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On q-extended eigenvectors of the integral and finite Fourier transforms

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Abstract
Mehta has shown that eigenvectors of the $N \times N$ *finite* Fourier transform can be written in terms of the standard Hermite eigenfunctions of the quantum harmonic oscillator (1987 *J. Math. Phys.* **28** 781). Here, we construct a one-parameter family of q-extensions of these eigenvectors, based on the continuous q-Hermite polynomials of Rogers. In the limit when $q \to 1$ these q-extensions coincide with Mehta's eigenvectors, and in the continuum limit as $N \to \infty$ they give rise to q-extensions of eigenfunctions of the Fourier *integral* transform.

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1. Introduction

The finite Fourier transform is represented by an $N \times N$ unitary matrix $\mathbf{A}^{(N)} = \|A^{(N)}_{j,k}\|$ with elements

$$A^{(N)}_{m,m'} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi\mathrm{i}}{N} mm'\right), \qquad m, m' \in \{0, 1, \ldots, N-1\}. \tag{1}$$

(*Note.* In the physics literature it is more common to find the definition with a minus sign in the exponent; of course, our results do not depend on which convention is used.)

For any given complex-valued function $f(m)$, defined on the N integers $m \in \{0, 1, \ldots, N-1\}$, its *finite Fourier transform*,

$$\widetilde{f}(m) := \sum_{m'=0}^{N-1} A^{(N)}_{m,m'} f(m'),$$

is also a complex-valued function, well defined on those integers. As is well known, the fourth power of the Fourier matrix $\mathbf{A}^{(N)}$ in (1) is the unit matrix, so (for $N > 4$) it has only four distinct eigenvalues $\lambda_n \in \{1, \mathrm{i}, -1, -\mathrm{i}\}$. Its eigenvectors $\mathbf{f}^{(n)} = \| f^{(n)}(k)\|$ are the solutions of the standard equations

$$\sum_{m'=0}^{N-1} A^{(N)}_{m,m'} f^{(n)}(m') = \lambda_n f^{(n)}(m), \qquad n \in \{0, 1, \ldots, N-1\}. \tag{2}$$

The finite Fourier transform appears in various physical and mathematical settings and has been studied from several directions for applications in signal analysis [1, 2], as foundation for finite models in quantum mechanics [3, 4] and optics [5] and for its intrinsic mathematical interest [6, 7]. In continuous models, the Fourier integral transform is of great importance because it interconnects the coordinate and momentum/frequency representations of a wavefunction; we expect it to have a parallel role in finite models.

In this paper, we consider the eigenvalue problem (2) for the finite Fourier transform with the strategy of Mehta [8], who explicitly found sets of N *analytic* eigenvectors of the finite Fourier transform of dimension N. Here, we find a manifold of q-special functions $f^{(n)}(m; q)$ in the interval $0 < q \leqslant 1$ (or equivalently $1 \leqslant q < \infty$) that are analytic in their argument m, which can be continued into the complex plane, and for which (2) holds as a summation formula. Mehta's results are recovered in the limit $q \to 1$, and in the independent continuous limit $N \to \infty$ we obtain a new set of eigenfunctions of the Fourier integral transform.

In section 2, we recall the main results of Mehta [8] with Hermite polynomials and formulate the q-extension of this method in section 3, with an integral property of the q-Hermite polynomials of Rogers. From these we build the finite Fourier q-extended eigenbasis in section 4, and in section 5 we derive a new q-extended eigenbasis for the Fourier integral transform. In section 7, we offer some concluding remarks.

2. Mehta's basis with Hermite functions

We call here *Hermite functions* the (non-normalized) quantum wavefunctions of the linear harmonic oscillator,

$$\mathcal{H}_n(x) := H_n(x)\exp(-x^2/2), \tag{3}$$

where $H_n(x)$ $(n \in \{0, 1, 2, \ldots\})$ are the Hermite polynomials of degree n in the variable x. These are well known to be reproduced under the Fourier integral transform with eigenvalues i^n,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}xy}\mathcal{H}_n(x) = \mathrm{i}^n\mathcal{H}_n(y). \tag{4}$$

In particular, the subset $\{\mathcal{H}_n(x)\}$ with $n \equiv 0 \pmod 4$ consists of the so-called *self-Fourier* functions, which are their own Fourier transforms.

