning is not yet clear. The n-levels of this pyramid are composed of *two* subsets of distinct parities (α_1, α_2) that alternate: $(-1)^{\alpha_1+\alpha_2} = (-1)^n$. Thus, for even n, (0, 0), and (1, 1), while for odd n, (0, 1), and (1, 0),

$$\begin{array}{llll} n=0, & \Upsilon^{(0,0)}_{0,0}, & & d_0=1,\\ n=1, & \Upsilon^{(1,0)}_{0,0}, & \Upsilon^{(0,1)}_{0,0} & d_1=2,\\ n=2, & \Upsilon^{(0,0)}_{1,\mu=0,1}, & \Upsilon^{(1,1)}_{0,0}, & d_2=3,\\ n=3, & \Upsilon^{(1,0)}_{1,\mu=0,1}, & \Upsilon^{(0,1)}_{1,\mu=0,1}, & d_3=4,\\ n=4, & \Upsilon^{(0,0)}_{2,\mu=0,1,2}, & \Upsilon^{(1,1)}_{1,\mu=0,1}, & d_4=5,\\ n=5, & \Upsilon^{(1,0)}_{2,\mu=0,1,2}, & \Upsilon^{(0,1)}_{2,\mu=0,1,2}, & d_5=6,\\ \ldots & \ldots & \ldots & \ldots\\ n \text{ even}, & \Upsilon^{(0,0)}_{N=\frac12 n,\mu|_0^N}, & \Upsilon^{(1,1)}_{N=\frac12 n-1,\mu|_0^N}, & d_n=n+1,\\ n \text{ odd}, & \Upsilon^{(1,0)}_{N=\frac12(n-1),\mu|_0^N}, & \Upsilon^{(0,1)}_{N=\frac12(n-1),\mu|_0^N}, & d_n=n+1. \end{array} \tag{52}$$

So let us now go back to the solutions to the Zernike equation (1) on the disk $(x, y) \in \mathcal{D}$, which are now characterized by the same set of quantum labels borne by the functions in (42)–(45) and were related through (8),

$$\begin{aligned}\Psi^{(\alpha_1,\alpha_2)}_{n,\mu}(x,y) \equiv c^{(\alpha_1,\alpha_2)}_{N,\mu}\,\xi_3^{-\frac12}\,\Upsilon^{(\alpha_1,\alpha_2)}_{N,\mu}(a;\xi_1,\xi_2), \quad \mu|_0^N,\\ n = 2N + \alpha_1 + \alpha_2,\end{aligned} \tag{53}$$

with $\xi_3^2 := 1 - x^2 - y^2$. Yet note that the Heun functions $Hp^{(\alpha_1,\alpha_2)}_{n,\mu}(a;\rho)$ in (37)–(40) also contain non-polynomial pre-factors in ρ. These are easily expressed in the coordinates on the disk, $\mathbf{r} = (x, y)$ in (12), because

$$\begin{aligned}\big((a_3-\rho_1)(a_3-\rho_2)\big)^{1/4} &= (1-x^2-y^2)^{1/4}\big((a_3-a_1)(a_3-a_2)\big)^{1/4},\\ \big((a_2-\rho_1)(\rho_2-a_2)\big)^{1/2} &= y\big((a_2-a_1)(a_3-a_2)\big)^{1/2},\\ \big((\rho_1-a_1)(\rho_2-a_1)\big)^{1/2} &= x\big((a_2-a_1)(a_3-a_1)\big)^{1/2}\end{aligned} \tag{54}$$

and translates into a factor of 1, x, y, or xy according to the parities. We can thus express the elliptic-separated Zernike solutions on the disk (53) using the polynomials defined in (36) as

$$\Psi^{(\alpha_1,\alpha_2)}_{n,\mu}(a;x,y) = c^{(\alpha_1,\alpha_2)}_{N,\mu}\,x^{\alpha_1}y^{\alpha_2}\,Q^{(\alpha_1,\alpha_2)}_{N,\mu}(a;\rho_1)\,Q^{(\alpha_1,\alpha_2)}_{N,\mu}(a;\rho_2), \tag{55}$$

with $\rho_i(a; x, y)$ given by (14)–(16) and a normalization constant $c^{(\alpha_1,\alpha_2)}_{N,\mu}$.

