Canonical transformations and accidental degeneracy. II. The isotropic oscillator in a sector

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In this paper we discuss the accidental degeneracy in the problem of a particle in two dimensional oscillator potential constrained to move in a sector of angle $\pi/q, q$ integer. The degeneracy is due to both the Hamiltonian and the boundary conditions. The symmetry Lie group of canonical transformations is suggested by the explicit form of a complete nonorthonormal set of states expressed in terms of the creation operators. This group is complex and the corresponding representation in quantum mechanics is nonunitary. We discuss briefly the appearance of complex canonical transformations in physical problems.

1. INTRODUCTION

In the preceeding paper[1] we analyzed the symmetry groups of canonical transformations responsible for the accidental degeneracy of the anisotropic oscillators whose ratio of frequencies was rational. From the discussion of these problems we arrived at some general conclusions for the determination of the groups. There are, however, other problems in which accidental degeneracy is present which seem to require a different type of approach. One of these problems is the motion of a particle in a two dimensional configuration space under the action of an harmonic oscillator potential, but restricted to a sector of the plane of angle π/q, where q is a positive integer. This sector is drawn in Fig. 1 for $q = 3$ and the heavy lines indicate the infinite potential barriers that limit it. We shall analyze this problem in the present paper both because of its intrinsic interest and the insight it provides into the general problem of accidental degeneracy.

The classical trajectory is very easy to draw. The particle under an oscillator potential moving unconstrained in the full plane will have an elliptical trajectory centered at the origin of the potential. We can draw this trajectory on a transparent plastic napkin. Then folding the napkin in such a way that it sustains an angle π/q, we immediately see the orbit of the particle as modified by the barriers at the boundary of the sector. This orbit is periodic and nonergodic,[2] i.e., it does not fill all the phase space surface of constant energy. It is drawn in Fig. 1 for $q = 3$, where we also show the reflection of the orbit as if the barriers were mirrors.

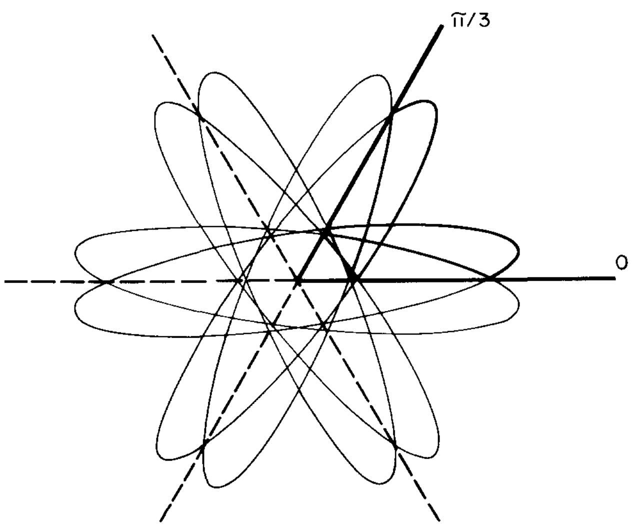

FIG. 1. Classical trajectory of a particle (bold lines) subject to an harmonic oscillator potential and restricted to a sector $\pi/3$ in the plane.

Does the corresponding quantum mechanical problem have accidental degeneracy? In polar coordinates r, φ the Schrödinger equation (in which $\hbar$, the mass of the particle and the frequency of the oscillator are taken as 1) has solutions, subject to the condition that the wave function vanishes at $\varphi = 0, \pi/q$, of the form[3]

$$\langle r\varphi \mid \nu_1\nu_2\rangle = [2(\nu_1 !)[(\nu_1 + \nu_2 q + q)!]^{-1/2} \times r^{(\nu_2+1)q} L_{\nu_1}^{(\nu_2+1)q}(r^2)\, e^{-r^2/2}\, \pi^{-1/2} \sin[q(\nu_2+1)\varphi]. \quad (1.1)$$

