JOURNAL OF MATHEMATICAL PHYSICS VOLUME 44, NUMBER 4 APRIL 2003

Wigner functions for curved spaces. II. On spheres

Miguel Angel Alonso, George S. Pogosyan,[a] and Kurt Bernardo Wolf
Centro de Ciencias Físicas, Universidad Nacional Autónoma de México, Apartado Postal 48-3, Cuernavaca, Morelos 62251, México

(Received 12 November 2002; accepted 10 December 2002)

The form of the Wigner distribution function for Hamiltonian systems in spaces of constant negative curvature (i.e., hyperboloids) proposed in M. A. Alonso, G. S. Pogosyan, and K. B. Wolf, "Wigner functions for curved spaces. I. On hyperboloids" [J. Math. Phys. **43**, 5857 (2002)], is extended here to spaces whose curvature is constant and positive, i.e., spheres. An essential part of this construction is the use of the functions of Sherman and Volobuyev, which are an overcomplete set of plane-wave-like solutions of the Laplace–Beltrami equation for this space. Rotations that displace the poles transform these functions with a multiplier factor, and their momentum direction becomes formally complex; the covariance properties of the proposed Wigner function are understood in these terms. As an example for the one-dimensional case, we consider the energy eigenstates of the oscillator on the circle in a Pöschl–Teller potential. The standard theory of quantum oscillators is regained in the contraction limit to the space of zero curvature. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1559644]

I. INTRODUCTION

In the first part of this series[1] we proposed a generic form for the Wigner quasiprobability distribution function defined in terms of the generalized basis of plane waves; this form may be extended in a natural way to curved configuration spaces, provided that an analogous basis of plane-wave-like solutions can be found on those manifolds; the new functions will correspondingly endow their argument and index with the physical meaning of position and momentum. Although one may think to generalize the Wigner function to any manifold, the hyperboloid and the sphere are the two simplest cases to start such a study. In Ref. 1 we considered spaces of constant negative curvature, i.e., the upper sheet of a two-sheeted hyperboloid, where the basic plane waves were the set of Shapiro functions.[2] That Wigner function has the desired marginal projections, and its properties of covariance under rotations and hyperbolic translations were shown to stem from those of the Shapiro functions. The goal of this second part is the study of the Wigner function on spaces of positive constant curvature, i.e., on spheres.

As was the case in Ref. 1, the generalization offered in our approach results from recognizing that the Wigner function on flat phase space $(\mathbf{p},\mathbf{x})\in\Re^{2D}$,[3] in addition to its usual expression as a single integral, can be written also in the following twofold integral form with a Dirac δ,

$$\begin{aligned}W_{\Re^D}(f,g|\mathbf{x},\mathbf{p}) &:= \frac{1}{(2\pi)^D}\int_{\Re^D} d^D\mathbf{z}\, f(\mathbf{x}-\tfrac{1}{2}\mathbf{z})^* e^{-i\mathbf{p}\cdot\mathbf{z}} g(\mathbf{x}+\tfrac{1}{2}\mathbf{z})\\ &= \frac{1}{(2\pi)^D}\int_{\Re^D} d^D\mathbf{x}'\int_{\Re^D} d^D\mathbf{x}''\, f(\mathbf{x}')^* g(\mathbf{x}'')\,\phi_{\mathbf{p}}(\mathbf{x}')\,\delta^D(\mathbf{x}-\tfrac{1}{2}(\mathbf{x}'+\mathbf{x}''))\,\phi_{\mathbf{p}}(\mathbf{x}'')^*,\end{aligned} \tag{1}$$

[a]Permanent address: Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia and International Center for Advanced Studies, Yerevan State University, Yerevan, Armenia.

where the function $\phi_{\mathbf{p}}(x)$ and its complex conjugate $\phi_{\mathbf{p}}(x)^*$, whose argument and index variables bind the position and momentum variables, are the plane waves

$$\phi_{\mathbf{p}}(\mathbf{x}) := \exp(i\mathbf{p}\cdot\mathbf{x}), \quad -\Delta\,\phi(\mathbf{x}) = p^2\phi(\mathbf{x}), \tag{2}$$

where $p := |\mathbf{p}|$, and which are solutions of the Helmholtz (Laplace–Beltrami) equation on flat space. Momentum $\mathbf{p}$ has units of inverse length when $\hbar = 1$; in optics, p is the wave number of light.