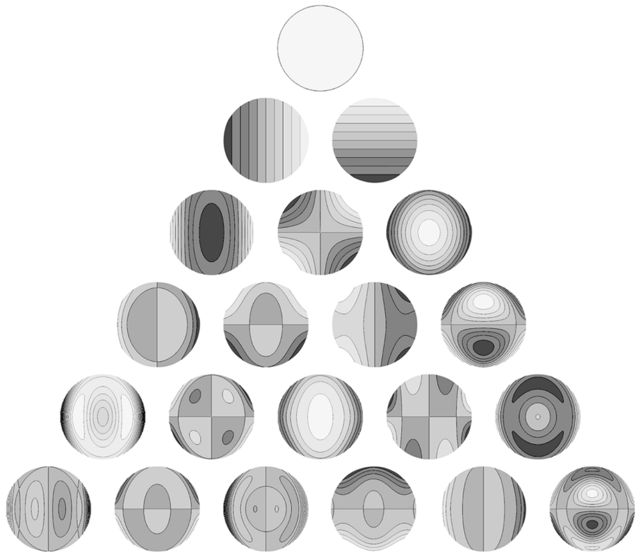

FIG. 2. Pyramid of elliptic Zernike solutions $\Psi^{(\alpha_1,\alpha_2)}_{n,\mu}$ for $0 \le n \le 5$, arranged as in (57). On each row n, the states are arranged by alternate parities (0, 0) and (1, 1) for n even or (1, 0) and (0, 1) for n odd. The values of μ number the solutions obtained from the order of the eigenvectors (35) afforded by the diagonalization algorithm.

The four lowest $N = 0$ Zernike solutions on the disk $\mathcal{D}$ that constitute the top *rhombus* in the pyramid scheme can be built easily since $Q^{(\alpha_1,\alpha_2)}_{N=0,\mu} = 1$, and their square-normalization can be determined directly. These are placed as

$$\begin{array}{llll} n=0: & & \Psi^{(0,0)}_{0,0}(x,y) = 1/\sqrt{\pi} & \\ n=1: & \Psi^{(1,0)}_{1,0}(x,y) = 2x/\sqrt{\pi} & & \Psi^{(0,1)}_{1,0}(x,y) = 2y/\sqrt{\pi}. \\ n=2: & \cdots & \Psi^{(1,1)}_{2,0}(x,y) = xy\sqrt{24/\pi} & \cdots \end{array} \tag{56}$$

These solutions are independent of the eccentricity parameter k in (33) and are common to all coordinate systems, I and II, in particular. Following this lead, we propose the following interdigitation of the two solution subsets in (52):

$$\begin{array}{lccccccccccc} n=0: & & & & & & \Psi^{(0,0)}_{0,0} & & & & & \\ n=1: & & & & & \Psi^{(1,0)}_{1,0} & & \Psi^{(0,1)}_{1,0} & & & & \\ n=2: & & & & \Psi^{(0,0)}_{2,1} & & \Psi^{(1,1)}_{2,0} & & \Psi^{(0,0)}_{2,0} & & & \\ n=3: & & & \Psi^{(1,0)}_{3,1} & & \Psi^{(0,1)}_{3,1} & & \Psi^{(1,0)}_{3,0} & & \Psi^{(0,1)}_{3,0} & & \\ n=4: & & \Psi^{(0,0)}_{4,2} & & \Psi^{(1,1)}_{4,1} & & \Psi^{(0,0)}_{4,1} & & \Psi^{(1,1)}_{4,0} & & \Psi^{(0,0)}_{4,0} & \\ n=5: & \Psi^{(1,0)}_{5,2} & & \Psi^{(0,1)}_{5,2} & & \Psi^{(1,0)}_{5,1} & & \Psi^{(0,1)}_{5,1} & & \Psi^{(1,0)}_{5,0} & & \Psi^{(0,1)}_{5,0} \end{array} . \tag{57}$$

In Fig. 2, we plot these functions on the disk, and in Appendix B we give their explicit expressions obtained from the matrix eigen-solutions in (35). The normalization constants in Appendix B were obtained numerically integrating the solutions over the disk.