Notice that the state (1.1) is normalized with respect to the surface element $rdrd\varphi$ over the whole plane. We denote the state in the full Dirac notation, though later when referring to it we shall abbreviate to the ket $|\nu_1\nu_2\rangle$; the $L_{\nu_1}^{(\nu_2+1)q}(r^2)$ are associated Laguerre polynomials[4] with ν_1, ν_2 being arbitrary nonnegative integers. We write the solution in terms $\nu_2 + 1$, rather than ν_2, so that the lowest energy state of this problem corresponds to $\nu_1 = \nu_2 = 0$. The eigenvalue of the Hamiltonian[3] for the state $|\nu_1\nu_2\rangle$ is

$$E_{\nu_1\nu_2} = 2\nu_1 + q\nu_2 + q + 1. \quad (1.2)$$

We now proceed to discuss separately the cases in which q is odd and even. In the first case we divide the set of states (1.1) into $2q$ subsets characterized by λ_1, λ_2 defined by

$$\nu_1 \equiv \lambda_1 \bmod q, \quad \lambda_1 = 0, 1, 2, \cdots, q-1, \quad (1.3a)$$

$$\nu_2 \equiv \lambda_2 \bmod 2, \quad \lambda_2 = 0, 1, \quad (1.3b)$$

which implies that we may write

$$\nu_1 = qn_1 + \lambda_1, \quad \nu_2 = 2n_2 + \lambda_2, \quad (1.3c)$$

where n_1, n_2 are nonnegative integers. The energy $E_{\nu_1\nu_2}$ of (1.2) satisfies the equation

$$(E_{\nu_1\nu_2} - q - 1)/(2q) = n_1 + n_2 + (\lambda_1/q) + (\lambda_2/2). \quad (1.4)$$

For q even we can write the energy (1.2) as

$$\tfrac{1}{2}(E_{\nu_1\nu_2} - q - 1) = \nu_1 + (q/2)\nu_2. \quad (1.5)$$

We then divide the set of states (1.1) into $q/2$ subsets characterized by

$$\nu_1 \equiv \lambda_1 \bmod(q/2), \quad \lambda_1 = 0, 1, \cdots, (q/2) - 1, \quad (1.6a)$$

$$\nu_2 \equiv \lambda_2 \bmod 1, \quad \lambda_2 = 0, \quad (1.6b)$$

which implies that we may write

$$\nu_1 = (q/2)n_1 + \lambda_1, \quad \nu_2 = n_2. \tag{1.6c}$$

Thus, for q even we have

$$(E_{\nu_1\nu_2} - q - 1)/q = n_1 + n_2 + (2\lambda_1/q). \tag{1.7}$$

For both q odd and even, states corresponding to different (λ_1, λ_2) have different energies, but for a given (λ_1, λ_2) and a fixed value $n_1 + n_2 = N$ we have states that are degenerate in the energy $N + 1$ times.

In so far as the energy spectrum is concerned and the degeneracy of the states, the problem with q odd looks very similar to an anisotropic oscillator[1] whose ratio of frequencies is $(k_2/k_1) = (2/q)$, while for q even the ratio is $(k_2/k_1) = (1/[q/2])$. As the spectrum of the isotropic oscillator appears in $2q$ (q odd) or $q/2$ (q even) copies, we suspect that the group responsible for the accidental degeneracy in the present problem can be derived from $SU(2)$ by some canonical transformation.[1] Unfortunately, we cannot obtain this $SU(2)$ group by mapping the Hamiltonian of our problem on an isotropic oscillator, as the restriction on the states comes both from the Hamiltonian in the Schrödinger equation and the boundary conditions at $\varphi = 0, \pi/q$. We seem to require, then, a completely new approach and one is suggested in the next section when we express the states of the oscillator in a sector in terms of creation and annihilation operators.

2. CREATION AND ANNIHILATION OPERATORS AND THE STATES OF THE OSCILLATOR IN A SECTOR

When dealing with the two dimensional isotropic quantum oscillator it is convenient to introduce the spherical components of coordinate and momenta by the definition

$$X_\pm = (1/\sqrt{2})(X_1 \pm iX_2), \quad P_\pm = (1/\sqrt{2})(P_1 \pm iP_2), \tag{2.1}$$

where $p_i = -i\partial/\partial x_i$. From them we can in turn construct the creation operators

$$\eta_\pm = (1/\sqrt{2})(X_\pm - iP_\pm), \tag{2.2}$$

which in polar coordinates, where $x_\pm = re^{\pm i\varphi}$, take the form

$$\eta_\pm = \tfrac{1}{2}\, e^{\pm i\varphi}\left(r - \frac{\partial}{\partial r} \mp \frac{i}{r}\,\frac{\partial}{\partial\varphi}\right). \tag{2.3}$$

