# Deformations of inhomogeneous classical Lie algebras to the algebras of the linear groups

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We study a new class of deformations of algebra representations, namely, $i_2so(n) \Rightarrow sl(n, \mathbb{R})$, $i_2u(n) \Rightarrow sl(n,\mathbb{C}) \oplus u(1)$ and $i_2sp(n) \oplus sp(1) \Rightarrow sl(n, Q) \oplus sp(1)$. The new generators are built as commutators between the Casimir invariant of the maximal compact subalgebra and a second-rank mixed tensor. These algebra deformations are related to multiplier representations and manifold mappings of the corresponding Lie groups. Behavior of the representations under Inönü–Wigner contractions is exhibited. Through the use of these methods we can construct a principal degenerate series of representations of the linear groups and their algebras.

## 1. INTRODUCTION

The closely related concepts of expansion and deformation of Lie algebras has been developed in both the physics and mathematics literature. In physics, expansions first appeared as a way of building symplectic algebras $sp(n, \mathbb{R})$ from the position and momentum operators with the canonical commutation relations,[1] and later by deforming the Poincaré algebra to the de Sitter algebra[2] as well as obtaining possible dynamical algebras for various physical systems.[3,4] Indeed, these latter types of deformations have been performed for inhomogeneous orthogonal,[2,5,6] unitary[5,6] and symplectic[6,7] Lie algebras using a specific type of deformation, i.e., $iso(n) \Longrightarrow so(n,1)$, $iu(n) \oplus u(1) \Longrightarrow u(n,1)$, $isp(n) \oplus sp(1) \Longrightarrow sp(n,1)$ and other noncompact forms. These deformations have then been applied to various problems in representation theory[5,8] and shown by Gilmore[9] to constitute a well-defined family of deformations in which the coset space of the deformed algebra in the Cartan decomposition is of rank one.

In this article we present a family of deformations of representations of Lie algebras on homogeneous spaces of rank one (spheres), but where the rank of the coset spaces of the deformed algebras in the Cartan decomposition is higher. Specifically, in Sec. 2 we treat the cases $i_2so(n) \Longrightarrow sl(n, \mathbb{R})$, $i_2u(n) \Longrightarrow sl(n, \mathbb{C}) \oplus u(1)$ and $i_2sp(n) \oplus sp(1) \Longrightarrow sl(n, \mathbb{Q}) \oplus sp(1)$. We develop these cases separately so as to provide a clearer formulation for the reader who is not so familiar with the complications of the quaternionic field, which would be necessary in a general formulation. We then discuss in Sec. 3 the connection between the deformations of these algebras and the existence of corresponding multiplier representations[10,11] of the groups $SL(n, \mathbb{R})$, $SL(n, \mathbb{C}) \otimes U(1)$ and $SL(n, \mathbb{Q}) \otimes Sp(1)$ on the homogeneous spaces corresponding to the real, complex and quaternionic spheres. In Sec. 4 we show that the Inönü–Wigner contraction[12] of the representations of these groups with respect to the maximal compact subgroups are the groups $I_2SO(n)$, $I_2SU(n) \otimes U(1)$, and $I_2Sp(n) \otimes Sp(1)$.

## 2. DEFORMATIONS OF INHOMOGENEIZATIONS OF THE CLASSICAL LIE ALGEBRAS

Since we will be interested in deformations[12] and expansions[9] which are representation-dependent, we begin with suitable definitions of expansions and deformations of representations in which nothing is said about the abstract Lie algebra. Let $\phi$ be a representation of a Lie algebra $\mathfrak{A}$, i.e., a homomorphism of $\mathfrak{A}$ into some suitable defined vector space, which for our purposes can be taken as the space of infinitely differentiable functions over spheres. An *expansion* of the representation $\phi$ is a mapping $\phi \to \psi_\lambda$ such that the $\psi_\lambda$'s form a representation of a Lie algebra $\mathfrak{A}'$. Moreover, if the condition $\psi_\lambda \xrightarrow{\lambda \to 0} \phi$ is satisfied, the expansion is said to be a *deformation*. The deformation of an inhomogeneous algebra can be thought of as the inverse of contraction.[12] It is seen that the requirement that the $\psi_\lambda$'s form a Lie algebra places severe restrictions on the possible mappings $\psi_\lambda$. Such restrictions have an elegant formulation in terms of Lie algebra cohomology theory,[13] however, rather than attempt the general formulation here, we will discuss a family of specific examples of representation-dependent deformations of inhomogeneizations of the classical Lie algebras.

### A. $i_2so(n) \Rightarrow sl(n, \mathbb{R})$

Consider the Lie algebra $so(n)$ of the orthogonal group whose generators satisfy the well-known commutation relations[14]

$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\rho}M_{\nu\sigma} - \delta_{\nu\sigma}M_{\mu\rho} + \delta_{\mu\sigma}M_{\nu\rho}, \quad (2.1)$$

which preserve the usual metric in real $n$-space $\mathbb{R}^n$, so that the Greek indices take values $1, \ldots, n$. We adjoin now to this algebra a set of commuting $n$-dimensional second-rank symmetric tensors $P_{\mu\nu} = P_{\nu\mu}$. We thus arrive at a Lie algebra which we denote by $i_2so(n)$, which is characterized, along with Eq. (2.1) by

$$[M_{\mu\nu}, P_{\rho\sigma}] = \delta_{\nu\rho}P_{\mu\sigma} - \delta_{\mu\rho}P_{\nu\sigma} + \delta_{\nu\sigma}P_{\mu\rho} - \delta_{\mu\sigma}P_{\nu\rho}, \quad (2.2a)$$

$$[P_{\mu\nu}, P_{\rho\sigma}] = 0. \quad (2.2b)$$

The set of $\frac{1}{2}n(n+1)$ generators $P$ constitute the maximal Abelian ideal of $i_2so(n)$.

The technique for deformation now consists of taking the commutator of the Casimir operator $\Phi$ of the original algebra $so(n)$ with the $P$'s. Specifically, we consider the following members of the enveloping algebra of $i_2so(n)$:

$$N_{\mu\nu} \equiv \tfrac{1}{2}[\Phi, P_{\mu\nu}] + \tau P_{\mu\nu}, \quad (2.3)$$

where $\Phi \equiv -\frac{1}{2}M_{\mu\nu}M_{\nu\mu}$ and $\tau$ is an arbitrary complex number.

As $\Phi$ commutes with all of $so(n)$, it follows that the $N$'s transform under $so(n)$ as the $P$'s, i.e., they satisfy Eq. (2.2a) with $P_{\mu\nu}$ replaced by $N_{\mu\nu}$. However, if we consider the analog of Eq. (3.2b), that is, the commutator of two $N$'s, we find that in general, (for any choice of $\tau$ other than the contraction limit $\tau \to \infty$), the $N$'s do not close into a finite-dimensional Lie algebra. This is to be contrasted with the better-known expansion[2,9] $iso(n) \Longrightarrow so(n, 1)$ where the algebra closes modulo a normalization factor. We can, however, obtain a representation of a Lie algebra if we impose some further restrictions. We choose the following representation[15] for $P_{\mu\nu}$ and $M_{\mu\nu}$: $P_{\mu\nu} = x_\mu x_\nu/x^2$ where the $x_\mu$'s commute, $x^2 = x_\mu x_\mu$ and $M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu + \sigma_{\mu\nu}$ with $[\sigma_{\mu\nu}, x_\lambda] = 0$, where we have introduced $\partial_\mu \equiv \partial/\partial x_\mu$. It is then found that the algebra will close if and only if $\sigma_{\mu\nu}$ vanishes. This means that, in contrast with the $iso(n) \Longrightarrow so(n, 1)$ expansion, we do not have the freedom to construct an additional vector space structure upon the representations described by Eq. (2.3), i.e., vector bundles over the sphere. This limits the possible representations one can construct to degenerate ones.[16] In a straightforward manner one can then verify that

$$[N_{\mu\nu}, N_{\rho\sigma}] = \delta_{\nu\rho} M_{\mu\sigma} + \delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\mu\rho} + \delta_{\mu\sigma} M_{\nu\rho}. \quad (2.4)$$

