Phase reconstruction from intensity measurements in linear systems

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The phase of a signal at a plane is reconstructed from the intensity profiles at two close parallel screens connected by a small *abcd* canonical transform; this applies to propagation along harmonic and repulsive fibers and in free media. We analyze the relationship between the local spatial frequency (the signal phase derivative) and the derivative of the squared modulus of the signal under a one-parameter canonical transform with respect to the parameter. We thus generalize to all linear systems the results that have been obtained separately for Fresnel and fractional Fourier transforms.

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1. INTRODUCTION

Phase retrieval and local frequency estimation of a signal from intensity profiles are important problems in radio location, optical signal processing, quantum mechanics, and other fields. Several successful iterative algorithms for phase reconstruction from the squared modulus of the signal and its power spectrum, or its Fresnel spectrum, have been proposed recently,[1–4] and related techniques are applied in various regions of the electromagnetic spectrum and in quantum mechanics.[5–7] The development of noniterative procedures for generic systems remains an attractive research topic.

A noniterative approach for phase retrieval, based on the so-called transport-of-intensity equation in optics, was proposed by Teague[8] and then further developed by others.[9–11] It was shown that the longitudinal derivative of the Fresnel spectrum is proportional to the transversal derivative of the product of the instantaneous power and the instantaneous frequency of the signal. A similar procedure was proposed for the fractional Fourier transform.[12,13]

In this paper we show that a noniterative formulation applies for general one-parameter canonical transforms.[14–17] We show that the local frequency (the first derivative of the phase of the signal) is directly related to the derivative of the squared modulus of the one-parameter canonical transform with respect to the parameter and is given by the evolution Hamiltonian of the optical medium. From this relationship we conclude that the phase of the signal can be reconstructed by letting it propagate in such systems and measuring the intensity profiles of the signal for two close values of the parameter.

After a short reminder of definitions in Section 2, in Section 3 we find the parametric derivative of the intensity under canonical transforms; this is used in Section 4 to derive our main result; in Section 5 we offer some concluding remarks.

2. PHASE AND LOCAL FREQUENCY

The local spatial frequency $\bar{p}(x)$ of a one-dimensional, complex, and coherent signal of amplitude $\psi(x)$ is defined as the derivative of its phase $\phi(x)$,

$$\bar{p}(x) := \phi'(x) := \frac{\mathrm{d}\phi(x)}{\mathrm{d}x}, \qquad \psi(x) = |\psi(x)|\exp[i\phi(x)]. \tag{1}$$

This local frequency can be expressed in terms of the signal itself, $\psi(x)$, by writing

$$\begin{aligned}\bar{p}(x) &= \frac{\mathrm{d}\phi(x)}{\mathrm{d}x} = \mathrm{Im}\frac{\mathrm{d}\ln\psi(x)}{\mathrm{d}x}\\ &= \mathrm{Im}\frac{\psi'(x)}{\psi(x)}\\ &= \frac{1}{2}i\left(\frac{\psi'^*(x)}{\psi^*(x)} - \frac{\psi'(x)}{\psi(x)}\right)\\ &= \frac{1}{2}i\,\frac{\psi'^*(x)\psi(x) - \psi^*(x)\psi'(x)}{\psi^*(x)\psi(x)}.\end{aligned}$$

Thus we obtain a relation that we shall use below:

$$\bar{p}(x)|\psi(x)|^2 = \tfrac{1}{2}i[\psi'^*(x)(\psi)(x) - \psi^*(x)\psi'(x)]. \tag{2}$$

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3. ONE-PARAMETER CANONICAL OPERATORS