V. CONCLUSIONS

The study of the Zernike system, defined by the differential equation (1) on the disk and realized by wavefronts on a circular pupil whose aberrations one seeks to minimize,[1] has yielded a host of polynomial and special function properties and relations. This is due to the superintegrability of this remarkable system, which provides closed elliptical orbits in the classical model,[2] and in the quantum/wave model,[3] its separation in a manifold of the now seen generally elliptic coordinate systems, which are orthogonal on the half-sphere. Whereas previous papers[3,4] dealt with polar coordinates and hypergeometric polynomials,[18] the generic elliptic case involves the higher Heun functions that solve equations with four regular singular points.[11] We thank the reviewer who brought to our attention that the separation of variables on the *complex* half-sphere occurs in four coordinate systems, as shown in Ref. 22, Table B.2 and labeled as $S3$. However, for the *real* half-sphere, the horospheric and degenerate elliptic-1 coordinate systems defined there do not apply. These may be useful to describe separation of variables of a different generalized Zernike system defined *outside* the disk $\mathcal{D}$, which we may cover in a future paper.

The generic elliptic coordinate separation that we have examined here can reliably be seen to lead to further relations of interest such as interbasis expansions, the validation of the limits $k \to 0$ and 1 to polar coordinates, and the explicit realization of the separating operators in the Higgs algebra that are expected to lead to recursion relations among its solutions. These are topics to cover in following papers.

As we stressed in Sec. III, the Zernike system is *two*-dimensional, and in this respect it falls under the same *caveata* as the two dimensional hydrogen atom and oscillator in two-dimensional elliptic coordinates,[10,19–21] where both the energy and the separation constants are present in *both* the equations that separate for each of the two coordinates.[15] An elementary analog applies when we consider two simultaneous linear algebraic equations in x and y that yield their two values; if one of these equations is erased, we remain with a single two-variable equation whose solutions are a line in the x–y plane, which is qualitatively different from the original system. In the same vein, when the solution of the defining equation pair (22) and (23) is simply replaced by the single Heun-type equation (26), the solution of the latter will not be rich enough to describe the two-dimensional system. As stressed for other such two-dimensional systems (hydrogen atom and harmonic oscillator in elliptic coordinates in Ref. 17), the $\mathcal{L}^2(\mathcal{D})$ solutions are necessarily of polynomial type.

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APPENDIX A: NECESSITY OF POLYNOMIAL SOLUTIONS

In this appendix, we shall give a detailed discussion of the explicit form for the solutions of (26). We first separate the factor containing $(\rho - a_3)^{-\frac{1}{4}}$ and introduce a new function

$$F(a;\rho) := (\rho - a_3)^{-\frac{1}{4}}\, P(a;\rho), \tag{A1}$$

which, according to the Zernike condition (27), satisfies at the boundary

$$\lim_{\rho\to a_3} F(a;\rho) = \text{const} \neq 0. \tag{A2}$$

Inserting $F(a;\rho)$ in Eq. (26), we get

$$\begin{aligned}\frac{d^2F(a;\rho)}{d\rho^2} + \frac{1}{2}\left(\frac{1}{\rho-a_1} + \frac{1}{\rho-a_2} + \frac{2}{\rho-a_3}\right)\frac{dF(a;\rho)}{d\rho}\\ - \frac{1}{4(\rho-a_1)(\rho-a_2)(\rho-a_3)}\left(E\rho + \tfrac{1}{4}(a_1+a_2+a_3) + \Lambda\right)F(a;\rho) = 0.\end{aligned} \tag{A3}$$

After a change of the independent variable $\rho = a_1 + (a_2 - a_1)x$, the equation (A3) transforms into the canonical form of Heun's equation for the function

$$G(a;x) := F(a;\, a_1 + (a_2-a_1)\, x\,), \tag{A4}$$

namely,

$$\begin{aligned}\frac{d^2G(a;x)}{dx^2} + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{2}{x-t}\right)\frac{dG(a;x)}{dx}\\ - \frac{1}{4x(x-1)(x-t)}\left(Ex + \frac{\frac{1}{4}(a_1+a_2+a_3) + \Lambda + Ea_1}{a_2 - a_1}\right)G(a;x) = 0,\end{aligned} \tag{A5}$$

where $t = (a_3 - a_1)/(a_2 - a_1) > 1$, so the singularities in x will now be at the points $(0, 1, t, \infty)$. Also, for $\rho = \rho_1$, the "physical region" is $x \in [0, 1]$, while for $\rho = \rho_2$, it is $x \in [1, t]$.