We note the following symmetry properties of these operators: If we have a reflection across the $X_2 = 0$ line in the plane, i.e.,

$$\varphi \to -\varphi, \quad \text{then } \eta_\pm \to \eta_\mp. \tag{2.4a}$$

If we carry out a rotation by an angle π/q, i.e.,

$$\varphi \to \varphi + (\pi/q), \quad \text{then } \eta_\pm \to e^{\pm i\pi/q}\,\eta_\pm, \tag{2.4b}$$

and thus, in particular, we have that when

$$\varphi \to \varphi + (\pi/q), \quad \text{then } \eta_\pm^q \to -\eta_\pm^q \tag{2.4c}$$

The Hamiltonian of the two dimensional oscillator can now be written as

$$H = \eta_+\xi_+ + \eta_-\xi_- + 1, \tag{2.5}$$

where $\xi_\pm$ is the annihilation operator

$$\xi_\pm = \eta_\pm^\dagger = \frac{1}{\sqrt{2}}(X_\mp + iP_\mp) = \tfrac{1}{2}\, e^{\mp i\varphi}\left(r + \frac{\partial}{\partial r} \mp \frac{i}{r}\,\frac{\partial}{\partial\varphi}\right). \tag{2.6}$$

In terms of $\eta_\pm, \xi_\pm$ the angular momentum takes the form

$$L = X_1P_2 - X_2P_1 = \frac{1}{i}\frac{\partial}{\partial\varphi} = \eta_+\xi_+ - \eta_-\xi_-. \tag{2.7}$$

The state (1.1) is an eigenstate of the Hamiltonian (2.5) with eigenvalue (1.2) and of the square of the angular momentum L^2 with eigenvalue $(\nu_2 + 1)^2q^2$. Thus, it can also be written in terms of creation operators as

$$\langle r\varphi \mid \nu_1\nu_2\rangle = 2^{-1/2}\{[\nu_1 + (\nu_2 + 1)q]!\,\nu_1!\}^{-1/2} \times (\eta_+\eta_-)^{\nu_1}[\eta_+^{(\nu_2+1)q} - \eta_-^{(\nu_2+1)q}]\,|0\rangle, \tag{2.8}$$

where the symmetry properties (2.4) of $\eta_\pm$ guarantee that the wave function vanishes at $\varphi = 0, \pi/q$. The ket $|0\rangle$ is the ordinary ground state $\pi^{-1/2}\exp(-\frac{1}{2}r^2)$.

We note immediately one basic difference between the states (2.8) and those of (I.2.7) for the anisotropic oscillator. The latter can be written as

$$|n_1k_1 + \lambda_1, n_2k_2 + \lambda_2\rangle = [(n_1k_1 + \lambda_1)!\,(n_2k_2 + \lambda_2)!]^{-1/2} \times (\eta_1^{k_1})^{n_1}(\eta_2^{k_2})^{n_2}\eta_1^{\lambda_1}\eta_2^{\lambda_2}\,|0\rangle, \tag{2.9}$$

and thus almost immediately suggest the classical canonical transformation (I.3.3) [or its quantum mechanical version (I.4.2)] as, for example, the creation operator η_1' when applied to (2.9) transforms it into a state in which $n_1 \to n_1 + 1, n_2 \to n_2$.

The states (2.8) are differences of monomial products of creation operators and not just a single product of powers of basic operators as (2.9). We can though express our states in the latter form if we are willing to settle for a complete, linearly independent, but not orthonormal set of states. For this purpose let us write

$$\langle r\varphi \mid \nu_1\nu_2) = (\nu_1!\,\nu_2!)^{-1/2}(\eta_+\eta_-)^{\nu_1}(\eta_+^q + \eta_-^q)^{\nu_2}(\eta_+^q - \eta_-^q)\,|0\rangle, \tag{2.10}$$

where we use a round bracket for the ket $|\nu_1\nu_2)$ to distinguish the state from the one defined by $|\nu_1\nu_2\rangle$ in (2.8). As the polynomial in η_+, η_- appearing in (2.10) is homogeneous the ket $|\nu_1\nu_2)$ is an eigenstate of the Hamiltonian (2.5) with eigenvalues given by (1.2). It remains then to prove that it vanishes at $\varphi = 0, \pi/q$. We note from (2.4a) that under a change $\varphi \to -\varphi$, $|\nu_1\nu_2) \to -|\nu_1\nu_2)$ and thus $(r, 0 \mid \nu_1\nu_2) = 0$. Furthermore, from (2.4), for $\varphi \to \varphi + (\pi/q)$, $|\nu_1\nu_2) \to (-1)^{q(\nu_2+1)}|\nu_1\nu_2)$ so that

$$\langle r, \pi/q \mid \nu_1\nu_2) = (-1)^{q(\nu_2+1)}\langle r, 0 \mid \nu_1\nu_2) = 0. \tag{2.11}$$