In the paraxial regime of two-dimensional geometric optics, a quasi-homogeneous medium of refractive index $n(x) \approx n_o - \nu x^2$ is characterized by the Hamiltonian function

$$h(x, k) = \frac{1}{2n_o}k^2 + \nu x^2. \tag{3}$$

Harmonic fibers have $\nu > 0$, a free medium is described by $\nu = 0$, and repulsive fibers correspond to $\nu < 0$. Quadratic functions of phase space (x, k) have a unique quantization (or wavization); we will generalize our considerations to the generic Hamiltonian operators

$$\mathcal{H} := \tfrac{1}{2}A\mathcal{P}^2 + \tfrac{1}{2}B(\mathcal{QP} + \mathcal{PQ}) + \tfrac{1}{2}C\mathcal{Q}^2 = \mathcal{H}^\dagger, \tag{4}$$

using the well-known Schrödinger realization of the momentum ($k \mapsto \mathcal{P}$) and position ($x \mapsto \mathcal{Q}$) by

$$\mathcal{P} = -i\frac{\partial}{\partial x}, \qquad \mathcal{Q} = x, \qquad \mathcal{P}^2 = -\frac{\partial^2}{\partial x^2},$$

$$\mathcal{QP} + \mathcal{PQ} = -2i\left(x\frac{\partial}{\partial x} + \frac{1}{2}\right), \qquad \mathcal{Q}^2 = x^2, \tag{5}$$

and acting on the Hilbert space of square-integrable signals $\psi(x)$.

The Hamiltonian operator [Eq. (4)] generates the evolution of the signals along the z axis of the fibers or free space. This evolution is given by a one-parameter group of unitary canonical transform operators, which act on signals through the canonical integral transform [Ref. 17, Eqs. (9.25), (9.73), and (9.74)],

$$\mathcal{C}(\alpha) := \exp(i\alpha\mathcal{H}) = \mathcal{C}(-\alpha)^\dagger, \qquad \alpha \in \mathrm{Re}, \tag{6}$$

$$\mathcal{C}(\alpha)\psi(x) := \psi_\alpha(x) = \int_{-\infty}^{\infty} C(\alpha; x, x')\psi(x')\mathrm{d}x' \tag{7}$$

$$\mathcal{H}\psi(x) = -i\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{-\infty}^{\infty} C(\alpha; x, x')\psi(x')\mathrm{d}x'|_{\alpha=0}$$
$$= -i\frac{\mathrm{d}\psi_\alpha(x)}{\mathrm{d}\alpha}\bigg|_{\alpha=0}. \tag{8}$$

The fractional Fourier transformation is produced in a harmonic fiber whose Hamiltonian [Eq. (4)] has parameter values $A = C = 1$ and $B = 0$ and is

$$C_\mathrm{e}(\alpha; x, x') = \frac{1}{(2i\pi\sin\alpha)^{1/2}} \times \exp\left(-i\frac{x^2\cos\alpha - 2xx' + x'^2\cos\alpha}{2\sin\alpha}\right), \tag{9}$$

with α counted modulo 4π to cover the two values of the metaplectic sign [for *abcd* parameters near the identity system, we understand that $i^{-1/2} := \exp(-i\pi/4)$]. The Fresnel transformation of free flight in a homogeneous medium has the Hamiltonian [Eq. (4)] with $A = 1$ and $B = C = 0$:

$$C_\mathrm{p}(\alpha; x, x') = \frac{1}{(2i\pi\alpha)^{1/2}}\exp\left(-i\frac{x^2 - 2xx' + x'^2}{2\alpha}\right). \tag{10}$$

Finally, a repulsive fiber corresponds to $A = -C = 1$ and $B = 0$; its integral kernel is

$$C_\mathrm{h}(\alpha; x, x') = \frac{1}{(2i\pi\sinh\alpha)^{1/2}} \times \exp\left(-i\frac{x^2\cosh\alpha - 2xx' + x'^2\cosh\alpha}{2\sinh\alpha}\right). \tag{11}$$

The subindex e,p,h of the three previous kernels distinguishes between the elliptic, parabolic, and hyperbolic one-parameter subgroups of all paraxial optical systems. Any other system can be obtained from them through similarity; for example, multiplication by a quadratic phase factor ($A = B = 0$ and $C = 1$) is obtained through Fourier transformation of the parabolic kernel [Eq. (10)] and scaling ($A = C = 0$ and $B = 1$) by the square root of the Fourier transform of the hyperbolic kernel [Eq. (11)]. Yet these two canonical transforms are not *integral* transforms; their kernels reduce to Dirac δs, and the signal is multiplied by the factor and rescaled.