Recall that the canonical form of the Heun's differential equation[12] is

$$\frac{\mathrm{d}^2 W(x)}{\mathrm{d}x^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-t}\right)\frac{\mathrm{d}W(x)}{\mathrm{d}x} + \frac{\alpha\beta x - q}{x(x-1)(x-t)}W(x) = 0, \tag{A6}$$

where the coefficients $\alpha, \beta, \gamma, \delta, \varepsilon$ are subject to the constraint $\alpha + \beta + 1 = \gamma + \delta + \varepsilon$. This equation is of Fuchsian type, with regular singularities at $x = 0, 1, t, \infty$; the exponents of these singularities are, respectively, $\{0, 1-\gamma\}$, $\{0, 1-\delta\}$, $\{0, 1-\varepsilon\}$, and $\{\alpha, \beta\}$. The parameters in (A6) play different roles: t is a *singular* parameter, $(\alpha, \beta, \gamma, \delta, \varepsilon)$ are *exponent* parameters, and q is called the *accessory* parameter. Thus the substitution (A1) transforms (26) to the canonical form of Heun's differential equation (A6) with the exponent parameters $\gamma = \delta = \frac{1}{2}$, $\varepsilon = 1$, and

$$\alpha = \tfrac{1}{2}(1 - \sqrt{E+1}), \qquad \beta = \tfrac{1}{2}(1 + \sqrt{E+1}). \tag{A7}$$

The accessory parameter q is connected to the elliptic separation constant by the relation

$$q = \frac{1}{4(a_2 - a_1)}\Big(\Lambda + Ea_1 + \tfrac{1}{4}(a_1 + a_2 + a_3)\Big). \tag{A8}$$

Note that since each singularity at $x = 0, 1, t$, and ∞ is regular, by the usual theory of Heun's equation, in the neighbourhood of any one of these singularities there exist *two* linearly independent solutions of (A5), one associated with each of its two exponents. In particular, there are the *four* classes of local (*Frobenius*) solutions around the two singularities at $x = 0$ and $x = 1$ which correspond to exponent pairs: $(0, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$, and $(\frac{1}{2}, \frac{1}{2})$. So, if we write the function $G(a; x)$ in (A4) as

$$G(a; x) = x^{\frac{1}{2}\alpha_1}(x-1)^{\frac{1}{2}\alpha_2}\, Z^{(\alpha_1,\alpha_2)}(x), \tag{A9}$$

then the function $Z^{(\alpha_1,\alpha_2)}(a; x)$, $\alpha_i \in \{0, 1\}$, will satisfy Eq. (29) which, in terms of the variable x, is

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}x^2} + \frac{1}{2}\left(\frac{2\alpha_1+1}{x} + \frac{2\alpha_2+1}{x-1} + \frac{2}{x-t}\right)\frac{\mathrm{d}Z}{\mathrm{d}x} - \frac{\widetilde{E}x + \widetilde{q}}{4x(x-1)(x-t)}Z = 0, \tag{A10}$$

with

$$\widetilde{E} = E - (\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + 2), \tag{A11}$$

$$\widetilde{q} = 2\alpha_1 + (\alpha_1 + \alpha_2)^2 t + \frac{\Lambda + Ea_1 + \frac{1}{4}(a_1 + a_2 + a_3)}{a_2 - a_1}. \tag{A12}$$

Let us now expand the wave function $Z^{(\alpha_1,\alpha_2)}(a; x)$ around the singular point $x = 0$,

$$Z^{(\alpha_1,\alpha_2)}(a; x) = \sum_{s=0}^{\infty} b_s x^s. \tag{A13}$$

Inserting this power series into the differential equation (A10) leads to the three-term recurrence relation

$$b_{s+1} = A_s b_s + B_s b_{s-1}, \qquad b_{-1} = 0, \quad b_0 = 1, \quad s \in \{0, 1, \ldots\}, \tag{A14}$$

with

$$A_s = \frac{s\Big(2(t+1)s + 2t(\alpha_1 + \alpha_2) + 2\alpha_1 + 1\Big) + \frac{1}{2}\widetilde{q}}{t(s+1)(2s + 2\alpha_1 + 1)}, \tag{A15}$$

$$B_s = -\frac{2(s-1)(s + \alpha_1 + \alpha_2) - \frac{1}{2}\widetilde{E}}{t(s+1)(2s + 2\alpha_1 + 1)}. \tag{A16}$$

To find the convergence of the power series (A13), we note that the coefficients A_s and B_s for $s \to \infty$ behave asymptotically as

$$A_s \sim \frac{t+1}{t} - \frac{2 + t(3 - 2\alpha_2)}{2ts}, \qquad B_s \sim -\frac{1}{t} + \frac{\frac{5}{2} - \alpha_2}{ts}. \tag{A17}$$