The energy spectrum (1.2) was analyzed in the previous section and thus, again, we see that the states (2.10) are degenerate $N + 1$ times for a given (λ_1, λ_2), if $n_1 + n_2 = N$ and the relation between n_1, n_2 and ν_1, ν_2 is given by (1.3c) when q is odd, or (1.6c) when q is even.

We note that the states (2.10) corresponding to different energies are, of course, orthogonal as H of (2.5) is Hermitian. On the other hand the states of the same energy in the $N + 1$ degenerate multiplet are not orthonormal as seen from their scalar product using the commutation relations $[\xi_\pm, \eta_\pm] = 1$, $[\xi_\mp, \eta_\pm] = 0$. Thus, we still have to prove that they are linearly independent. We shall do this for q odd and a similar analysis holds for q even. From (1.3c), and as (λ_1, λ_2) is fixed, we conclude that the part of the polynomial in (2.10) that changes with each state of multiplet of energy $2qN + (2\lambda_1 + q\lambda_2 + q + 1)$ is given by

$$(\eta_+\eta_-)^{qn_1}(\eta_+^q + \eta_-^q)^{2n_2}. \qquad (2.12)$$

As $n_1 + n_2 = N$, we see from (2.12) that the highest power that η_+ can take appears in the term

$$\eta_+^{q(N+n_2)}\eta_-^{qn_1}. \qquad (2.13)$$

Thus, for $n_2 = N$, $n_1 = 0$ the term η_+^{2qN} is present in (2.12). For any $n_2 < N$, $n_1 = N - n_2$, this term cannot appear and thus the state (2.10) in which (for q odd) $\nu_1 = \lambda_1$, $\nu_2 = 2N + \lambda_2$, is independent from all the others corresponding to a given (λ_1, λ_2) and N. But, clearly, we can show in the same way that for $n_2 = N - 1$, $n_1 = 1$, the term $\eta_+^{q(2N-1)}\eta_-^q$ does not appear for any $n_2 < N - 1$, $n_1 = N - n_2$, and continuing in this fashion prove that all the states (2.10) are linearly independent.

The states $|\nu_1\nu_2)$ of (2.10) now have a form very similar to those of (I.2.7) in the sense that they are given by a single product of certain simple polynomial functions of the creation operators. We shall take advantage of this fact to derive, first classically and then quantum mechanically, the Lie Algebra and Lie group responsible for the accidental degeneracy of the problem of the oscillator in a sector.

3. CLASSICAL LIE ALGEBRA AND SYMMETRY GROUP FOR THE HARMONIC OSCILLATOR IN A SECTOR

In this section we shall think of $\eta_\pm, \xi_\pm$ not as operators but as classical functions of X_i, P_i as defined through (2.1), (2.2) and (2.6). From these functions we see that the Poisson bracket of any two variables F, G can now be expressed as

$$\{F, G\} = i\left(\frac{\partial F}{\partial\eta_+}\frac{\partial G}{\partial\xi_+} - \frac{\partial F}{\partial\xi_+}\frac{\partial G}{\partial\eta_+}\right) + i\left(\frac{\partial F}{\partial\eta_-}\frac{\partial G}{\partial\xi_-} - \frac{\partial F}{\partial\xi_-}\frac{\partial G}{\partial\eta_-}\right), \qquad (3.1)$$

which implies $\{\eta_\pm, \xi_\pm\} = i$, $\{\eta_\mp, \xi_\pm\} = 0$.