In order to see more clearly the structure of the algebra spanned by the $M$'s and $N$'s it is convenient to construct the traceless operators

$$\begin{aligned} X_{\mu\nu} &\equiv \frac{1}{2}(M_{\mu\nu} + N_{\mu\nu}) - \frac{1}{2n}\delta_{\mu\nu}\,\mathrm{Tr}N \\ &= x_\mu \partial_\nu - x_\mu x_\nu (x\cdot\partial - \sigma) - \frac{\sigma}{n}\delta_{\mu\nu}, \end{aligned} \quad (2.5)$$

where $x \cdot \partial \equiv x_\mu \partial_\mu$, $\sigma = \frac{1}{2}(-n + \tau)$ and where we have taken the $x_\mu$ to be the Cartesian coordinates on the $(n-1)$-dimensional real sphere $S_{n-1}$, i.e., $x^2 = 1$. One then obtains the Lie algebra in the form

$$[X_{\mu\nu}, X_{\rho\sigma}] = \delta_{\nu\rho} X_{\mu\sigma} - \delta_{\mu\sigma} X_{\rho\nu}, \quad (2.6)$$

and $\mathrm{Tr}X = 0$. By identifying the $(n-1)$ independent commuting $X_{\mu\mu}$'s (no sum) as the Cartan subalgebra and the $X_{\mu\nu}$ $(\mu \neq \nu)$ as the root vectors, one identifies[14] the Cartan class $A_{n-1}$.

In order to see what type of representations are allowed in our constructions, we notice the following relation in the enveloping algebra of $sl(n, \mathbb{R})$,

$$X_{\mu\lambda} X_{\lambda\nu} = X_{\mu\nu}[(n+\sigma)(n-1) - \sigma]/n + \delta_{\mu\nu}(n-1)\sigma(\sigma+n)/n^2, \quad (2.7)$$

which can be used to express all the higher-order Casimir operators in terms of the second-order operator $C_2 = X_{\mu\nu}X_{\nu\mu}$; hence we have only a most degenerate series of representation. This reflects the fact that our representations are built on a rank one homogeneous space $S_{n-1}$. Contracting over $\mu$ and $\nu$ in (2.7), we have $C_2 = (n-1)\sigma(\sigma+n)/n$.

The fact that we have an $sl(n, \mathbb{R})$ form of $A_{n-1}$ is indicated by Eq. (2.6) and the form (2.5) with the specification of the hermiticity properties which must await the introduction of a Hilbert space structure which will be discussed in Sec. 3. Suffice it now to say that all the generators $X_{\mu\nu}$ will be anti-Hermitian under the usual scalar product on the sphere $S_{n-1}$ with the choice $\sigma = -\frac{1}{2}n + i\rho$, $\rho$ real, i.e., for $\tau = 2i\rho$. Indeed, with this choice of $\sigma$, $C_2$ is $-n(n-1)/4 - (n-1)\rho^2/n$. These representations are reducible as can be seen from the fact that the generators $X_{\mu\nu}$ are all even functions of $x_\mu$. An extra parity label $\epsilon$ is thus needed to specify irreducible representations. Then we can say that the parameters $(\rho, \epsilon)$ label the representations of a principal most degenerate series of $sl(n, \mathbb{R})$ built on the space of square-integrable functions on the sphere.

### B. $i_2\, u(n) \Rightarrow sl(n,\mathbb{C}) \oplus u(1)$

In analogy with the previous section, we consider the algebra $u(n)$, the usual metric-preserving algebra for the complex space $\mathbb{C}^n$ and adjoin an ideal formed by the set of commuting second-rank mixed tensors $Z_{\mu\nu}$ with the symmetry property $\overline{Z}_{\mu\nu} = Z_{\nu\mu}$ (the bar denotes complex conjugation). The $i_2 u(n)$ algebra is then defined through the commutation relations[14]

$$[C_{\mu\nu}, C_{\rho\sigma}] = \delta_{\nu\rho} C_{\mu\sigma} - \delta_{\mu\sigma} C_{\rho\nu}, \quad (2.8a)$$

$$[C_{\mu\nu}, Z_{\rho\sigma}] = \delta_{\nu\rho} Z_{\mu\sigma} - \delta_{\mu\sigma} Z_{\rho\nu}, \quad (2.8b)$$

and the two $Z$'s commute.

The extension proposed in (2.3) is, for the unitary groups,

$$D_{\mu\nu} \equiv \tfrac{1}{4}[\Psi, Z_{\mu\nu}] + \tau Z_{\mu\nu}, \quad (2.9)$$

where $\Psi = -2C_{\mu\nu}C_{\nu\mu}$ is the $u(n)$ second-order Casimir invariant. Again we are unable to find an expansion for a completely general $Z_{\mu\nu}$ and again one does not have the freedom to add an additional vector space structure. The particular choice of representation for which the deformation can be carried through is[15] $C_{\mu\nu} = Z_\mu \partial_\mu - \bar{z}_\nu \bar{\partial}_\mu$, $Z_{\mu\nu} = z_\mu \bar{z}_\nu/|z|^2$, $|z|^2 = z_\mu \bar{z}_\mu$, where we have used the notation $\partial_\mu \equiv \partial/\partial z_\mu$ and $\bar{\partial}_\mu \equiv \partial/\partial \overline{Z}_\mu$. It is then straightforward to verfiy that

$$[D_{\mu\nu}, D_{\rho\sigma}] = \delta_{\nu\rho} C_{\mu\sigma} - \delta_{\mu\sigma} C_{\rho\nu}, \quad (2.10)$$

while the $D$'s obey the same transformation properties (2.8b) under the $C$'s as the $Z$'s. Moreover, the trace of $C_{\mu\nu}$ which we denote as $C \equiv C_{\mu\mu}(sum)$, provides a $u(1)$ subalgebra which not only commutes with the $C$'s but also with the $D$'s, thus providing the direct sum algebra $sl(n, \mathbb{C}) \oplus u(1)$. The existence of this $u(1)$ subalgebra arises from the fact that the generators $D_{\mu\nu}$ as well as each $|z_\mu|^2$, ($\mu$ fixed) remains invariant under $z_\mu \to e^{i\phi} z_\mu$. This is the analog of the parity in the last section, and definite $u(1)$ transformation properties must be specified in order to get irreducible representations of $sl(n, \mathbb{C})$. Indeed, it will be seen shortly how this provides us with an additional Casimir operator.