The convergence of the infinite series (A13) is determined by the behavior of the quotient b_{s+1}/b_s for large s. From (A17), it follows, by the Poincaré-Perron theorem, that the limit

$$c_0 := \lim_{s\to\infty} \frac{b_{s+1}}{b_s} \tag{A18}$$

exists Ref. 16, p. 182. Let us now study the asymptotic behavior of the series (A13) for large s, as in Ref. 14. Dividing the recurrence relations (A14) by b_{s-1},

$$\left(\frac{b_{s+1}}{b_s}\right)\left(\frac{b_s}{b_{s-1}}\right) = A_s\left(\frac{b_s}{b_{s-1}}\right) + B_s, \tag{A19}$$

we may assume that for large s, the following approximations hold:

$$\frac{b_{s+1}}{b_s} \approx c_0 + \frac{c_1}{s} + \frac{c_2}{s^2}, \qquad \frac{b_s}{b_{s-1}} \approx c_0 + \frac{c_1}{s-1} + \frac{c_2}{(s-1)^2} \approx c_0 + \frac{c_1}{s} + \frac{c_1 + c_2}{s^2}. \tag{A20}$$

Putting these expressions into the three-term recurrence relation (A14) and writing the coefficients at s^2 and s, we find equations for the coefficients c_0 and c_1,

$$\begin{aligned} &tc_0^2 - (t+1)c_0 + 1 = 0, \\ &4tc_0c_1 - 2(t+1)c_1 = 5 - 2\alpha_2 - c_0\big(2 + t(3 - 2\alpha_2)\big) \end{aligned} \tag{A21}$$

and obtain two solutions

$$\text{case (a)}: \qquad c_0^{(1)} = 1, \quad c_1^{(1)} = -\tfrac{3}{2} + \alpha_2, \tag{A22}$$

$$\text{case (b)}: \qquad c_0^{(2)} = 1/t, \quad c_1^{(2)} = -1/t. \tag{A23}$$

Because $t > 1$, the case (b) presents the so-called "minimal" solution, while case (a) is the "maximal" one.

Let us consider first the maximal solution (a), where we have

$$\frac{b_{s+1}}{b_s} \approx 1 - (\tfrac{3}{2} - \alpha_2)\frac{1}{s} \tag{A24}$$

and consequently

$$b_s \approx \prod_{\ell=1}^{s}\left(1 - \frac{\frac{3}{2} - \alpha_2}{\ell}\right) \tag{A25}$$

$$= \frac{1}{s!}(-\tfrac{1}{2}+\alpha_2)(-\tfrac{1}{2}+\alpha_2+1)\cdots(-\tfrac{1}{2}+\alpha_2+s-1) = \frac{(-\frac{1}{2}+\alpha_2)_s}{s!},$$

with the Pochhammer symbol $(z)_s := \Gamma(z+s)/\Gamma(z)$. Thus we get

$$Z^{(\alpha_1,\alpha_2)}(x) \approx \sum_{s=0}^{\infty}\frac{(-\frac{1}{2}+\alpha_2)_s}{s!}x^s = (1-x)^{\frac{1}{2}-\alpha_2}, \tag{A26}$$

which means that the function $G(a;x)$ in (A9) converges to a finite limit at the singular point $x = 1$. Therefore the points $x = 0$ and $x = 1$ are regular; for $\rho = a_1 + (a_2 - a_1)x$, these points correspond to $\rho = a_1$ and $\rho = a_2$ and the solutions of the equation (A3) in the interval $\rho \equiv \rho_1 \in [a_1, a_2]$ are square integrable. Thus, we have no need to *assume* that the equation (A10) has polynomial solutions for $\rho = \rho_1$ as the only useful ones: there are no other.

Consider now the function $G(a;x)$ in (A9) for the interval $x \in [1, t]$ (or $\rho \equiv \rho_2 \in [a_2, a_3]$). The minimal solution (b) in (A23) leads to

$$\frac{b_{s+1}}{b_s} \approx \frac{1}{t}\left(1 - \frac{1}{s}\right), \qquad \text{hence} \qquad b_s \approx \frac{1}{t^s}\prod_{k=2}^{s}\left(1 - \frac{1}{k}\right) = \frac{1}{s}\,\frac{1}{t^s}. \tag{A27}$$