The form (1) of the Wigner function again suggests its generalization to the sphere $\mathcal{S}^D$ through replacing the integration over flat space $(\int_{\mathfrak{R}^D} d^D\mathbf{x})$ by an integration over the new D-dimensional manifold $(\int_{\mathcal{S}^D} dx)$, replacing the plane waves $\phi_{\mathbf{p}}(\mathbf{x})$ of flat space by *plane-wave-like* solutions of the Laplace–Beltrami equation on that manifold, and replacing the Dirac delta $\delta^D(\mathbf{x} - \frac{1}{2}(\mathbf{x}' + \mathbf{x}''))$ in (1) by an appropriate distribution on the sphere. The new reproducing kernel should guarantee that, if $\mathbf{x}'$ and $\mathbf{x}''$ are on the manifold, then $\mathbf{x}$ should lie halfway along a geodesic.

In flat space, the transformation between the position and momentum representations arises from the basis of plane wave functions (2) that defines the Fourier transform; on the hyperboloid, it is a Mellin transform. Here, this transform will relate wave functions on the sphere with functions over a momentum space, through a summation over the discrete values that the wave number can have on the sphere, and an integral over the directions of the plane waves. Both the hyperboloid and the sphere are characterized by the radius R (curvature $\pm 1/R$), which will serve as the contraction parameter whose limit $R \to \infty$ represents flat space, and where the traditional phase space and Wigner function are recovered.

Let us stress that, unlike previous studies where the sphere is the symplectic manifold on which the Wigner function is drawn, as in the cases for spin[4] and finite systems,[5,6] or of the Wigner function defined on the coadjoint orbits of a Lie algebra[7] which may have a similar or more complicated topology, this Wigner function describes wave fields whose configuration space is the sphere. Also, we distinguish the present case from other previous definitions describing Helmholtz wave fields in flat free space, where momentum is constrained to the so-called Descartes sphere of ray directions.[8]

In Sec. II we concentrate the necessary definitions and relevant properties of these plane-wave-like solutions, and our understanding of the momentum space conjugate to the sphere. In Sec. III we develop the new Wigner function on the direct product phase space, making explicit its covariance properties and its contraction limit. As in Ref. 1, we illustrate some of these results in Sec. IV with an example: the harmonic oscillator analog on the circle ($D=1$) that corresponds to the bounded-interval Pöschl–Teller potential. In Sec. V we recapitulate our results in the context of other approaches in the literature.

II. SPHERICAL SPACES AND MOMENTUM

We follow the plan of Ref. 1 to present the Laplace–Beltrami operator on the curved space—here a D-dimensional spherical manifold—and its corresponding basis of plane-wave functions.[9,10] This is the basis we choose to define the momentum manifold that will appear in the definition of the Wigner function in the next section.

A. Laplace–Beltrami operator on the sphere

Consider the D-dimensional manifold of a sphere $\mathcal{S}^D$ of radius $R > 0$, embedded in the ambient space $x \in \mathfrak{R}^{D+1}$,

$$|x|^2 := x_0^2 + \mathbf{x}^2 = R^2, \quad \mathbf{x}^2 := x_1^2 + x_2^2 + \cdots + x_D^2. \tag{3}$$

The isometry group of the manifold of $\mathbf{x}$'s is the real orthogonal group in $D+1$ dimensions; for simplicity we disregard reflections and use the proper rotation group $\mathrm{SO}(D+1)$. This will replace the Euclidean isometry $\mathrm{ISO}(D)_+$ of flat configuration space. The standard realization of the Lie algebra $\mathrm{so}(D+1)$ by generators of rotations of the ambient $(D+1)$-dimensional space (3), is

$$M_{j,k} := x_j \partial_{x_k} - x_k \partial_{x_j}, \quad j,k = 0,1,2,\dots,D. \tag{4}$$

The Laplace–Beltrami operator on $\mathcal{S}_+^D$ is (R^{-2} times) the second-order invariant Casimir operator, namely,

$$\Delta_{\mathrm{LB}} := \frac{1}{R^2}\mathcal{C} = \frac{1}{R^2}\sum_{0\leqslant j<k\leqslant D} M_{j,k}^2 . \tag{5}$$

The spectrum of the Casimir operator of so$(D+1)$ is well known to be the lower bound, discrete but infinite set of values

$$\Sigma(\mathcal{C}) = \{\ell(\ell+D-1) \mid \ell \in \mathcal{Z}_0^+\}, \quad \mathcal{Z}_0^+ := \{0,1,2,\dots\}. \tag{6}$$