A convenient form to display the $sl(n, \mathbb{C})$ structure is obtained by constructing the traceless combinations

$$X^{\pm}_{\mu\nu} \equiv \tfrac{1}{2}[C_{\mu\nu} \pm D_{\mu\nu} - \delta_{\mu\nu}(C \pm TrD)/n], \quad (2.11a)$$

given explicitly by

$$X^{+}_{\mu\nu} = z_\mu \partial_\nu - \tfrac{1}{2} z_\mu \bar{z}_\nu (z\cdot\partial + \bar{z}\cdot\bar{\partial} - \sigma) - \frac{C+\sigma}{2n}\delta_{\mu\nu}, \quad (2.11b)$$

$$X^{-}_{\mu\nu} = -\bar{z}_\nu \bar{\partial}_\mu + \tfrac{1}{2} z_\mu \bar{z}_\nu (z\cdot\partial + \bar{z}\cdot\bar{\partial} - \sigma) - \frac{C-\sigma}{2n}\delta_{\mu\nu}, \quad (2.11c)$$

where $\sigma = -n + \tau$ and we have set $|z|^2 = 1$, so that the $(2n-1)$ independent real numbers in $z$ are the complex Cartesian coordinates on the $(n-1)$-dimensional complex sphere $C_{n-1} \cong S_{2n-1}$. It is easy to check that all $X^+$'s commute with all $X^-$'s and hence we have explicitly a pair of commuting $sl(n, \mathbb{R})$ algebras given by (2.6). In this form the Cartan subalgebra is given by the $2(n-1)$ independent $X^{\pm}_{\mu\mu}$ (no sum) and one easily arrives at the Cartan structure $A_{n-1} \oplus A_{n-1}$. An additional advantage of the form (2.11) is the following convenient set of relations in the enveloping algebra of $sl(n, \mathbb{C})$

$$\begin{aligned} X^{\pm}_{\mu\lambda} X^{\pm}_{\lambda\nu} = &\left[N^{\pm} \mp 1 + \frac{n-2}{2n}(C \pm \sigma)\right] X^{\pm}_{\mu\nu} \\ &+ \frac{n-1}{2n}(C \pm \sigma)\left(\frac{C \pm \sigma}{n} \pm 1\right)\delta_{\mu\nu}, \end{aligned} \quad (2.12)$$

where $N^+ = n$ and $N^- = 0$.

As in the last section, these can be used to express all the higher-order Casimir invariants in terms of the second-order ones[17]. We are thus led to a class of degenerate representations, but not just a most degenerate series: the two Casimir operators obtainable from (2.12) are $C_2^{\pm} \equiv X^+_{\mu\nu}X^+_{\nu\mu} \pm X^-_{\mu\nu}X^-_{\nu\mu}$ and take the values $(n-1)[\sigma(2n+\sigma)+C^2]/2n$ and $(n-1)C(n+\sigma)/n$, respectively.

Now using the fact that $C$ is the generator of a $U(1)$ group and restricting the representations of this $U(1)$ to be single-valued, one finds that the eigenvalues of $C$ are integers $m$. In the next section we shall introduce a definite scalar product on $C_{n-1}$, with respect to which hermiticity will be defined. For the generators (2.8a) we will have $C_{\mu\nu}{}^\dagger = C_{\nu\mu}$, while for the choice $\sigma = -n + i\rho$, ($\rho$ real), i.e., $\tau$ imaginary in (2.9) $D_{\mu\nu}{}^\dagger = -D_{\mu\nu}$ and for (2.11), $(X^{\pm}_{\mu\nu})^\dagger = X^{\mp}_{\nu\mu}$. For this choice of $\sigma$ and $C$, the Casimir operators $C_2^{\pm}$ are $-(m^2+4\rho^2)/2n - n/2$ and $2i\rho m/n$, respectively. Thus we have Hermitian representations of the $sl(n,\mathbb{C})$ algebra described by two numbers, a real $\rho$ and an integer $m$.

In performing the previous deformation, we followed the analogy with the real sphere, making $x_\mu x_\nu \to z_\mu \bar{z}_\nu$ and using the metric-preserving algebra on the complex sphere. We could alternatively have decomposed $z_\mu \bar{z}_\nu$ into its real and imaginary parts and considered the corresponding deformations separately. Indeed, if we would have done this our $\sigma$ would be $-n+\frac{1}{2}\tau$ instead of $-n+\tau$ making it more compatible with both the real and quarternionic cases. In the next section, when we consider the corresponding quaternionic case, it will be expedient to work in terms of real components due to the noncommutativity of the quaternions themselves. We shall indicate there the corresponding restrictions which yield the $sl(n,\mathbb{C})$, $sl(n,\mathbb{R})$, $u(n)$, and $so(n)$ subalgebras.

### C. $i_2 sp(n) \oplus sp(1) \Rightarrow sl(n,\mathbb{Q}) \oplus sp(1)$

Since the symplectic algebra $sp(n)$ is the metric-preserving algebra for the $n$-dimensional quaternionic plane[18] $\mathbb{Q}^n$, it seems natural to carry the analogy with the last two sections one step further and look for the corresponding deformation to essentially $sl(n,\mathbb{Q})$, the special linear algebra over the noncommutative quaternionic field[14] (continuous division ring) $\mathbb{Q}$. Since the quaternions are perhaps not so well known, we present first a brief review of their properties[19].

The quaternions form a four-dimensional noncommutative algebra over the field of real numbers with a base composed of $\mathbf{e}_\alpha$ ($\alpha = 0,1,2,3$) whose multiplication table is

$$\mathbf{e}_0\mathbf{e}_i = \mathbf{e}_i\mathbf{e}_0 = \mathbf{e}_i, \quad \mathbf{e}_i{}^2 = -\mathbf{e}_0, \mathbf{e}_i\mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k, \tag{2.13}$$

where $i,j,k = 1,2,3$. We use the convention that the early Greek letters range from 0 to 3, whereas the middle Latin letters over 1, 2, 3, reserving the middle Greek letters for the tensor indices. A quaternion can thus be written as $\mathbf{q} = q^\alpha \mathbf{e}_\alpha$. The quaternionic conjugate is defined as $\mathbf{q}^* \equiv q^0\mathbf{e}_0 - q^i\mathbf{e}_i$ and one verifies that $|\mathbf{q}|^2 \equiv \mathbf{q}^*\mathbf{q} = \mathbf{q}\mathbf{q}^* = q^\alpha q^\alpha$ is a real nonnegative number which vanishes iff $\mathbf{q}$ itself vanishes. We can form the quaternionic $n$ plane $\mathbb{Q}^n$ by taking the $n$-fold tensor product of $\mathbb{Q}$, which forms a vector space endowed with a scalar product $\mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Q}$ given by $\mathbf{u}\cdot\mathbf{q} \equiv \mathbf{u}^*_\mu \mathbf{q}_\mu$. The norm induced by this scalar product is $|\mathbf{q}|^2 \equiv \mathbf{q}\cdot\mathbf{q} = \mathbf{q}^*_\mu\mathbf{q}_\mu = \mathbf{q}_\mu\mathbf{q}^*_\mu = q^\alpha_\mu q^\alpha_\mu$. The scalar product defined above is left invariant by the group of $n$-dimensional symplectic transformations whose infinitesimal generators $M^\alpha_{\mu\nu}$ can be given in two different ways depending on whether the group action is defined from the right or from the left. This will be detailed in Sec. 3. An arbitrary second-rank mixed tensor with components $Q^\alpha_{\mu\nu}$ transforms under the generators of $sp(n)$ as

$$[M^0_{\mu\nu}, Q^\alpha_{\rho\sigma}] = \delta_{\nu\rho}Q^\alpha_{\mu\sigma} - \delta_{\mu\rho}Q^\alpha_{\nu\sigma} + \delta_{\nu\sigma}Q^\alpha_{\rho\mu} - \delta_{\mu\sigma}Q^\alpha_{\rho\nu}, \tag{2.14a}$$