Mehta [8] used the integral relation (4) to prove that the infinite sum of translated Hermite functions

$$F^{(n)}_m := \sum_{k=-\infty}^{\infty} \mathcal{H}_n(x_k(m)), \qquad x_k(m) := \sqrt{\frac{2\pi}{N}}(kN + m), \tag{5}$$

for integer $m \in \{0, 1, \ldots, N-1\}$, satisfy the eigenvalue equation (2) with $\lambda_n = \mathrm{i}^n$. These *Mehta* functions are real and naturally periodic with period N, so we may equivalently let the finite coordinate $\mu := m - \frac{1}{2}(N-1)$ range over the symmetric interval of equally spaced points $\mu \in \left[-\frac{1}{2}(N-1), \frac{1}{2}(N-1)\right]$.

The usual inner product, norm and angle for complex N-vectors are

$$(\mathbf{f}, \mathbf{g}) := \sum_{m=0}^{N-1} f_m^* g_m, \qquad \|\mathbf{f}\| := \sqrt{(\mathbf{f}, \mathbf{f})},$$
$$\cos\theta_{f,g} := (\mathbf{f}, \mathbf{g}) \Big/ \sqrt{\|\mathbf{f}\|\|\mathbf{g}\|}, \tag{6}$$

and when $n_1 \not\equiv n_2 \pmod 4$, $(\mathbf{F}^{(n_1)}, \mathbf{F}^{(n_2)}) = 0$. The basis of Mehta functions is not orthogonal however [9], only *approximately* so, in the sense that for small $n_1 \equiv n_2 \pmod 4$, the overlap $(\mathbf{F}^{(n_1)}, \mathbf{F}^{(n_2)})$ is also *small*. For example, we have estimated that when $N = 33$ and for $n_1 < 16$, the largest deviation from orthogonality occurs for $n_2 = 31$, with the angle $90^\circ - \theta_{15,31} \approx 1.5^\circ$.

The Mehta basis has served to define an *approximate* fractional discrete Fourier transform of power α modulo 4, whose matrix kernel is defined as the bilinear generating matrix of the vectors $F_m^{(n)}$, summed over n, with the phase power $\exp\left(\mathrm{i}\frac{1}{2}\pi n\alpha\right)$. This, together with other orthonormal and 'approximately orthonormal' Fourier eigenbases, was computed numerically in [10].

3. Bases with q-Hermite functions

It turns out that many q-extensions of the well-known classical polynomial families enjoy remarkably simple transformation properties with respect to the Fourier integral transform (see [11] and references therein), and in the appendix we give the definition and a résumé of properties of the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers [12–14]. It was shown in [15] that for these polynomials the following Fourier integral transform formula holds:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}xy} H_n(\sin\kappa x|q)\, \mathrm{e}^{-x^2/2} = \mathrm{i}^n q^{n^2/4} h_n(\sinh\kappa y|q)\, \mathrm{e}^{-y^2/2}$$
$$= q^{n^2/4} H_n(\mathrm{i}\sinh\kappa y|q^{-1})\, \mathrm{e}^{-y^2/2}, \tag{7}$$

where $q := \exp(-2\kappa^2)$, $0 \leqslant \kappa < \infty$ and

$$h_n(x|q) := \mathrm{i}^{-n} H_n(\mathrm{i}x|q^{-1}) \tag{8}$$

are the so-called q^{-1}-Hermite polynomials [16]. In the appendix, we provide the explicit expression and some of the properties of these functions that make them useful for finite signal analysis. (We employ the standard notations of the theory of special functions as in [17, 18].)