Corresponding to each value of ℓ there is a unitary irreducible representation belonging to the most degenerate (also called most symmetric) series, which is of finite dimension $[2\ell+1$ in so(3) for $D=2]$. The free wave functions on the sphere are the solutions to the Laplace–Beltrami equation characterized by those eigenvalues (6), that we choose to write as

$$\Delta_{\mathrm{LB}} f(\mathbf{x}) = -\frac{\ell(\ell+D-1)}{R^2} f(\mathbf{x}) = -\left[p^2 - \left(\frac{D-1}{2R}\right)^2\right] f(\mathbf{x}), \tag{7}$$

$$p := [\ell + \tfrac{1}{2}(D-1)]/R, \quad \ell = -\tfrac{1}{2}(D-1) + pR \in \mathcal{Z}_0^+ . \tag{8}$$

B. Sherman–Volobuyev functions on the sphere

In Ref. 1 we used the Shapiro functions, introduced by Gel'fand, Graev, and Shapiro in Ref. 2 as Fourier-type plane waves on a D-dimensional space of negative curvature (the upper sheet of the hyperboloid $\mathcal{H}_+^D$). Close analogs to these functions on the (compact) space of positive curvature—the sphere $\mathcal{S}^D \subset \mathfrak{R}^{D+1}$, were given by Sherman in Ref. 9 and were independently used by Volobuyev in Ref. 10, who wrote his work in the context of a phase space model where momentum space is the hyperboloid of Kadyshevsky and Mir–Kasimov,[11] and translated this to a spherical case with the Laplace–Beltrami equation on this manifold. In contrast to the denumerable basis of spherical harmonics, which are orthonormal and complete on $\mathcal{S}^D$, the generalized basis of Sherman–Volobuyev functions (as is the case with coherent states on flat space) are neither. Thus, this basis must be complemented by a distinct *dual* basis. In the following, we keep the notation in direct correspondence with that used in Ref. 1.

By vertical projection, the upper and lower hemispheres of a sphere $\mathcal{S}^D \subset \mathfrak{R}^{D+1}$ map on the same open equatorial disk $\mathcal{D}^D \subset \mathfrak{R}^D$ (and the equator on its common closure—a lower-dimensional $\mathcal{S}^{D-1}$ manifold). For convenience, functions $f(x)$ on the sphere $x \in \mathcal{S}^D$, $|x|^2 = R^2$, will be sometimes written as functions on $[\{-1,1\} \otimes \mathcal{D}^D] \oplus \mathcal{S}^{D-1}$ with colatitude angle χ as

$$f(x) \equiv f(x_0, \mathbf{x}) \equiv f_\sigma(\mathbf{x}),$$

$$x_0 = \sigma\sqrt{R^2 - \mathbf{x}^2} = R\cos\chi, \quad \sigma \in \{-1,1\}, \ 0 \leqslant \chi < \pi, \quad \text{or} \ \ \sigma = 0, \ \chi = \frac{1}{2}\pi, \tag{9}$$

$$\mathbf{x} = R\boldsymbol{\xi}\sin\chi \in \mathcal{D}^D \subset \mathfrak{R}^D, \quad \boldsymbol{\xi} \in \mathcal{S}^{D-1}.$$

The $\sigma=0$ submanifold is the equator of the sphere, but its explicit inclusion is not crucial to our work. Integration over the sphere will be written as

$$\int_{\mathcal{S}^D} dx\, f(x) := R \sum_{\sigma=-1,+1} \int_{\mathcal{D}^D} \frac{d\mathbf{x}}{\sqrt{R^2 - \mathbf{x}^2}} f_\sigma(\mathbf{x}), \tag{10}$$

and the $\sigma=0$ submanifold will be normally ignored.

The Sherman–Volobuyev functions and their duals are complex functions on the sphere that are solutions to the Laplace–Beltrami equation (7); they are classified according to (6) by the index $\ell\in\mathcal{Z}_0^+$ of completely symmetric representations in SO(D), or equivalently, by the discrete *wave number* p in (7), whose values are spaced by $1/R$. The functions in the Sherman–Volobuyev generalized basis of plane waves are characterized by a real *momentum* vector

$$\mathbf{p}:=p\mathbf{n},\quad p=\frac{1}{R}[\tfrac{1}{2}(D-1)+\ell],\quad \ell\in\mathcal{Z}_0^+,\quad \mathbf{n}\in\mathcal{S}^{D-1}. \tag{11}$$