$$[M^i_{\mu\nu}, Q^0_{\rho\sigma}] = \delta_{\nu\rho}Q^i_{\mu\sigma} - \delta_{\mu\rho}Q^i_{\nu\sigma} - \delta_{\nu\sigma}Q^i_{\rho\mu} + \delta_{\mu\sigma}Q^i_{\rho\nu}, \tag{2.14b}$$

$$[M^i_{\mu\nu}, Q^i_{\rho\sigma}] = -\delta_{\nu\rho}Q^0_{\mu\sigma} - \delta_{\mu\rho}Q^0_{\nu\sigma} + \delta_{\nu\sigma}Q^0_{\rho\mu} + \delta_{\mu\sigma}Q^0_{\rho\nu} \quad \text{(no sum on } i) \tag{2.14c}$$

$$[M^i_{\mu\nu}, Q^j_{\rho\sigma}] = \epsilon_{ijk}(\delta_{\nu\rho}Q^k_{\mu\sigma} + \delta_{\mu\rho}Q^k_{\nu\sigma} + \delta_{\nu\sigma}Q^k_{\rho\mu} + \delta_{\mu\sigma}Q^k_{\rho\nu}). \tag{2.14d}$$

The symplectic algebra[14] $sp(n)$ spanned by the $M$'s satisfies (2.14) with the additional specification that $M^0_{\mu\nu} = -M^0_{\nu\mu}$ and $M^i_{\mu\nu} = M^i_{\nu\mu}$. We can realize this algebra[15] on $\mathbb{Q}^n$ as

$$M^0_{\mu\nu} = q^\alpha_\mu \partial^\alpha_\nu - q^\alpha_\nu \partial^\alpha_\mu, \tag{2.15a}$$

$$M^{i\pm}_{\mu\nu} = \pm(q^i_\mu\partial^0_\nu + q^i_\nu\partial^0_\mu - q^0_\mu\partial^i_\nu - q^0_\nu\partial^i_\mu) - \epsilon_{ijk}(q^j_\mu\partial^k_\nu + q^j_\nu\partial^k_\mu), \tag{2.15b}$$

where $\partial^\alpha_\mu \equiv \partial/\partial q^\alpha_\mu$ and where (+) or (−) designates action from the left or right with respect to quaternionic multiplication. One sees then that $\mathbf{q}_\rho\mathbf{q}^*_\sigma$ transforms as a mixed second-rank tensor under the $M^+$'s but not under the $M^-$'s. Conversely, the quaternionic conjugate $\mathbf{q}^*_\sigma\mathbf{q}_\rho$ transforms as a mixed second-rank tensor under the $M^-$'s only. Furthermore, the commutator of the $M^+$'s with the $M^-$'s does not close to form a Lie algebra of finite dimension. It can be seen, however, that the traces

$$M^{i\mp} \equiv \tfrac{1}{4}M^{i\mp}_{\mu\mu} = \pm\tfrac{1}{2}(q^0_\mu\partial^i_\mu - q^i_\mu\partial^0_\mu) - \tfrac{1}{2}\epsilon_{ijk}q^j_\mu\partial^k_\mu, \tag{2.15c}$$

commute with the $M^{i\pm}_{\mu\nu}$'s, each forming the algebra $sp(1)^\mp \cong su(2)$. Moreover, $M^{i-}$ commutes with $\mathbf{q}_\rho\mathbf{q}^*_\sigma$ and $M^{i+}$ commutes with $\mathbf{q}^*_\sigma\mathbf{q}_\rho$. Hence, we finally arrive at two possible isomorphic algebras which we denote as $i_2sp(n)^+ \oplus sp(1)^-$ and $i_2sp(n)^- \oplus sp(1)^+$ spanned by $\{M^{\alpha+}_{\mu\nu}, M^{i-}, \mathbf{q}_\mu\mathbf{q}^*_\nu\}$ and $\{M^{\alpha-}_{\mu\nu}, M^{i+}, \mathbf{q}^*_\mu\mathbf{q}_\nu\}$, respectively.

Using either of these algebras we are now in a position to write down the deformation formulas which are a generalization of Eqs. (2.3) and (2.9) to the quaternionic case. We use the first set of operators spanning $i_2sp(n)^+ \oplus sp(1)^-$ and consider

$$N^\alpha_{\mu\nu} \equiv \tfrac{1}{2}[\Omega^+, (\mathbf{q}_\nu\mathbf{q}^*_\mu)^\alpha/|\mathbf{q}|^2] + \tau(\mathbf{q}_\nu\mathbf{q}^*_\mu)^\alpha/|\mathbf{q}|^2, \tag{2.16}$$

where $\Omega^\pm = -\frac{1}{2}M^0_{\mu\nu}M^0_{\nu\mu} + \frac{1}{2}M^{i\pm}_{\mu\nu}M^{i\pm}_{\nu\mu}$ is the second-order Casimir invariant for $sp(n)^\pm$. If we consider the combinations

$$X^0_{\mu\nu} \equiv \frac{1}{2}(M^0_{\mu\nu} + N^0_{\mu\nu}) - \frac{1}{2n}\delta_{\mu\nu}\,\mathrm{Tr}N^0, \tag{2.17a}$$

$$X^i_{\mu\nu} \equiv \tfrac{1}{2}(M^i_{\mu\nu} + N^i_{\mu\nu}), \tag{2.17b}$$

where the $X^0$ are built to be traceless, and place $|\mathbf{q}|^2 = 1$, the $(n-1)$-dimensional quaternionic sphere $Q_{n-1} \cong S_{4n-1}$ where the $4n-1$ independent real $q^\alpha_\mu$ are the Cartesian coordinates, and set $\sigma = -2n + \frac{1}{2}\tau$, we arrive, after a fairly tedious calculation, at the explicit form

$$X^0_{\mu\nu} = q^\alpha_\mu\partial^\alpha_\nu - (\mathbf{q}_\nu\mathbf{q}^*_\mu)^0(\mathbf{q}\cdot\partial - \sigma) - \delta_{\mu\nu}\sigma/n, \tag{2.18a}$$

$$X^i_{\mu\nu} = q^i_\mu \partial^0_\nu - q^0_\mu \partial^i_\nu - \epsilon_{ijk} q^j_\mu \partial^k_\nu$$
$$- (\mathbf{q}_\nu \mathbf{q}^*_\mu)^i (\mathbf{q}\cdot\partial - \sigma), \qquad \mathbf{q}\cdot\partial \equiv q^\alpha_\mu \partial^\alpha_\mu. \qquad (2.18\text{b})$$

It can be checked that the $X$'s generate the Lie algebra[14]

$$[X^0_{\mu\nu}, X^\alpha_{\rho\sigma}] = \delta_{\nu\rho} X^\alpha_{\mu\sigma} - \delta_{\mu\sigma} X^\alpha_{\rho\nu}, \qquad (2.19\text{a})$$

$$[X^i_{\mu\nu}, X^i_{\rho\sigma}] = -\delta_{\nu\rho} X^0_{\mu\sigma} + \delta_{\mu\sigma} X^0_{\rho\nu}, \qquad (2.19\text{b})$$

$$[X^i_{\mu\nu}, X^j_{\rho\sigma}] = \epsilon_{ijk}(\delta_{\nu\rho} X^k_{\mu\sigma} + \delta_{\mu\sigma} X^k_{\rho\nu}), \qquad (2.19\text{c})$$

identified as $sl(n, \mathbb{Q})$.