Looking now at the states (2.10) and using as an analogy the analysis of the previous paper for the states (I.2.7), it seems appropriate to define new creation variables as

$$\eta_1 \equiv \eta_+\eta_-, \quad \eta_2 \equiv \eta_+^q + \eta_-^q. \qquad (3.2)$$

Note that the creation variables defined by (3.2) are not to be confused with those given in the previous paper in terms of coordinates and momenta in the directions $i = 1, 2$. The annihilation variables ξ_1, ξ_2 corresponding to them must be canonically conjugate, i.e.,

$$\{\eta_i, \xi_j\} = i\delta_{ij}, \quad i, j = 1, 2. \qquad (3.3)$$

From the standpoint of commutators, this implies

$$[\xi_j, \eta_i] = \delta_{ij} \qquad (3.4)$$

and we can represent ξ_j as $\partial/\partial\eta_j$. We shall use this representation to derive in a simple fashion the ξ_j. From (3.2) we have that

$$\eta_- = \eta_1/\eta_+, \ \eta_+ = \{\tfrac{1}{2}[\eta_2 + (\eta_2^2 - 4\eta_1^q)^{1/2}]\}^{1/q}, \qquad (3.5)$$

and, thus,

$$\frac{\partial}{\partial\eta_j} = \frac{\partial\eta_+}{\partial\eta_j}\frac{\partial}{\partial\eta_+} + \left(\frac{1}{\eta_+}\delta_{1j} - \frac{\eta_-}{\eta_+}\frac{\partial\eta_+}{\partial\eta_j}\right)\frac{\partial}{\partial\eta_-}. \qquad (3.6)$$

Interpreting now $\partial/\partial\eta_\pm$ as $\xi_\pm$ and making use of the fact that from (3.5)

$$\eta_+^q - \eta_-^q = (\eta_2^2 - 4\eta_1^q)^{1/2}, \qquad (3.7)$$

we obtain

$$\xi_1 = (\eta_+^{q-1}\xi_- - \eta_-^{q-1}\xi_+)(\eta_+^q - \eta_-^q)^{-1}, \qquad (3.8a)$$

$$\xi_2 = q^{-1}(\eta_+\xi_+ - \eta_-\xi_-)(\eta_+^q - \eta_-^q)^{-1}. \qquad (3.8b)$$

We can now easily check that the η_i of (3.2) and ξ_i of (3.8) satisfy the Poisson bracket relation (3.3).

We note that ξ_1, ξ_2 are *not* the complex conjugates of η_1, η_2 (nor Hermitian conjugates in the quantum case) and, thus, if we make use of the customary relations between creation and annihilation variables and new coordinates and momenta, we are *not* led to real canonical transformations. As we shall show in the next section, this seems to be related with the fact that the quantum mechanical representation of the symmetry group of canonical transformations is not unitary. We furthermore show in Sec. 5 that complex canonical transformations are involved in several important problems in physics. Thus, their appearance in the symmetry group of the oscillator in a sector is not an isolated event.

From (3.2), (3.8) we can immediately check that

$$2\eta_1\xi_1 + q\eta_2\xi_2 = \eta_+\xi_+ + \eta_-\xi_-. \qquad (3.9)$$

If q is odd we can divide both sides by $2q$ and the left hand side has the form (I.3.4) of the anisotropic oscillator with $k_1 = q$, $k_2 = 2$. If q is even we divide by q and again the left hand side has the form (I.3.4), but with $k_1 = (q/2)$, $k_2 = 1$.

To arrive now at the generators of the classical Lie algebra and the symmetry group for the problem of the sector we need still to transform the Hamiltonian of the anisotropic oscillator appearing in (3.9) into that of the isotropic one. As shown in the preceeding paper, we can do this if we carry out the canonical transformation

$$\eta_i' = k_i^{-1/2}(\eta_i\xi_i)^{(1-k_i)/2}\eta_i^{k_i}, \quad \xi_i' = k_i^{-1/2}\xi_i^{k_i}(\eta_i\xi_i)^{(1-k_i)/2}, \qquad (3.10)$$

where k_i, $i = 1, 2$ takes the values indicated in the previous paragraph for q odd and even. Under this transformation the Hamiltonian in (3.9) becomes proportional to

$$\mathcal{H} = \eta_1'\xi_1' + \eta_2'\xi_2', \qquad (3.11)$$

and thus the generators of the Lie algebra of its symmetry group[1] are given by

$$T_+ = \eta_1'\xi_2', \quad T_3 = \tfrac{1}{2}(\eta_1'\xi_1' - \eta_2'\xi_2'), T_- = \eta_2'\xi_1'. \qquad (3.12)$$

We can immediately check that the Poisson brackets (3.1) of the variables $T_\pm, T_3$ and the Hamiltonian $\mathcal{H}$ are zero, while among themselves they lead to the Lie algebra of $SU(2)$.