which has the direction indicated by the unit vector on the sphere in the equatorial subspace. Using the relations (8) between the representation index ℓ and the absolute value of the momentum vector, $p=|\mathbf{p}|>0$ (for $D>1$), these functions and their duals are

$$\Phi_{\mathbf{p}}^{(D)}(x):=\left(\frac{x_0+i\mathbf{n}\cdot\mathbf{x}}{R}\right)^{\ell}=(\cos\chi+i\mathbf{n}\cdot\boldsymbol{\xi}\sin\chi)^{\ell}=\Phi_{p(-\mathbf{n})}^{(D)}(x)^*, \tag{12}$$

$$\begin{aligned}\overline{\Phi}_{\mathbf{p}}^{(D)}(x)&:=(\operatorname{sign}\mathbf{n}\cdot\mathbf{x})^{D-1}\left(\frac{x_0+i\mathbf{n}\cdot\mathbf{x}}{R}\right)^{1-D-\ell}\\&=(\operatorname{sign}\mathbf{n}\cdot\mathbf{x})^{D-1}(\cos\chi+i\mathbf{n}\cdot\mathbf{x}\sin\chi)^{1-D-\ell}\\&=(\operatorname{sign}\mathbf{n}\cdot\mathbf{x})^{D-1}/\Phi_{((D-1)/R+p)\mathbf{n}}^{(D)}(x)=\overline{\Phi}_{p(-\mathbf{n})}^{(D)}(x)^*.\end{aligned} \tag{13}$$

In Fig. 1 we show Sherman–Volobuyev functions for the case $D=2$, which can be readily plotted on the sphere $\mathcal{S}^2$. The functions (12) can be equivalently characterized as the highest-weight solid $\mathcal{S}^D$-hyperspherical harmonics $\mathcal{Y}_{\ell,\ldots,\ell}(x)\sim(x_1+ix_2)^{\ell}$ (which are solutions of the Laplace equation in the ambient space), rotated so as to bring the x_1-x_2 plane to the plane x_0-$\mathbf{n}$, for each equatorial direction $\mathbf{n}\in\mathcal{S}^{D-1}$. Their dual functions (13) are the second solutions of the Laplace equation, which are obtained by replacing $\ell\to 1-D-\ell$, and formally correspond to the same eigenvalues (6) of the Casimir operator on the sphere; they are singular on the $\mathcal{S}^{D-2}$ submanifold orthogonal to the x_0-$\mathbf{n}$ plane. In the $D=2$ case, these are the two points at right angles to the wavetrain.

C. Properties and limits

The Sherman–Volobuyev functions satisfy the following completeness and orthogonality relations:

$$\frac{1}{(2\pi)^D}\sum_{\ell=0}^{\infty}N^{(D)}(p)\int_{\mathcal{S}^{D-1}}d\mathbf{n}\,\Phi_{p\mathbf{n}}^{(D)}(x)\overline{\Phi}_{p\mathbf{n}}^{(D)}(x')=\delta_{\mathcal{S}^D}(x,x'), \tag{14}$$

$$\frac{1}{(2\pi)^D}\int_{\mathcal{S}^D}dx\,\overline{\Phi}_{p\mathbf{n}}^{(D)}(x)\Phi_{p'\mathbf{n}'}^{(D)}(x)=\frac{1}{N^{(D)}(p)}\delta_{p,p'}d_p(\mathbf{n},\mathbf{n}'), \tag{15}$$

where the Plancherel weight of the irreducible representations is

$$\begin{gathered}N^{(D)}(p):=pR\Gamma(\tfrac{1}{2}(D-1)+pR)/\Gamma(-\tfrac{1}{2}(D-3)+pR)=\tfrac{1}{2}(D-1)!\,\Delta_{\ell}^{(D)},\\ \Delta_{\ell}^{(D)}:=\dim\ \text{irrep}\ \ \ell\ \ \text{of}\ \ \mathrm{SO}(D+1).\end{gathered} \tag{16}$$

Writing $|\mathcal{S}^{D-1}|=2\pi^{\frac{1}{2}D}/\Gamma(\tfrac{1}{2}D)$ for the surface of the sphere, the $\delta_{\mathcal{S}^D}(x,x')$ on the ambient sphere $\mathcal{S}^D$, and the $d_p(\mathbf{n},\mathbf{n}')$ on the momentum direction spheres $\mathbf{n},\mathbf{n}'\in\mathcal{S}^{D-1}$, are

$$\delta_{\mathcal{S}^D}(x,x')=\delta_{\sigma,\sigma'}\sqrt{R^2-\mathbf{x}^2}\,\delta^D(\mathbf{x}-\mathbf{x}'),\quad \mathbf{x},\mathbf{x}'\in\mathcal{D}^D, \tag{17}$$