When $x \to t$ (or $\rho \to a_3$),

$$Z^{(\alpha_1,\alpha_2)}(x)\Big|_{x\to t} = \sum_{s=0}^{\infty} b_s x^s\Big|_{x\to t} \approx \sum_{s=1}^{\infty}\frac{1}{s}, \tag{A28}$$

which logarithmically diverges at the singular point $x = t$ (or $\rho \equiv \rho_2 = a_3$). Thus according to the condition (A9), the function in (A9), $G(a;x) = x^{\frac{1}{2}\alpha_1}(x-1)^{\frac{1}{2}\alpha_2}Z^{(\alpha_1,\alpha_2)}(x)$, also diverges at the singular point $x = t$, which contradicts Zernike's condition (A2). The case (a) gives a more divergent solution. Therefore, to obtain a regular solution of (A5), the series (A13) must terminate. This explicit example thus explains why all of the four solutions (37)–(40) of Eqs. (22) and (23) are only Heun-type polynomials, expressed in terms of terminating series.

APPENDIX B: HEUN ZERNIKE SOLUTIONS

The first four $N = 0$ states $\Psi^{(\alpha_1,\alpha_2)}_{0,0}$ were given in (56), with Frobenius series (37)–(40) that have single terms $C^{0;\alpha_1,\alpha_2}_{0,0} := 1$—but with undetermined normalization coefficients $c^{(\alpha_1,\alpha_2)}_{0,0}$. For higher-$n$ states, let us fix the Jacobi parameters to the simple values $\bar{a} := \{a_1 = 1, a_2 = 2, a_3 = 3\}$, which correspond to the eccentricity $k^2 = \frac{1}{2}$ in (33), so the angle between the ellipse foci on the sphere is $\frac{1}{2}\pi$, as in Fig. 1. Keeping k as a generic parameter lengthens the formulas needlessly. The argument of the Heun polynomials in (37)–(40) is then $(\rho - a_2)/(a_3 - a_1) = \frac{1}{2}\rho - 1$, and the Zernike solutions on the disk $(x, y) \in \mathcal{D}$ are separated polynomials in (ρ_1, ρ_2), where

$$\begin{aligned} \rho_1(\bar{a}; x, y) &= \tfrac{1}{2}(\bar{S} + \bar{T}), \quad \bar{S}(x, y) = 2x^2 + y^2 + 3, \\ \rho_2(\bar{a}; x, y) &= \tfrac{1}{2}(\bar{S} - \bar{T}), \quad \bar{T}(x, y) = \sqrt{(2x^2 + y^2)^2 - 2(2x^2 - y^2) + 1}. \end{aligned} \tag{B1}$$

There are eight solutions for $N = 1$, corresponding on the pyramid (57) to $n = 2N + \alpha_1 + \alpha_2$ in the rungs $n = 2$, 3, and 4. The matrices in (35) are 2×2, thus having *two* two-dimensional eigenvectors $\mathbf{C}^{N;\alpha_1,\alpha_2}_{\mu}$ whose scale we have set by setting the first Frobenius coefficient to be $C^{N;\alpha_1,\alpha_2}_{\mu,m=0} = 1$. The Heun functions $Hp^{(\alpha_1,\alpha_2)}_{N=1,\mu}(\bar{a}; \rho)$ are $N = 1$ degree polynomials with pre-factors of $(\rho - a_i)^{\nu_i}$ in (55) that yield the elliptic basis of $N = 1$ Zernike solutions

$$\Psi^{(0,0)}_{2,0}(x, y) = c^{(0,0)}_{2,0}\Big(1 + \tfrac{16}{1-\sqrt{17}}(\tfrac{1}{2}\rho_1 - 1)\Big)\Big(1 + \tfrac{16}{1-\sqrt{17}}(\tfrac{1}{2}\rho_2 - 1)\Big), \quad c^{(0,0)}_{2,0} = 0.102\,91, \tag{B2}$$

$$\Psi^{(0,0)}_{2,1}(x, y) = c^{(0,0)}_{2,1}\Big(1 + \tfrac{16}{1+\sqrt{17}}(\tfrac{1}{2}\rho_1 - 1)\Big)\Big(1 + \tfrac{16}{1+\sqrt{17}}(\tfrac{1}{2}\rho_2 - 1)\Big), \quad c^{(0,0)}_{2,1} = 0.541\,23; \tag{B3}$$