To obtain the classical symmetry group associated with this Lie algebra of $SU(2)$ we must proceed as in Sec. 3 of the preceeding paper. We shall only outline the steps as their algebraic implementation is trivial. We relate the new creation and annihilation variables $\bar{\eta}_\pm, \bar{\xi}_\pm$ with

$\eta_\pm, \xi_\pm$ in the following way: First we invert the expressions (3.2), (3.8) to determine $\bar{\eta}_\pm, \bar{\xi}_\pm$ in terms of $\bar{\eta}_i, \bar{\xi}_i$, $i = 1, 2$. Then we invert (3.10) to obtain $\bar{\eta}_i, \bar{\xi}_i$ in terms $\bar{\eta}'_i, \bar{\xi}'_i$. The $\bar{\eta}'_i, \bar{\xi}'_i$ are related to η'_i, ξ'_i by the $U(2)$ transformation (I.3.7b). Finally, we can express η'_i, ξ'_i, $i = 1, 2$ in terms of $\eta_\pm, \xi_\pm$ through (3.10) and then (3.2), (3.8).

Having analyzed the classical Lie algebra and the symmetry group, we turn now our attention to the quantum picture.

4. THE GENERATORS AND THE REPRESENTATION OF THE SYMMETRY GROUP IN THE QUANTUM PICTURE

In the quantum picture the creation and annihilation variables return to their roles as operators, but then we must also express η_i, ξ_i, $i = 1, 2$ of (3.2), (3.8) as operators that act on the state (2.10) without ambiguities. We have no problem for the effect of η_1, η_2 of (3.2) on the state $|\nu_1\nu_2)$ of (2.10) as they are polynomial functions of η_+, η_- only and these commute. For ξ_1, ξ_2 we have both η_+, η_- and ξ_+, ξ_- in (3.8) which do not commute and, furthermore, ξ_1, ξ_2 contain the factor $\eta_+^q - \eta_-^q$ to a negative power. Yet we shall assume that ξ_1, ξ_2 as operators are given by (3.8) in the order in which η_+, η_-, ξ_+, ξ_- appear.

Due to the presence of the factor $(\eta_+^q - \eta_-^q)^{-1}$ in (3.8), the determination of the matrix elements of ξ_1, ξ_2 with respect to a complete set of orthonormal states in the sector, such as $|\nu_1\nu_2\rangle$ of (2.8), seems impossible. We note though that the states $|\nu_1\nu_2\rangle$ can be expanded in terms of the complete but not orthonormal set $|\nu_1\nu_2)$ of (2.10) with the help of transformation brackets that will be discussed below. Thus, we need only to see whether the application of ξ_1, ξ_2 to the states $|\nu_1\nu_2)$ can be carried out. As all the states $|\nu_1\nu_2)$ have a factor $(\eta_+^q - \eta_-^q)$, the $(\eta_+^q - \eta_-^q)^{-1}$ in ξ_1, ξ_2 just cancels it. Furthermore, as the commutators $[\xi_\pm, \eta_\pm] = 1$, $[\xi_\mp, \eta_\pm] = 0$, when applying the operators $\xi_\pm$ to polynomials in the creation operators $\eta_\pm$, we can replace the former by $\partial/\partial\eta_\pm$. Using these considerations, we obtain from (3.2), (3.8) and the explicit form (2.10) for the state $|\nu_1\nu_2)$ that

$$\eta_1 \,|\, \nu_1\nu_2) = (\nu_1 + 1)^{1/2} \,|\, \nu_1 + 1, \nu_2), \tag{4.1a}$$

$$\eta_2 \,|\, \nu_1\nu_2) = (\nu_2 + 1)^{1/2} \,|\, \nu_1, \nu_2 + 1), \tag{4.1b}$$