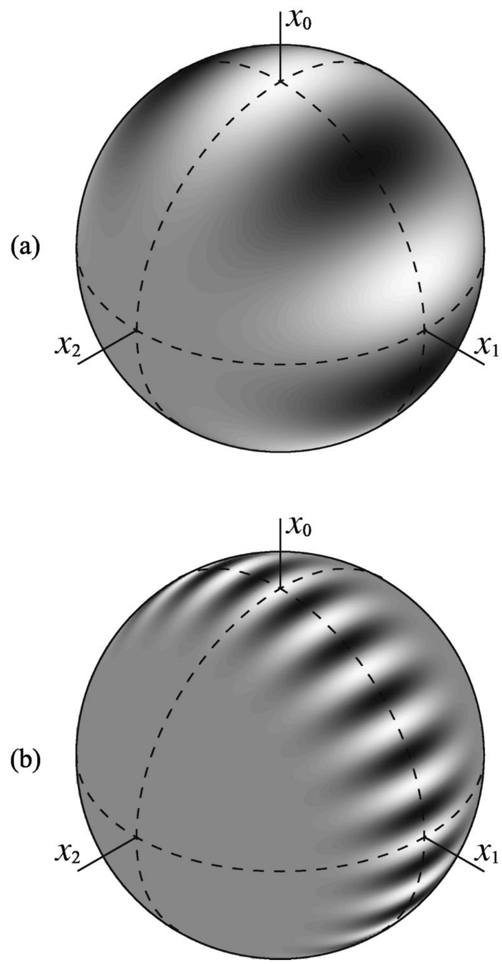

FIG. 1. Sherman–Volobuyev functions for the case $D=2$, $\Phi_{\mathbf{p}}^{(2)}(x)$ on the sphere $x\in\mathcal{S}^2$. The real part is shown for $\ell=5$ and 20, for momenta $\mathbf{p}=p\,\mathbf{n}$ with $p=11/2R$ and $41/2R$, in the same direction $\mathbf{n}(\theta)\in\mathcal{S}^1$ along 1-axis ($\theta=0$). White and black correspond to values $+1$ and -1 of the function; the 2ℓ extrema occur along the meridian at the x_0-x_1 plane; at the two points on the x_2 axis of the sphere the complex functions are zero. The imaginary part is identical to the real part except for a rotation of $\pi/2\ell$ around the x_2 axis, i.e., by a displacement of $\frac{1}{4}$ wavelength.

$$d_p(\mathbf{n},\mathbf{n}')=\frac{1}{|\mathcal{S}^{D-1}|}\left(C_\ell^{D/2}(\mathbf{n}\cdot\mathbf{n}')+C_{\ell-1}^{D/2}(\mathbf{n}\cdot\mathbf{n}')\right),\quad \mathbf{n},\mathbf{n}'\in\mathcal{S}^{D-1}, \tag{18}$$

where $\delta^D(\mathbf{x}-\mathbf{x}')$ is the D-dimensional Dirac delta on the disk $\mathcal{D}^D$, and there is the Kronecker delta $\delta_{p,p'}:=\delta_{\ell,\ell'}$ between spheres of discrete radii p and p'. The $C_\ell^{1/2\,D}(\kappa)$ are the Gegenbauer polynomials of degree ℓ in $\kappa=\mathbf{n}\cdot\mathbf{n}'$, i.e., the cosine of the angle between the two momentum vectors, $\mathbf{p}$ and $\mathbf{p}'$. In particular, note that for $\ell=0$, $N^{(D)}=\frac{1}{2}\Gamma(D)$.

As pointed out by Sherman and by Volobuyev,[9,10] the last $d_p(\mathbf{n},\mathbf{n}')$ in (18) is not a true Dirac δ, but a reproducing kernel in the $\Delta_\ell^{(D)}$-dimensional vector space spanned by the functions $\{\Phi_{p\mathbf{n}}^{(D)}(x)\}_{\mathbf{n}\in\mathcal{S}^{D-1}}$ of fixed wave number $p\leftrightarrow\ell$,

$$\int_{\mathcal{S}^{D-1}}d\mathbf{n}'\;d_p(\mathbf{n},\mathbf{n}')\Phi_{p\mathbf{n}'}^{(D)}(x)=\Phi_{p\mathbf{n}}^{(D)}(x), \tag{19}$$

and the same property holds for the duals $\{\overline{\Phi}_{p\mathbf{n}}^{(D)}(x)\}_{\mathbf{n}\in\mathcal{S}^{D-1}}$. In the limit of large wave numbers, $\lim_{pR\to\infty}d_p(\mathbf{n},\mathbf{n}')=\delta_{\mathcal{S}^{D-1}}(\mathbf{n},\mathbf{n}')$.