Observe at this point that the integral Fourier transform (7) interrelates the continuous q-Hermite polynomials $H_n(x|q)$ with the q^{-1}-Hermite polynomials $h_n(x|q)$. This change of q into q^{-1} is the price to formulate a meaningful q-extension of the classical Fourier transform (4). In the limit as $q \to 1$ ($\kappa \to 0$), the Fourier integral transform (7) of course reduces to (4). This property of (7) is a direct consequence of the known limit relations for the continuous q-Hermite polynomials $H_n(x|q)$,

$$\lim_{q\to 1^-} \kappa^{-n} H_n(\sin\kappa y|q) = H_n(y) = \lim_{q\to 1^-} \kappa^{-n} h_n(\sinh\kappa y|q), \tag{9}$$

for $0 < q \leqslant 1$ on the left-hand side and $1 \leqslant q < \infty$ on the right-hand side of (9), respectively.

Having the Fourier integral transform (7), it is natural to consider its finite analogue. In the proof of (2) for Mehta's eigenvectors $F_m^{(k)}$ under the finite Fourier transform it was essential to use the simple transformation property (4) of the Hermite functions (3) under the Fourier integral transform. The authors of [19] employed Mehta's technique to prove (2), but their basic objective was (7) rather than (4). They showed that the q-extensions of Mehta's eigenvectors $F_m^{(k)}$ given by the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers, namely

$$F_m^{(n)}(q) := \sum_{k=-\infty}^{\infty} \exp\left[-\frac{1}{2}(x_k(m))^2\right] H_n(\sin\kappa x_k(m)|q), \tag{10}$$

$$F_m^{(n)}(q^{-1}) := \mathrm{i}^n \sum_{k=-\infty}^{\infty} \exp\left[-\frac{1}{2}(x_k(m))^2\right] h_n(\sinh\kappa x_k(m)|q), \tag{11}$$

at the points $x_k(m)$ defined in (5), also enjoy simple transformation properties with respect to the *finite* Fourier transform,

$$\sum_{m=0}^{N-1} A_{j,m}^{(N)} F_m^{(n)}(q) = q^{n^2/4} F_j^{(n)}(q^{-1}), \tag{12}$$

$$\sum_{m=0}^{N-1} A_{j,m}^{(N)} F_m^{(n)}(q^{-1}) = (-1)^n q^{-n^2/4} F_j^{(n)}(q). \tag{13}$$

Since the parameters q and N are *independent* in the foregoing formulae [19, 20], from (10) and (11) it is readily verified, by using (9), that the Mehta eigenvectors are regained in the limit $q \to 1^-$,

$$\lim_{q\to 1^-} \kappa^{-n} F_m^{(n)}(q) = F_m^{(n)} = \mathrm{i}^{-n} \lim_{q\to 1^-} \kappa^{-n} F_m^{(n)}(q^{-1}), \tag{14}$$

and both relations reduce to (2). Also, the continuous limit when $N \to \infty$ returns the Fourier integral transform (7) and its inverse, respectively.

4. Finite Fourier *q*-extended eigenbases

We now show that relations (12) and (13) enable us to find extensions of the Mehta eigenvectors for the finite Fourier transform, which depend on the deformation parameter q and which are their own finite Fourier transforms (times a phase).

We thus evaluate the action of the Fourier matrix $\mathbf{A}^{(N)}$ of (1), on the linear combination of (10) and (11),

$$f_m^{(n)}(q) := a_n(q) F_m^{(n)}(q) + b_n(q) F_m^{(n)}(q^{-1}), \tag{15}$$

with some nonzero constants $a_n(q)$ and $b_n(q)$, which we need to determine. Then, upon using (12) and (13), one obtains that

$$\sum_{j=0}^{N-1} A_{k,j}^{(N)} \left(a_n(q) F_j^{(n)}(q) + b_n(q) F_j^{(n)}(q^{-1})\right) = q^{n^2/4} a_n(q) F_k^{(n)}(q^{-1}) + (-1)^n q^{-n^2/4} b_n(q) F_k^{(n)}(q)$$
$$= \mathrm{i}^n \left(a_n(q) F_k^{(n)}(q) + b_n(q) F_k^{(n)}(q^{-1})\right). \tag{16}$$

provided that

$$b_n(q) = \mathrm{i}^{-n} q^{n^2/4} a_n(q). \tag{17}$$

When this relation holds for each $n \in \{0, 1, \ldots, N-1\}$, the linear combinations $f_m^{(n)}(q)$ in (15) are eigenvectors of the finite Fourier transform $\mathbf{A}^{(N)}$, with the eigenvalue $\lambda_n = \mathrm{i}^n$.