$$\xi_1 \,|\, \nu_1\nu_2) = \nu_1^{1/2} \,|\, \nu_1 - 1, \nu_2), \tag{4.1c}$$

$$\xi_2 \,|\, \nu_1\nu_2) = \nu_2^{1/2} \,|\, \nu_1, \nu_2 - 1). \tag{4.1d}$$

The behavior of the η_i, ξ_i with respect to the states $|\nu_1\nu_2)$ is then entirely similar to that of the creation and annihilation operators in the two directions $i = 1, 2$ of the anisotropic oscillator with respect to the corresponding state (I.2.6). Just as in the case of the anisotropic oscillator, we can now divide the set of states $|\nu_1\nu_2)$ of (2.10) into subsets characterized by (λ_1, λ_2). As indicated in the introduction, there will be $2q$ subsets for q odd and $(q/2)$ for q even. For each one of these subsets of states we can pass, again as in the anisotropic oscillator, from the operators η_i, ξ_i to η'_i, ξ'_i by the transformation (I.4.2), where $k_1 = q$, $k_2 = 2$ for q odd, $k_1 = q/2$, $k_2 = 1$ for q even. The quantum mechanical generators of the symmetry group of the oscillator in a sector continue to be given by (3.12), but now the η'_i, ξ'_i, $i = 1, 2$ in it, are obtained for each subset (λ_1, λ_2) of states (2.10) in terms of $\eta_\pm, \xi_\pm$ through (I.4.2) and (3.2), (3.8).

To see what is the effect of a finite group transformation of the form $R(\alpha, \beta, \gamma)$ of (I.4.2) on the states $|\nu_1\nu_2)$ of (2.10), we first rewrite them as

$$|\nu_1\nu_2) \equiv |jm\}_{\lambda_1\lambda_2}, \tag{4.2}$$

where $j = \frac{1}{2}(n_1 + n_2)$, $m = \frac{1}{2}(n_1 - n_2)$ and $n_1, n_2, \lambda_1, \lambda_2$ are related to ν_1, ν_2 by (1.3c) when q is odd and (1.6c) for q even. It is immediately clear then that, as in (I.4.7),

$$R(\alpha, \beta, \gamma) \,|\, jm\}_{\lambda_1\lambda_2} = \sum_{m'} |\, jm'\}_{\lambda_1\lambda_2} \mathfrak{D}^j_{m'm}(\alpha\beta\gamma). \tag{4.3}$$

From this result we wish now to obtain the representation of the $SU(2)$ transformation with respect to the set of orthonormal states $|\nu_1\nu_2\rangle$ of (2.8). We require first the development of the states $|jm\}_{\lambda_1\lambda_2}$ of (4.2) and (2.10) in terms of $|\nu_1\nu_2\rangle$. Using the notation (4.2), we can write

$$|jm\}_{\lambda_1\lambda_2} = \sum_{\nu_1\nu_2} |\nu_1\nu_2\rangle\langle\nu_1\nu_2 \,|\, jm\}_{\lambda_1\lambda_2}. \tag{4.4}$$

The summation extends over the finite set of states corresponding to the same energy which implies that ν_1, ν_2 correspond to the same values of λ_1, λ_2 appearing in the round ket $|\nu_1\nu_2)$ of (2.10). The transformation brackets in (4.4) can be easily obtained from the expansion of the polynomial in (2.10), and the matrix

$$\|\langle \nu_1\nu_2 \,|\, jm\}\|, \tag{4.5}$$

where we suppressed λ_1, λ_2 for a clearer notation, is invertible as the set of states (2.10) is linearly independent. Denoting by $\langle\nu_1\nu_2 \,|\, jm\}^{-1}$ the elements of the inverse matrix, we have now that

$$\begin{aligned}\langle \nu'_1\nu'_2 \,|\, R(\alpha,\beta,\gamma) \,|\, \nu_1\nu_2\rangle &= \langle \nu'_1\nu'_2 \,|\, R(\alpha,\beta,\gamma) \sum_m |\, jm\}_{\lambda_1\lambda_2}\langle\nu_1\nu_2 \,|\, jm\}^{-1} \\ &= \sum_{m',m} \langle\nu'_1\nu'_2 \,|\, jm'\}\, \mathfrak{D}^j_{m'm}(\alpha\beta\gamma)\langle\nu_1\nu_2 \,|\, jm\}^{-1}.\end{aligned} \tag{4.6}$$

We note that, again as in the case of the anisotropic oscillator, the matrix elements are different from zero only when $|\nu_1\nu_2\rangle, |\nu'_1\nu'_2\rangle$ belong to the same subset of states characterized by a given (λ_1, λ_2). Furthermore, the corresponding n_1, n_2 and n'_1, n'_2 related to ν_1, ν_2 and ν'_1, ν'_2 by (1.3c) or (1.6c), must satisfy $n_1 + n_2 = n'_1 + n'_2$ due to the invariance of the Hamiltonian under the transformation.