It is not difficult to see that the $X^0$'s span an $sl(n, \mathbb{R})$ subalgebra while the $X^0$'s and $X^i$'s for one fixed $i$, span an $sl(n, \mathbb{C}) \oplus u(1)$ subalgebra. By taking the $X^\alpha_{\mu\nu}$'s given by Eq. (2.18) and the traces $M^{i-}$ in (2.15c), we obtain an algebra $sl(n, \mathbb{Q})^+ \oplus sp(1)^-$. It is easy to check that, indeed, $M^{i-}$ commutes with all the $X^{\alpha+}_{\mu\nu}$'s. Alternatively, we could have constructed the algebra $sl(n, \mathbb{Q})- \oplus sp(1)^+$ by starting from the $i_2 sp(n)^- \oplus sp(1)^+$ algebra. The net effect of this on Eq. (2.18) is to reverse the sign of the non-epsilon terms in the expression for the $X^i_{\mu\nu}$'s. The $sl(n, \mathbb{Q})$ structure of Eqs. (2.19) can be brought out by taking $X^0_{\mu\mu}$ and, say, $X^1_{\mu\mu}$ (no sum) as the Cartan subalgebra. The root vectors are then given by $X^0_{\mu\nu} \pm iX^1_{\mu\nu}$ and $X^2_{\mu\nu} \pm iX^3_{\mu\nu}$. This identifies the Cartan class $A_{2n-1}$.

The role played by the $sp(1)$ is analogous to that of $u(1)$ in the complex case. Both the set of generators $X^\alpha_{\mu\nu}$ and each term $|\mathbf{q}_\mu|^2$ of the quaternionic sphere are invariant under $sp(1)^-$. Again, definite $sp(1)^-$ transformation properties must be specified in order to get irreducible representations of $sl(n, \mathbb{Q})^+$. This will become clearer in the group theoretical context in the following section.

In the enveloping algebra of $sl(n, \mathbb{Q})$ we were able to derive one analog of Eqs. (2.7) and (2.12)

$$X^0_{\mu\lambda} X^0_{\lambda\nu} - X^{i+}_{\mu\lambda} X^{i+}_{\lambda\nu} = -X^{i-}_{\mu\nu} M^{i-} + \left[4(n-1) + \frac{n-2}{n}\right] X^0_{\mu\nu} + \frac{n-1}{n^2}\sigma(\sigma + 4n)\delta_{\mu\nu}. \qquad (2.20)$$

It seems, however, that this relation is not by itself sufficient to reduce all higher-order Casimir operators to the second-order one. Indeed, we expect more non-independent Casimir invariants due to the existence of the $sp(1)^-$ algebra. These invariants will be of higher order than second, since in contrast to the $sl(n, \mathbb{C})$ case, $sl(n, \mathbb{Q})$ is a simple real Lie algebra. Due to the complexity in deriving such relations, however, we have thus far been unable to find them. Relation (2.20) does provide the second-order Casimir invariant $C_2 \equiv X^0_{\mu\nu} X^0_{\nu\mu} - X^{i+}_{\mu\nu} X^{i+}_{\nu\mu} = -M^{i-}M^{i-} + (n-1)\sigma(\sigma + 4n)/n$.

The $sp(1)^-$ invariant $(M^-)^2 \equiv M^{i-}M^{i-}$ can be chosen to define a basis where its eigenvalues are $l(l+1)$ ($l$ integer on half-integer). In the next section we shall introduce a definite scalar product on $Q_{n-1}$, with respect to which all the operators used in this section are anti-Hermitian if we choose $\sigma = -2n + i\rho$ ($\rho$ real). For these values of $\sigma$, the eigenvalue of the Casimir invariant $C_2$ is the real number $-l(l+1) - (n-1)(4n^2 + \rho^2)/n$.

## 3. HOMOGENEOUS FUNCTIONS AND MULTIPLIER REPRESENTATIONS

In this section we shall relate the expressions obtained in the previous sections by the deformation of inhomogeneous algebras to the corresponding technique of constructing multiplier representations for the group from certain classes of homogeneous functions similar to those introduced by Bargmann[10] and Gel'fand.[11] Indeed, from the expressions for the generators given by Eqs. (2.5), (2.11), and (2.18), the terms in $\sigma$ indicate that they should upon integration give rise to multiplier representations.[8] Rather than exponentiate these expressions directly, however, we prefer to construct the multipliers by Gel'fand's method of homogeneous functions[11] and obtain the corresponding generators for the one-parameter subgroups. This procedure lends insight into the expansions of the form (2.3) on the global group level. All the known expansions of the form (2.3) display this correspondence to homogeneous functions.[8]

It is not difficult to see that the spheres $S_{n-1}, C_{n-1}$, and $Q_{n-1}$ introduced in the last section correspond to homogeneous spaces of the groups $SL(n, \mathbb{F})$ of $n \times n$ matrices $G = \|g_{\mu\nu}\|$, $(\mu, \nu = 1, \ldots, n)$, $\det G = 1, g_{\mu\nu} \in \mathbb{F}$, where $\mathbb{F}$ indicates the real, complex and quaternion fields. Indeed, consider the Iwasawa decomposition[4] of $SL(n, \mathbb{F}) = KAN$ where $K$ is $SO(n), SU(n)$, and $Sp(n)$, respectively, $A$ is the $(n-1)$-dimensional Abelian subgroup of diagonal matrices of $SL(n, \mathbb{F})$, and $N$ is the nilpotent subgroup of lower-triangular matrices. Then in each case, if $K'$ is the canonical subgroup $SO(n-1)$, $SU(n-1)$, and $Sp(n-1)$, respectively. $K'AN$ is the subgroup of $n \times n$ matrices $G' = \|g'_{\mu\nu}\|$ such that the elements $g'_{in} = 0$ $(i = 1, \ldots, n-1)$, and $\det G' = 1$. The homogeneous spaces $\Omega \equiv KAN/K'AN$ are then, respectively, the spheres $S_{n-1}, C_{n-1}$, and $Q_{n-1}$. The action of the group element $g \in G$ on the Cartesian coordinates $s_\mu (\mu = 1, \ldots, n)$, $s_\mu \in \mathbb{F}$, $s^*_\mu s_{\mu=1}$ from the *left* is given by

$$s_\mu \xrightarrow{g^L} s'_\mu = \frac{r}{r'} g^{-1}_{\mu\nu} s_\nu, \qquad \frac{r}{r'} = [s^*_\lambda g^{-1*}_{\rho\lambda} g^{-1}_{\rho\sigma} s_\sigma]^{-1/2}, \qquad (3.1\text{a})$$

and if the field $\mathbb{F}$ is $\mathbb{Q}$ we can also have a distinct action from the *right* as given by

$$s_\mu \xrightarrow{g^R} s''_\mu = \frac{r}{r''} s_\nu g^{-1*}_{\mu\nu}, \qquad \frac{r}{r''} = [g^{-1}_{\rho\lambda} s^*_\lambda s_\sigma g^{-1*}_{\rho\sigma}]^{-1/2}, \qquad (3.1\text{b})$$

where it should be understood that the involutive automorphism $*: s_\mu \to s_\mu{}^*$ is the identity for $\mathbb{F} = \mathbb{R}$, complex conjugation for $\mathbb{F} = \mathbb{C}$, and quaternionic conjugation for $\mathbb{F} = \mathbb{Q}$. The subgroup $G' = K'AN$ is then the stability group of the point $(s_\mu) = (0, \ldots, 0, 1)$ on $\Omega$. The transformations of $K$ [$SO(n), SU(n)$, and $Sp(n)$, respectively] are the largest group of *rigid* transformations of the sphere $\Omega$ since they leave the measure $d\Omega$ on the sphere invariant. The rest of the transformations $g \in G$ will produce a "*deformation*" of the surface of $\Omega$, where the Jacobian is

$$(d\Omega/d\Omega')^L = (r'/r)^p, \qquad (3.2\text{a})$$

$$(d\Omega/d\Omega'')^R = (r''/r)^p, \qquad (3.2\text{b})$$

where $p = n \dim\mathbb{F}$, i.e., $p = n, 2n$, and $4n$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{Q}$, respectively. In the former two cases, (3.2a) and (3.2b) are equal.