With the choice for the overall constant $a_n(q) := \frac{1}{2} q^{-n^2/8}$ in (17) to be justified below, we write the q-extended Fourier eigenvector components as

$$\psi_m^{(n)}(q) := \frac{1}{2} q^{-n^2/8} \big(F_m^{(n)}(q) + \mathrm{i}^{-n} q^{n^2/4} F_m^{(n)}(q^{-1}) \big) \tag{18}$$

$$= \frac{1}{2} \sum_{k=-\infty}^{\infty} (q^{-n^2/8} H_n(\sin\kappa x_k(m)|q) + q^{n^2/8} h_n(\sinh\kappa x_k(m)|q))\, \mathrm{e}^{-(x_k(m))^2/2}, \tag{19}$$

where the equally spaced set of points $x_k(m)$ was given in (5), the integer coordinate being $m \in \big[-\frac{1}{2}(N-1), \frac{1}{2}(N-1)\big]$. In the second line of (19) we introduced the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers. This equation shows that the eigenvectors $\psi_m^{(n)}(q)$ will exhibit some symmetry with respect to the exchange $q \Rightarrow q^{-1}$. With the choice of $a_n(q)$ in (19), it is straightforward to verify that

$$\psi_m^{(n)}(q^{-1}) = \mathrm{i}^n \psi_m^{(n)}(q). \tag{20}$$

The vectors $\boldsymbol{\psi}^{(n)}(q) = \big\| \psi_m^{(n)}(q) \big\|$ will thus be the eigenvectors of the $N \times N$ finite Fourier transform matrix,

$$\mathbf{A}^{(N)} \boldsymbol{\psi}^{(n)}(q) = \mathrm{i}^n \boldsymbol{\psi}^{(n)}(q) = \boldsymbol{\psi}^{(n)}(q^{-1}). \tag{21}$$

We understand these q-extended vectors to form the finite wavefunction analogue of a thus-defined harmonic oscillator, that we may write on occasion as $\psi_n(m; q) \equiv \psi_m^{(n)}(q)$.

In particular, the 'ground' state among the q-extended Fourier eigenfunctions is a Gaussian sum on $x_k(m)$ (since $H_0(x|q) = 1 = h_0(x|q)$), which is *independent* of q, and hence equal to the $q = 1$ undeformed Mehta ground state $F_m^{(0)}$ in (5). It can be written in terms of the Jacobi ϑ_3-function [20]:

$$\psi_0(m) := \psi_m^{(0)}(q) = \sum_{k=-\infty}^{\infty} \exp\left(-\frac{\pi}{N} (kN + m)^2 \right) \tag{22}$$

$$= \mathrm{e}^{-\pi m^2/N} \vartheta_3(\pi \mathrm{i} m, \mathrm{e}^{-\pi N}) = \frac{1}{\sqrt{N}} \vartheta_3 \left(\frac{\pi m}{N}, \mathrm{e}^{-\pi/N} \right). \tag{23}$$

These all have Gaussian shape and converge to the Gaussian $\sim \exp\left(-\frac{1}{2}x^2\right)$ in the continuous limit when $N \to \infty$.

The overall coefficients $a_n(q)$ were chosen as $\frac{1}{2} q^{-n^2/8}$ to normalize the q-extended eigenvectors $\boldsymbol{\psi}^{(n)}(q)$, so that they exhibit the natural limit property to the Mehta eigenvectors,

$$\lim_{q \to 1} \kappa^{-n} \psi_m^{(n)}(q) = F_m^{(n)}, \tag{24}$$

consequence of $\lim_{q \to 1} a_n(q) = \frac{1}{2}$.