It is important to notice that the representation of the $SU(2)$ group in the quantum mechanical picture is no longer unitary due to the transformation brackets in (4.6). This seems related to the complex character of the canonical transformation as indicated in the previous section.

5. CONCLUSIONS

We can draw the following conclusions from our procedure of deriving the Lie algebra and symmetry group of a plane oscillator in a sector of angle π/q. We note first that in this problem we required the expression of the wave function in terms of creation operators acting on the lowest energy state. The states that proved useful for our purpose were the nonorthonormal ones (2.10) given as powers of certain simple polynomials in the creation operators. The form of these states then suggested the group of complex canonical transformations responsible for accidental degeneracy.

If we can decompose the states of other problems where accidental degeneracy is present in terms of powers of some basic operators acting on a ground state, we may hope that a similar procedure could give us an insight into their symmetry group. A problem with this structure is the one proposed by Calogero,[5] where particles in one dimension interact through a quadratic and inverse quadratic potentials in their relative distances. In the case when we have only three particles, and after eliminating the center of mass, we get a problem in the plane. Perelomov[6] has shown how the states of this problem can be written as products of powers of two operators acting on the ground state. The situation resembles very much that in the expression (2.10), but now the two operators are not only functions of η_+, η_- but of the coordinates r, φ as well. Thus, if we identify these two operators with η_1, η_2 as in (3.2), it is considerably more difficult to find the corresponding ξ_1, ξ_2. The problem is being studied at present and we hope to present it in a third article in this series.

The procedure followed in the present paper leads to a symmetry group which is a group of complex canonical transformations. Now normally in mechanics we are concerned with real canonical transformations and so the question arises whether the complex variety appears elsewhere than in the present problem. We wish to indicate that the simple group of complex linear canonical transformations

$$\begin{pmatrix} \bar{x} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} a & ib \\ -ic & d \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \text{ real} \tag{5.1}$$

has a number of interesting applications.

We note first that the matrices appearing in (5.1) form a group as a product of two of the type leads to another of the same form. It is also a group of canonical transformations as $\{\bar{x}, \bar{p}\} = 1$. The representation, which is nonunitary, can be derived from the results obtained in the paper of Moshinsky and Quesne[7] for real linear canonical transformations when we replace b by ib and so, when $b \neq 0$, it takes the form

$$\langle x' \mid U \mid x'' \rangle = (2\pi \mid b \mid)^{-1} \exp[-(2b)^{-1}(ax'^2 - 2x'x'' + dx''^2)]. \tag{5.2}$$

When $a = d = 0$, $b = -c = 1$ we get of the kernel of the Laplace transform, while in the corresponding real case, i.e., $\bar{x} = p, \bar{p} = -x$, the representation, which is unitary, gives the kernel of the Fourier transform.[7]

When we have

$$a = d = b = -c = 1/\sqrt{2}, \tag{5.3}$$

$\bar{x}$ is just the annihilation operator and the representation (5.2) corresponds to the states[7] for which $\bar{x}$ is diagonal, i.e., the coherent states of optics.[8]

When $a = d = 1$, $b \neq 0$ and $c = 0$, the representation is a Gaussian and so the transformation[7]

$$\mid x'' \rangle = \int \mid x' \rangle \, dx' \langle x' \mid U \mid x'' \rangle \tag{5.4}$$

provides a Gaussian transform of the type used in clustering theory by Brink.[9] Its inverse can then be determined purely from the fact that it corresponds to a representation of the transformation (5.1).

The expression (5.2) also appears in a very important fashion as a kernel in clustering theory as was shown by Kramer.[10] The realization that it is a representation (5.1) is very important for the factorization and products of such kernels.

Thus, the complex linear canonical transformation (5.1) and its nonunitary representation plays an important role in several branches of physics. It is, therefore, not surprising that other complex canonical transformations and their nonunitary representations appear in relation with problems such as the symmetry group of the plane oscillator in a sector of angle π/q.

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