In comparing this approach with the one used for[4,8,20] $SO(n,1) \supset SO(n)$, $SU(n,1) \supset SU(n)$, and[21] $Sp(n,1) \supset Sp(n)$, we notice that there is one essential difference with the above and that is that in the case of these groups the subgroup $K'$ is the centralizer of $A$ and the normalizer of $N$ in $K$. This has the consequence that the irreducible representations of the subgroup $K'AN$ are just direct products of irreducible representations of $K'$ and the irreducible representations (characters) of $A$. Hence, one can induce all these representations to the full group $KAN$. In the case of $SL(n, \mathbb{F})$, however, $K'$ is no longer the centralizer of $A$ and only a "most degenerate" representation of $K'AN$ labeled by a character of $A$ can be induced[22] to irreducible representations of $SL(n, \mathbb{F})$. These are just the representations described in the previous sections by the deformation of the corresponding representations of the inhomogeneous Lie algebras.

### A. $SL(n, \mathbb{R})$

Consider the space of homogeneous functions over the $n$-dimensional real plane $\mathbb{R}^n$ which are infinitely differentiable (except possibly at the origin) and satisfy[11,23]

$$F(ay_\mu) = a^\sigma \operatorname{sgn}^\epsilon a F(y_\mu), \tag{3.3}$$

where $a, y_\mu \in \mathbb{R}$ and $\epsilon = 0, 1$. Now representations of $SL(n, \mathbb{R})$ can be constructed over this space as representations by left action

$$\tilde{T}_g F(y_\mu) = F(y'_\mu) = F(g^{-1}_{\mu\nu} y_\nu), \quad g \in SL(n, \mathbb{R}). \tag{3.4}$$

Since the functions $F$ satisfy (3.3), i.e., are homogeneous functions, the representation (3.4) gives rise to a representation on the unit sphere $S_{n-1}$ in the following way: From (3.3) we see that we can define a function on $S_{n-1}$ through $F(y_\mu) = r^\sigma f(x_\mu)$, with $x_\mu \in S_{n-1}$ and $r \geqslant 0$. A simple calculation shows that (3.4) induces the representation

$$T^\sigma_g f(x_\mu) = (r'/r)^\sigma f(x'_\mu) \tag{3.5}$$

over functions on $S_{n-1}$, where $T^\sigma_g = r^{-\sigma}\tilde{T}_g r^\sigma$ and the group action is given by (3.1a). Furthermore, from (3.4) and the infinite differentiability of the $F$'s, it follows that the $f$'s span the space $\mathfrak{D}^\epsilon$ of infinitely differentiable functions on $S_{n-1}$ which satisfy

$$f(-x_\mu) = (-i)^\epsilon f(x_\mu). \tag{3.6}$$

The function $(r'/r)^\sigma$ is a *multiplier* which trivially satisfies the condition[10,11] $(r''/r')^\sigma (r'/r)^\sigma = (r''/r)^\sigma$ and hence Eq. (3.5) is indeed a representation of $SL(n, \mathbb{R})$.

We obtain the infinitesimal generators of $SL(n, \mathbb{R})$ by considering the one-parameter subgroups $g_{\mu\nu}(t)$ which to first order are $g_{\nu\mu}(t) \simeq \delta_{\nu\mu} - t\alpha_{\nu\mu}$, $T^\sigma_g \simeq 1 + t\alpha_{\nu\mu} X_{\mu\nu}$, where we leave implicit the dependence of $X_{\mu\nu}$ on $\sigma$. As it is well known that demanding $\det\|g_{\mu\nu}\| = 1$ imposes the tracelessness conditions on the generators. We can use (3.1a) and (3.5) to arrive exactly at the generators (2.5) of $sl(n, \mathbb{R})$ obtained in Sec. 2A.

One can obtain unitary representations of $SL(n, \mathbb{R})$ over $S_{n-1}$ taking the vector spaces $\mathfrak{D}^\epsilon$, $\epsilon = 0, 1$, and completing them with respect to the norm induced by the inner product

$$(f^\epsilon_1, f^\epsilon_2)_S = \int d\Omega(x) \overline{f^\epsilon_1(x)} f^\epsilon_2(x), \tag{3.7}$$

where $d\Omega(x)$ is the $SO(n)$-invariant measure on $S_{n-1}$. Then one can see that the representation (3.6) is unitary with respect to (3.7) with $\sigma = -\frac{1}{2}n + i\rho$ ($\rho$ real). The multiplier in (3.5) is just what is needed to offset the transformation (3.2a) of the measure $d\Omega(x)$ under $SL(n, \mathbb{R})$.

### B. $SL(n, \mathbb{C})$

Consider the space of functions $F(\omega_\mu)$ over the complex $n$ plane $\mathbb{C}^n$, infinitely differentiable in $\omega_\mu$ and $\overline{\omega}_\mu$ (except possibly at the origin), which satisfy[11,24]

$$F(a\omega_\mu) = a^{\sigma_1} \bar{a}^{\sigma_2} F(\omega_\mu), \tag{3.8}$$

where $a, \omega_\mu, \sigma_1, \sigma_2 \in \mathbb{C}$. Furthermore, we note that $F(e^{i\psi}\omega_\mu) = \exp[i(\sigma_1 - \sigma_2)\psi] F(\omega_\mu)$, thus providing a representation of $U(1)$. Requiring this representation to be single-valued implies that $\sigma_1 - \sigma_2 = m$ is an integer. Then the functions $F$ are said to be *homogeneous* of degree $(\sigma, m)$ where $\sigma = \sigma_1 + \sigma_2$. Now representations of $SL(n, \mathbb{C})$ can be constructed through left action as

$$\tilde{T}_g F(\omega_\mu) = F(\omega'_\mu) = F(g^{-1}_{\mu\nu}\omega_\nu), \quad g \in SL(n, \mathbb{C}). \tag{3.9}$$

The homogeneity of the functions $F(\omega)$ allows us to construct functions over $C_{n-1}$ as $F(\omega_\mu) = r^\sigma f(z_\mu)$ with $z \in C_{n-1}, r \geqslant 0$. The representation (3.9) induces the multiplier representation

$$T^\sigma_g f(z_\mu) = (r'/r)^\sigma f(z'_\mu) \tag{3.10}$$

over functions on $C_{n-1}$, where $T^\sigma_g = r^{-\sigma}\tilde{T}_g r^\sigma$ and (3.1a) for $C_{n-1}$. It then follows that the functions $f(z)$ are infinitely differentiable in $z_\mu$ and $\bar{z}_\mu$ with the auxiliary condition

$$f(e^{i\psi}z_\mu) = e^{im\psi} f(z_\mu). \tag{3.11}$$

We denote this space of functions as $\mathfrak{D}^m$. Actually (3.11) defines a representation of the $U(1)$ subgroup of $SL(n, \mathbb{C}) \otimes U(1)$ as $T_\mu f(z_\mu) = f(e^{i\psi}z_\mu) = e^{im\psi} f(z_\mu)$.