$$\Psi^{(0,1)}_{3,0}(x, y) = c^{(0,1)}_{3,0}\, y\Big(1 + \tfrac{24}{1-\sqrt{73}}(\tfrac{1}{2}\rho_1 - 1)\Big)\Big(1 + \tfrac{24}{1-\sqrt{73}}(\tfrac{1}{2}\rho_2 - 1)\Big), \quad c^{(0,1)}_{3,0} = 0.654\,53, \tag{B4}$$

$$\Psi^{(0,1)}_{3,1}(x, y) = c^{(0,1)}_{3,1}\, y\Big(1 + \tfrac{24}{1+\sqrt{73}}(\tfrac{1}{2}\rho_1 - 1)\Big)\Big(1 + \tfrac{24}{1+\sqrt{73}}(\tfrac{1}{2}\rho_2 - 1)\Big), \quad c^{(0,1)}_{3,1} = 1.013\,86; \tag{B5}$$

$$\Psi^{(1,0)}_{3,0}(x, y) = c^{(1,0)}_{3,0}\, x\Big(1 + 6(\tfrac{1}{2}\rho_1 - 1)\Big)\Big(1 + 6(\tfrac{1}{2}\rho_2 - 1)\Big), \quad c^{(1,0)}_{3,0} = 0.317\,50, \tag{B6}$$

$$\Psi^{(1,0)}_{3,1}(x, y) = c^{(1,0)}_{3,1}\, x\Big(1 - 4(\tfrac{1}{2}\rho_1 - 1)\Big)\Big(1 - 4(\tfrac{1}{2}\rho_2 - 1)\Big), \quad c^{(1,0)}_{3,1} = 0.211\,95; \tag{B7}$$

$$\Psi^{(1,1)}_{4,0}(x, y) = c^{(1,1)}_{4,0}\, xy\Big(1 - \tfrac{32}{1-\sqrt{97}}(\tfrac{1}{2}\rho_1 - 1)\Big)\Big(1 - \tfrac{32}{1-\sqrt{97}}(\tfrac{1}{2}\rho_2 - 1)\Big), \quad c^{(1,1)}_{4,0} = 2.403\,27, \tag{B8}$$

$$\Psi^{(1,1)}_{4,1}(x, y) = c^{(1,1)}_{4,1}\, xy\Big(1 - \tfrac{32}{1+\sqrt{97}}(\tfrac{1}{2}\rho_1 - 1)\Big)\Big(1 - \tfrac{32}{1+\sqrt{97}}(\tfrac{1}{2}\rho_2 - 1)\Big), \quad c^{(1,1)}_{4,1} = 1.130\,54. \tag{B9}$$

The normalization constants $c^{(\alpha_1,\alpha_2)}_{n,u}$ were computed through numerical integration. The placement of these eight solutions in the Zernike pyramid (57) shows that they complete the $n = 2$ level, cover the $n = 3$ level, and provide two of the five solutions at the $n = 4$ level.

The following twelve $N = 2$ states involve the diagonalization of 3×3 matrices which can best be handled numerically to find its 3 eigenvectors labeled by $\mu = 0, 1, 2$, with coefficients $C^{2;\alpha_1,\alpha_2}_{\mu,m}$ whose first $m = 0$ component in (35) is set to unity. They provide solutions in the pyramid rungs of principal quantum numbers $n = 4, 5, 6$; these yield 3, 6, and 3 functions, respectively,

$$\Psi^{(0,0)}_{4,0}(x,y) = c^{(0,0)}_{4,0}\Big(1 - 18.1802(\tfrac{1}{2}\rho_1{-}1) + 41.0267(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 - 18.1802(\tfrac{1}{2}\rho_2{-}1) + 41.0267(\tfrac{1}{2}\rho_2{-}1)^2\Big), \quad c^{(0,0)}_{4,0} = 0.014\,280, \tag{B10}$$

$$\Psi^{(0,0)}_{4,1}(x,y) = c^{(0,0)}_{4,1}\Big(1 + 12.9930(\tfrac{1}{2}\rho_1{-}1) + 24.4675(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 + 12.9930(\tfrac{1}{2}\rho_2{-}1) + 24.4675(\tfrac{1}{2}\rho_2{-}1)^2\Big), \quad c^{(0,0)}_{4,1} = 0.040\,73, \tag{B11}$$

$$\Psi^{(0,0)}_{4,2}(x,y) = c^{(0,0)}_{4,2}\Big(1 - 0.8128(\tfrac{1}{2}\rho_1{-}1) - 8.1608(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 - 0.8128(\tfrac{1}{2}\rho_2{-}1) - 8.1608(\tfrac{1}{2}\rho_2{-}1)^2\Big), \quad c^{(0,0)}_{4,2} = 0.244\,19; \tag{B12}$$