In Eq. (8) we differentiated $\psi_\alpha(x)$ with respect to α at the origin ($\alpha = 0$) and obtained up to second derivatives of $\psi(x)$. We now differentiate its absolute square,

$$\frac{\partial|\psi_\alpha(x)|^2}{\partial\alpha} = \frac{\partial\psi_\alpha^*\psi_\alpha}{\partial\alpha} = \frac{\partial\psi_\alpha^*}{\partial\alpha}\psi_\alpha + \psi_\alpha^*\frac{\partial\psi_\alpha}{\partial\alpha}$$
$$= (i\mathcal{H}\psi_\alpha)^*\psi_\alpha + \psi_\alpha^*(i\mathcal{H}\psi_\alpha)$$
$$= -i[-\tfrac{1}{2}A\psi_\alpha''^* + iB(x\psi_\alpha'^* + \tfrac{1}{2}\psi_\alpha^*)$$
$$+ \tfrac{1}{2}Cx^2\psi_\alpha^*]\psi_\alpha + i\psi_\alpha^*[-\tfrac{1}{2}A\psi_\alpha''$$
$$- iB(x\psi_\alpha' + \tfrac{1}{2}\psi_\alpha) + \tfrac{1}{2}Cx^2\psi_\alpha]$$
$$= \tfrac{1}{2}iA(\psi_\alpha''^*\psi_\alpha - \psi_\alpha^*\psi_\alpha'') + Bx(\psi_\alpha^*\psi_\alpha + \psi_\alpha^*\psi_\alpha')$$
$$+ B\psi_\alpha^*\psi_\alpha,$$

and arrive at

$$\frac{\partial|\psi_\alpha(x)|^2}{\partial\alpha} = \frac{1}{2}iA\frac{\mathrm{d}}{\mathrm{d}x}[\psi_\alpha'^*(x)\psi_\alpha(x) - \psi_\alpha^*(x)\psi_\alpha'(x)] + B\frac{\mathrm{d}}{\mathrm{d}x}[x\psi_\alpha^*(x)\psi_\alpha(x)]. \tag{12}$$

4. RECONSTRUCTION OF THE PHASE

We recognize in Eqs. (2) and (12) the common subexpression $\psi_\alpha'^*(x)\psi_\alpha(x) - \psi_\alpha^*(x)\psi_\alpha'(x)$, so we combine them to obtain

$$\frac{\mathrm{d}[A\bar{p}(x) + Bx]|\psi(x)|^2}{\mathrm{d}x} = \frac{\partial|\psi_\alpha(x)|^2}{\partial\alpha}\bigg|_{\alpha=0}, \tag{13}$$

This equation relates the transversal derivative of the signal intensity with respect to the space variable x and the derivative of this intensity with respect to the canonical transform parameter α.

Assuming that $[A\bar{p}(x) + Bx]|\psi(x)|^2 \to 0$ for $x \to \pm\infty$, and noting that owing to the unitarity unitary of the canonical transformation $\mathcal{C}(\alpha)$ the Parseval relation holds [Ref. 17, Eq. (9.11)],

$$\int_{-\infty}^{\infty} \frac{\partial|\psi_\alpha(x')|^2}{\partial\alpha}\bigg|_{\alpha=0} \mathrm{d}x' = \frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty}|\psi_\alpha(x')|^2\mathrm{d}x'\bigg|_{\alpha=0} = \frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty}|\psi(x')|^2\mathrm{d}x'\bigg|_{\alpha=0} = 0, \tag{14}$$