5. Integral Fourier q-extended eigenbases

We recall that the parameters q and N are independent in the relations of the previous section. Indeed, the continuous limit $N \to \infty$ of (24) and (20) gives rise to q-extended eigenfunctions of the Fourier *integral* transform,

$$\psi_n(x; q) := \lim_{N \to \infty} \psi_{m(x)}^{(n)}(q)$$

$$= \tfrac{1}{2} (q^{-n^2/8} H_n(\sin\kappa x|q) + q^{n^2/8} h_n(\sinh\kappa x|q))\, \mathrm{e}^{-x^2/2} \tag{25}$$

where the interval and density of points $x_k(m)$ defined in (5) tend to infinity as $\sqrt{N}$ and are replaced by real x. The functions (25) then satisfy

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}xy}\psi_n(x;q) = \mathrm{i}^n\psi_n(y;q). \tag{26}$$

This Fourier integral transform is a q-extension of the classical result (4).

It should be noted that from (25), the q-extended ground states

$$\psi_0(x;q) = \mathrm{e}^{-x^2/2}, \tag{27}$$

all coincide with the ordinary, non-extended ground state: the basic Gaussian.

6. Concluding remarks

Note that our closing formula (26) may also be derived directly from the Fourier integral transform (7) and its inverse, by repeating *mutatis mutandis* the same steps that led from (10) and (11) to (24). Then, having derived the continuous case (26), one could have tried to work out the finite case (24). It is difficult to tell which way is shorter: the one outlined above or the route that we have followed in this study. We also remark that the authors of [15], in which the Fourier integral transform (7) was first formulated, did not see the possibility of taking one step further: to deduce (26) from (7) and its inverse.

We must leave for further study some of the uses of the q-extended Fourier eigenfunctions. As was brought out in [10], there is a large freedom in choosing the subgroup $\mathsf{U}(1)$ of unitary fractional Fourier transform matrices $(\mathbf{A}^{(N)})^{\alpha}$ (α counted modulo 4): for every orthonormal basis in N-dimensional space that is also eigenbasis of the Fourier transform, there is such a subgroup. Of particular interest are the bases that derive from finite oscillator analogues, since they can be linearly combined into coherent states that should display a clean harmonic motion [21]. *Approximately* orthonormal bases, including the Mehta basis, can also be satisfactory according to this criterion [10]. The q-extended set of eigenfunctions studied here is not orthonormal, but with a sufficiently close analogy to the quantum harmonic oscillator, to warrant attention.

Finally, we underline the obvious fact that all functions in this paper are periodic in the discrete position coordinate m modulo N. Also *non*-periodic finite oscillators and q-oscillators have been studied (see [22, 23]); there, the finite Fourier transform is replaced by a Fourier–*Kravchuk* transform, whose fractionalization is natural in the context of an $\mathsf{SO}(3)$ rotation group. Here, however, we are concerned with the finite Fourier matrix.

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Appendix. q-Hermite and q^{-1}-Hermite polynomials

The standard Hermite polynomials are known to be generated by the three-term recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad H_0(x) = 1. \tag{A.1}$$

Their explicit form for degree n in x is given by

$$H_n(x) := \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n!}{k!(n-2k)!}(2x)^{n-2k}, \tag{A.2}$$

where $\lfloor \nu \rfloor$ is the integer part of ν.

On the other hand, the q-Hermite polynomials of Rogers, also of degree n in y, satisfy the three-term recurrence relation (cf equation (A.1)),

$$H_{n+1}(x|q) = 2x H_n(x|q) - (1-q^n) H_{n-1}(x|q), \qquad H_0(x|q) = 1, \tag{A.3}$$

and their explicit form is

$$H_n(\sin y|q) := \mathrm{i}^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \mathrm{e}^{\mathrm{i}y(n-2k)}, \tag{A.4}$$

where we use the q-binomial and q-shifted factorial,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \qquad (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k). \tag{A.5}$$

Since the q^{-1}-Hermite polynomials are defined as in (8), from the two last formulae it follows that

$$h_n(\sinh y|q) := \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} \, \mathrm{e}^{y(n-2k)}. \tag{A.6}$$

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Endnotes

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