In the same way as in the preceding section, the infinitesimal generators of both the representation of $SL(n, \mathbb{C})$ (3.10) and $U(1)$ (3.11) can be found with the parametrization

$$u \simeq 1 - i\psi, \quad T_u \simeq 1 + i\psi C,$$

$$g_{\nu\mu}(t) \simeq \delta_{\nu\mu} - t\alpha_{\nu\mu}, \quad T^\sigma_g \simeq 1 + t(\alpha_{\nu\mu} X^+_{\mu\nu} - \overline{\alpha_{\nu\mu}} X^-_{\mu\nu}),$$

By imposing the condition of tracelessness on these generators, we arrive at the expressions for $X^\pm_{\mu\nu}$ given by Eq. (2.11).

We endow the spaces $\mathfrak{D}^m$ with a Hilbert space structure by completion with respect to the norm induced by the inner product

$$(f^m_1, f^m_2)_C = \int d\Omega(z) \overline{f^m_1(z)} f^m_2(z), \tag{3.12}$$

where $d\Omega(z)$ is the $U(n)$-invariant measure on $C_{n-1}$. It follows that the representation (3.10) will be unitary with respect to the inner product (3.12) if we choose $\sigma = -n + i\rho$ ($\rho$ real), since the multiplier just cancels the change in the measure (3.2a).

### C. $SL(n, \mathbb{Q})$

The description of $SL(n, \mathbb{Q})$ follows those of $SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$, the major difference now being that the multiplication of quaternions in the representation can be taken from the left or right, giving rise to two different realizations of $SL(n, \mathbb{Q})$. Let $F(\mathbf{u}_\mu)$ be an infinitely differentiable complex-valued function on the quaternionic $n$ plane $\mathbb{Q}^n$ (except possibly at the origin). We can define representations by left and right group action as

$$\tilde{T}^L_g F(\mathbf{u}_\mu) = F(\mathbf{g}^{-1}_{\mu\nu}\mathbf{u}_\nu), \tag{3.13a}$$

$$\tilde{T}^R_g F(\mathbf{u}_\mu) = F(\mathbf{u}_\nu \mathbf{g}^{-1*}_{\mu\nu}), \tag{3.13b}$$

with $\mathbf{u}_\mu \in \mathbb{Q}$, $g \in SL(n, \mathbb{Q})$. Notice that we always have left multiplication with respect to the tensor indices. As in the previous cases, we want to restrict our class of functions $F$ to be homogeneous functions of $\mathbb{Q}^n$ in some sense. Due to the quaternion noncommutativity, there is an ambiguity in factoring out quaternions as done in Eq. (3.8) for $\mathbb{C}$. We thus consider "*homogeneous*" functions (in an expanded sense) of degree $(\sigma, l, m)$ which satisfy

$$F^l_m(a\mathbf{u}_\mu) = a^\sigma F^l_m(\mathbf{u}_\mu), \qquad a \in \mathbb{R}, \tag{3.14a}$$

$$F^l_m(\mathbf{u}_\mu \mathbf{s}) = \sum_{m'} F^l_{m'}(\mathbf{u}_\mu) D^l_{m'm}(\mathbf{s}(\alpha,\beta,\gamma)), \tag{3.14b}$$

where we have used the familiar Wigner $D$ function for $SU(2) \cong Sp(1)$, and $\mathbf{s}$ is a unit quaternion $|\mathbf{s}|^2 = 1$, parametrized by Euler angles $\mathbf{s}(\alpha,\beta,\gamma) = \exp(\mathbf{e}_3\alpha)\exp(\mathbf{e}_2\beta)\exp(\mathbf{e}_3\gamma)$.

We recognize that in order to write an equations such as (3.14b) we must consider vector-valued functions $F^l(\mathbf{u}_\mu)$ on $\mathbb{Q}^n$ of degree $l$. There is an expression analogous to (3.14b) obtained by multiplication from the left. As in previous sections, we construct functions $f$ on $Q_{n-1}$ through $F^l_m(\mathbf{u}_\mu) = r^\sigma f^l_m(\mathbf{q}_\mu)$, $\mathbf{q} \in Q_{n-1}$. These will constitute the space $\mathfrak{D}^{l,m}$ of infinitely differentiable functions over $Q_{n-1}$. We then construct a multiplier representation of $SL(n, \mathbb{Q})^L$ on $\mathfrak{D}^{l,m}$ as

$$T^{L\sigma}_g f^l_m(\mathbf{q}_\mu) = (r'/r)^\sigma f^l_m(\mathbf{q}'_\mu), \tag{3.15}$$

where $T^{L\sigma}_g = r^{-\sigma} \tilde{T}^L_g r^\sigma$ and $\mathbf{q}'_\mu$ is found in (3.1a). There is an expression similar to (3.17) for $T^R_g$ by using (3.1b). Now Eq. (3.14b) simply becomes

$$f^l_m(\mathbf{q}_\mu \mathbf{s}) = \sum_{m'} f^l_{m'}(\mathbf{q}_\mu) D^l_{m'm}(\mathbf{s}). \tag{3.16}$$

Notice that $\mathfrak{D}^{lm}$ is not invariant under $Sp(1)$. Indeed, it is seen that the $f^l_m$ transform as the components of a rank $l$ spherical tensor under $Sp(1)^R$ acting from the right. This equation also defines a representation of $Sp(1)^R$ by right action, i.e., $T^R_{\mathbf{s}} f^l_m(\mathbf{q}_\mu) = f^l_m(\mathbf{q}_\mu \mathbf{s})$. It can be shown easily that this action and (3.15) commute, leading to the structure $SL(n, \mathbb{Q})^L \otimes Sp(1)^R$. It is clear that one can similarly construct $SL(n, \mathbb{Q})^R \otimes Sp(1)^L$.

One can then obtain the infinitesimal generators by using the parametrization $\mathbf{g}_{\mu\nu} = \delta_{\mu\nu}\mathbf{e}_0 - t\boldsymbol{\alpha}_{\mu\nu}$ and imposing the condition of tracelessness on the $\alpha^0_{\mu\nu}$ term to arrive at the generators given in (2.15). We can similarly obtain the infinitesimal generators of $T^R_g$ by reversing the sign of all non-epsilon terms in the coefficient of $\alpha^i_{\mu\nu}$. Also from Eq. (3.16) we can obtain the generators of $Sp(1)^R$; they are the traces $M^{i-}$ in (2.15c).

We introduce the Hilbert space structure by completing $\mathfrak{D}^{lm}$ with respect to the norm induced by the inner product

$$(f_1{}^l_m, f_2{}^l_m)_Q = \int d\Omega(\mathbf{q}) \overline{f_1{}^l_m(\mathbf{q})} f_2{}^l_m(\mathbf{q}), \tag{3.17}$$

where $\mathbf{q} \in Q_{n-1}$ and $d\Omega(\mathbf{q})$ is the $Sp(n)^L \otimes Sp(1)^R$ invariant measure on $Q_{n-1}$. Notice that there is no sum over $m$ here since the space $\mathfrak{D}^{l,m}$ is invariant under the representation (3.15) of $SL(n, \mathbb{Q})^L$. This representation is unitary if we choose $\sigma = -2n + i\rho$ due to the $SL(n, \mathbb{Q})$ transformation of $d\Omega(\mathbf{q})$ in (3.2a).