$$\Psi^{(0,1)}_{5,0}(x,y) = c^{(0,1)}_{5,0}\, y\Big(1 - 8.9183(\tfrac{1}{2}\rho_1{-}1) + 15.6772(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 - 8.9183(\tfrac{1}{2}\rho_2{-}1) + 15.6772(\tfrac{1}{2}\rho_2{-}1)^2\Big), \quad c^{(0,1)}_{5,0} = 0.256\,72, \tag{B13}$$

$$\Psi^{(0,1)}_{5,1}(x,y) = c^{(0,1)}_{5,1}\, y\Big(1 + 7.3521(\tfrac{1}{2}\rho_1{-}1) + 11.2865(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 + 7.3521(\tfrac{1}{2}\rho_2{-}1) + 11.2865(\tfrac{1}{2}\rho_2{-}1)^2\Big), \quad c^{(0,1)}_{5,1} = 0.513\,94, \tag{B14}$$

$$\Psi^{(0,1)}_{5,2}(x,y) = c^{(0,1)}_{5,2}\, y\Big(1 - 0.4338(\tfrac{1}{2}\rho_1{-}1) - 6.4303(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 - 0.4338(\tfrac{1}{2}\rho_2{-}1) - 6.4303(\tfrac{1}{2}\rho_2{-}1)^2\Big), \tag{B15}$$

$$c^{(0,1)}_{5,2} = 1.084\,34, \tag{B16}$$

$$\Psi^{(1,0)}_{5,0}(x,y) = {}^{(1,0)}_{5,0}\, x\Big(1 + 20.0956(\tfrac{1}{2}\rho_1{-}1) + 49.9406(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 + 20.0956(\tfrac{1}{2}\rho_2{-}1) + 49.9406(\tfrac{1}{2}\rho_2{-}1)^2\Big), \quad c^{(1,0)}_{5,0} = 0.018\,09, \tag{B17}$$

$$\Psi^{(1,0)}_{5,1}(x,y) = c^{(1,0)}_{5,1}\, x\Big(1 - 14.9479(\tfrac{1}{2}\rho_1{-}1) + 31.5558(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 - 14.9479(\tfrac{1}{2}\rho_2{-}1) + 31.5558(\tfrac{1}{2}\rho_2{-}1)^2\Big), \quad c^{(1,0)}_{5,1} = 0.266\,05, \tag{B18}$$

$$\Psi^{(1,0)}_{5,2}(x,y) = c^{(1,0)}_{5,2}\, x\Big(1 + 0.8522(\tfrac{1}{2}\rho_1{-}1) - 10.8297(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 + 0.8522(\tfrac{1}{2}\rho_2{-}1) - 10.8297(\tfrac{1}{2}\rho_2{-}1)^2\Big), \quad c^{(1,0)}_{5,2} = 0.533\,46; \tag{B19}$$

$$\Psi^{(1,1)}_{6,0}(x,y) = c^{(1,1)}_{6,0}\, xy\Big(1 + 9.7443(\tfrac{1}{2}\rho_1{-}1) + 18.5364(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 + 9.7443(\tfrac{1}{2}\rho_2{-}1) + 18.5364(\tfrac{1}{2}\rho_2{-}1)^2\Big), \tag{B20}$$

$$\Psi^{(1,1)}_{6,1}(x,y) = c^{(1,1)}_{6,1}\, xy\Big(1 - 8.1898(\tfrac{1}{2}\rho_1{-}1) + 13.7598(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 - 8.1898(\tfrac{1}{2}\rho_2{-}1) + 13.7598(\tfrac{1}{2}\rho_2{-}1)^2\Big), \tag{B21}$$

$$\Psi^{(1,1)}_{6,2}(x,y) = c^{(1,1)}_{6,2}\, xy\Big(1 + 0.4455(\tfrac{1}{2}\rho_1{-}1) - 8.0296(\tfrac{1}{2}\rho_1{-}1)^2\Big) \times\Big(1 + 0.4455(\tfrac{1}{2}\rho_2{-}1) - 8.0296(\tfrac{1}{2}\rho_2{-}1)^2\Big). \tag{B22}$$

In Fig. 2, we have plotted these functions down to $n = 5$ in the pyramid scheme (57); the normalization constants for the three $n = 6$ states outside that figure are omitted.

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