we integrate Eq. (13) over x. This yields

$$\begin{aligned}[A\bar{p}(x) + Bx]|\psi(x)|^2 &= \int_{-\infty}^{x} \frac{\partial|\psi_\alpha(x')|^2}{\partial\alpha}\bigg|_{\alpha=0}\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} \frac{\partial|\psi_\alpha(x')|^2}{\partial\alpha}\bigg|_{\alpha=0} u(x - x')\mathrm{d}x' \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial|\psi_\alpha(x')|^2}{\partial\alpha}\bigg|_{\alpha=0} \mathrm{sgn}(x - x')\mathrm{d}x', \end{aligned} \tag{15}$$

where $u(\xi)$ is the unit step function [$u(\xi) = 1$ for $\xi > 0$ and $u(\xi) = 0$ for $\xi < 0$] related to the sign function through $\mathrm{sgn}(\xi) = 2u(\xi) - 1$; the difference between the two (-1) does not change the value of the integral due to Eq. (14).

We consider Eq. (15) to be our main result, because it relates the phase derivative of the signal, $\bar{p}(x)$ in Eq. (1), to a convolution integral of the derivative of the signal intensity $|\psi_\alpha(x)|^2$ with respect to the evolution parameter α in a medium characterized by the generic Hamiltonian [Eq. (4)]. As in Ref. 13, two signal intensities $|\psi_{\pm\epsilon}(x)|^2$ measured at two screens separated by 2ϵ will provide the numerical value of the right-hand side of Eq. (15) and of their average

$$|\psi(x)|^2 \approx \tfrac{1}{2}(|\psi_{+\epsilon}(x)|^2 + |\psi_{-\epsilon}(x)|^2), \qquad \epsilon \ll 1. \tag{16}$$

This allows the numerical determination of the local frequency by

$$\bar{p}(x) \approx -\frac{B}{A}x + \frac{1}{2|\psi(x)|^2} \times \int_{-\infty}^{\infty}\frac{|\psi_{+\epsilon}(x')|^2 - |\psi_{-\epsilon}(x')|^2}{2\epsilon}\mathrm{sgn}(x - x')\mathrm{d}x'. \tag{17}$$

The integration of Eq. (1) recovers the signal phase (up to an overall constant) through $\phi(x) = \int^x \bar{p}(x')\mathrm{d}x'$. In Ref. 13 the validity of approximation (17) in the neighborhood of the zeros of $|\psi(x)|^2$ is examined in several numerical examples. Since only square moduli appear in all expressions, the metaplectic sign of the canonical transforms is irrelevant.

We note the absence of C, the Hamiltonian parameter associated with the refractive index, in the left-hand side of Eq. (15); comparison with the classical form [Eq. (3)] shows that the determination of the signal phase is the same whether the medium be harmonic, repulsive, or free (Fourier, hyperbolic, or Fresnel transform intervening). The parameter B of the scaling term of the Hamiltonian does appear in that left-hand side; when it is present, relation (17) shows that it will only chirp the local frequency $\bar{p}(x)$ by $-Bx/A$. Thus we see that to determine the local frequency of a signal from two intensities it is necessary only that the canonical transform be generated by a Hamiltonian with a nonzero Ak^2 free-flight term.

5. CONCLUSION

In this paper we have established the relation [Eq. (15)] between the local frequency of a signal and the derivative of the squared modulus of any one-parameter subgroup line of canonical transforms of an optical signal, including the previously known Fourier and Fresnel cases. This allows us to reconstruct the phase of a signal by measuring two intensity profiles at two close values of the free-flight parameter in two-dimensional optical fibers of any refractive index—harmonic, repulsive, or free. Scaling terms in the ruling Hamiltonian lead to chirping in the frequency; multiplication of the signal by quadratic phases has of course no effect.

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