Also it can be seen that the representation (3.16) of $Sp(1)^R$ is unitary for $l$ integer or half-integer, upon introduction of the usual vector space inner product

$$(f^l_1, f^l_2)(\mathbf{q}) = \sum_m \overline{f_1{}^l_m(\mathbf{q})} f_2{}^l_m(\mathbf{q}). \tag{3.18}$$

## 4. CONTRACTIONS OF REPRESENTATIONS

### A. Of the algebra

The contraction[12] of the previous representations of the Lie algebras $sl(n, \mathbb{F}) \oplus a(\mathbb{F})$ is to the algebras $i_2k(\mathbb{F}) \oplus a(\mathbb{F})$. We will adopt the notation used in the beginning of Sec. 3 and treat the three cases $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{Q}$ together, and hence let $k(\mathbb{F})$ denote, respectively, $so(n), u(n)$, and $sp(n)$, while $a(\mathbb{F})$ denotes $0, u(1)$, and $sp(1)$. The generators of $k(\mathbb{F})$ are $M^\alpha_{\mu\nu}$, where $\alpha = 0$ for $\mathbb{F} = \mathbb{R}$; $\alpha = 0, 1$ for $\mathbb{F} = \mathbb{C}$; and $\alpha = 0, 1, 2, 3$ for $\mathbb{F} = \mathbb{Q}$; the remaining generators are

$$N^\alpha_{\mu\nu} = \tfrac{1}{2}[\Omega, (s_\nu, s^*_\mu)^\alpha] + \tau(s_\nu s^*_\mu)^\alpha \tag{4.1}$$

with the same ranges of $\alpha$. In order to perform the contraction one considers the generators $N^\alpha_{\mu\nu}/\tau$ as spanning along with the $M^\alpha_{\mu\nu}$ a sequence of representations denoted by $sl(n, \mathbb{F})_\tau$. Upon taking the limit as $|\tau| \to \infty$, one finds

$$\lim_{|\tau|\to\infty} \frac{1}{\tau} N^\alpha_{\mu\nu} = (s_\nu s^*_\mu)^\alpha, \tag{4.2}$$

whence we write $sl(n, \mathbb{F})_\tau \xrightarrow{|\tau|\to\infty} i_2k(\mathbb{F})$. Equation (2.7), (2.12), and (2.20) become identities in the contraction limit. We note that no role is played by $a(\mathbb{F})$ in the contraction procedure. The deformation performed in Sec. 2 and the above contraction are inverse operation.[6] We make note that although our representations were built as deformations of $i_2k(\mathbb{F}) \oplus a(\mathbb{F})$, they can also be viewed as expansions of the inhomogeneous algebra $ik(\mathbb{F}) \oplus a(\mathbb{F})$.

### B. Of the group

The contraction of the corresponding group representations (3.5), (3.10), and (3.15) proceeds in the standard way[8,12] by allowing the group transformation $g(t)$ to approach the identity ($t = 0$) as we let $\rho \to \infty$ in the sequence of representations $SL(n, \mathbb{F})_{i\rho}$ in such a way that $t\rho = \xi$, a real constant. Thus we see from (3.1) that $s'_\mu \xrightarrow{t\to\infty} s_\mu$ and

$$\lim\left(\frac{r'}{r}\right)^{-p/2+i\rho} = \lim\left(1 + \frac{\xi}{\rho} s^*_\nu \frac{\alpha^*_{\mu\nu} + \alpha_{\nu\mu}}{2} s_\mu\right)^{i\rho}$$
$$= \exp[i\xi\alpha^\alpha_{\nu\mu}(s_\nu s^*_\mu)^\alpha], \tag{4.3}$$

where the same remarks for the cases $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{Q}$ apply. It should be noticed that only the symmetric part of $\alpha^0_{\nu\mu}$ and the antisymmetric one of $\alpha^i_{\nu\mu}$ contribute to the multiplier; the representations therefore contract as

$$\lim T^\sigma_g f(s_\mu) = \exp[i\xi\alpha^\alpha_{\nu\mu}(s_\nu s^*_\mu)^\alpha] f(s_\mu), \tag{4.4}$$

showing that only the "boost" group elements generated by (4.1) have a finite contraction limit. We thus have found the representations built in Sec. 3 to contract as $SL(n, \mathbb{R}) \to I_2SO(n)$, $SL(n, \mathbb{C}) \otimes U(1) \to I_2SU(n) \otimes U(1)$, and $SL(n, \mathbb{Q}) \otimes Sp(1) \to I_2Sp(n) \otimes Sp(1)$.

## 5. CONCLUSION

We have exhibited deformations of inhomogeneizations of all the classical Cartan Lie algebras to those of the linear groups. These deformations are representation-dependent in that the procedure can be implemented only for representations which can be realized on rank one homogeneous spaces. While we have not been able to provide an 'if and only if' statement of this fact, we believe that we have indeed constructed the most general representations from our deformation procedure. Thus while our family of deformations falls outside the class studied by Gilmore[9] (i.e., rank one coset space in the Cartan decomposition), it does so only in a mild way, since a rank one homogeneous space is involved. These representations are precisely those which do not exhibit multiplicity problems when reduced to the maximum compact subgroup. This supports a conjecture by Mukunda[25] and Hermann[4] that the ability to use the deformation or expansion algorithm is intimately connected with the nonexistence of multiplicity problems.

On the group level, it was shown that these deformations are related to multiplier representations and "deformations" of the homogeneous space. In this regard, there seems to be a need to establish a more thoroughgoing connection between the infinitesimal and global approaches. A related approach is to perform the deformations not merely on homogeneous spaces, but on the whole group manifold.[21] In this way, cases where multiplicity problems appear might be incorporated.

The multiplier representations discussed in this paper can be used to calculate the finite group element representation matrix elements[8] in the basis obtained by the canonical decomposition $SL(n, \mathbb{F}) \supset K(\mathbb{F})$ [$K = SO(n), SU(n)$, or $Sp(n)$]. Although we only explicitly constructed a principal series, other series (e.g., supplementary and discrete) should be obtainable by allowing a nonlocal measure[10,11] as has been done[26] for $SO(n, 1) \supset SO(n)$. It is also to be remarked that other noncompact chains can be discussed as well through our deformation procedure and multiplier representations implemented on hyperboloids $s_1 s_1^* + \cdots + s_k s_k^* - s_{k+1} s_{k+1}^* - \cdots - s_n s_n^* = 1$, as well as spheres. This would allow one to discuss such decompositions as $SL(n, \mathbb{R}) \supset SO(n-k, k)$, $SL(n, \mathbb{C}) \supset SU(n-k, k)$, and $SL(n, \mathbb{Q}) \supset Sp(n-k, k)$, $(k = 1, \ldots, n-1)$, without multiplicity problems beyond the doubling encountered in the reduction[8] $SO(n, 1) \supset SO(n-1, 1)$.

In conclusion it can be said that our realizations for $n = 2, SL(2, \mathbb{R}) \overset{2-1}{\cong} SO(2, 1)$, $SL(2, \mathbb{C}) \overset{2-1}{\cong} SO(3, 1)$ yield all the principal series representations and reproduce the known results on these groups by Bargmann[10] and Gel'fand and collaborators.[11] One can use similar procedures to discuss the representations of $SL(2, \mathbb{Q}) \overset{2-1}{\cong} SO(5, 1)$.

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