The Inönu–Wigner contraction limit of the rotation to the Euclidean group SO$(D+1)$ $\to$ISO(D) is the limit $R\to\infty$ in our expressions for vectors with $x_0\approx R$, $\mathbf{x}^2\ll R^2$, and $\mathbf{p}=p\,\mathbf{n}$ as before with discrete values of p separated by a decreasing R^{-1}, i.e.,

$$\lim_{R\to\infty}\Phi_{\mathbf{p}}^{(D)}(x)=\lim_{R\to\infty}\left(\frac{x_0+i\mathbf{n}\cdot\mathbf{x}}{R}\right)^{-1/2(D-1)+pR}\approx\lim_{R\to\infty}\left(1+i\frac{\mathbf{n}\cdot\mathbf{x}}{R}\right)^{pR}=\exp(i\mathbf{x}\cdot\mathbf{p}),\tag{20}$$

$$\lim_{R\to\infty}\overline{\Phi}_{\mathbf{p}}^{(D)}(x)=\exp(-i\,\mathbf{x}\cdot\mathbf{p}).\tag{21}$$

Correspondingly, $\lim_{R\to\infty}N^{(D)}(p)=1$ and $\delta^D(x,x')\to\delta^D(\mathbf{x}-\mathbf{x}')$. Ordinary Fourier analysis and synthesis are thus recovered in the contraction limit; this justifies the name of plane waves for the Sherman–Volobuyev functions, as well as our expectation that they will provide the bridge between the position on the sphere and a physically appropriate momentum space.

D. Momentum space for the sphere

The basis of Sherman–Volobuyev functions is nonorthonormal and overcomplete, as can be seen from (15), (18), and (19), but allows the synthesis of functions $f(x)$ on the sphere $x\in\mathcal{S}^D$, with coefficients in a space that we recognize as the *momentum* manifold, $\mathbf{p}=p\mathbf{n}\in\mathcal{Z}_0^+\otimes\mathcal{S}^{D-1}$ of the D-dimensional system on configuration space $x\in\mathcal{S}^D$.

The Sherman–Volobuyev synthesis of a complex function $f(x)$ over the sphere $x\in\mathcal{S}^D$, involves a sum of integrals over spheres; the sum ranges over the radii $p=\frac{1}{2}(D-1)/R,\frac{1}{2}(D+1)/R,\ \frac{1}{2}(D+3)/R,\ldots$ (corresponding to $\ell=0,1,2,\ldots$), and the integrals over $\mathbf{n}\in\mathcal{S}^{D-1}$, with both the functions and their duals, as follows:[9,10]

$$f(x)=\frac{1}{(2\pi)^{D/2}}\sum_{\ell=0}^{\infty}N^{(D)}(p)\int_{\mathcal{S}^{D-1}}d\mathbf{n}\,\Phi_{p\mathbf{n}}^{(D)}(x)\tilde{f}(p\mathbf{n}),\tag{22}$$

$$f(x)^*=\frac{1}{(2\pi)^{D/2}}\sum_{\ell=0}^{\infty}N^{(D)}(p)\int_{\mathcal{S}^{D-1}}d\mathbf{n}\,\overline{\Phi}_{p\mathbf{n}}^{(D)}(x)\tilde{\tilde{f}}(p\mathbf{n}).\tag{23}$$

The coefficients are found by

$$\tilde{f}(p\mathbf{n})=\frac{1}{(2\pi)^{D/2}}\int_{\mathcal{S}^D}dx\,\overline{\Phi}_{p\mathbf{n}}^{(D)}(x)\,f(x),\tag{24}$$

$$\tilde{\tilde{f}}(p\mathbf{n})=\frac{1}{(2\pi)^{D/2}}\int_{\mathcal{S}^D}dx\,\Phi_{p\mathbf{n}}^{(D)}(x)\,f(x)^*.\tag{25}$$

This means that there are *two* (rather than a single) mutually dual momentum representations for any one wave function on the sphere. That both should be considered on equal footing is indicated by the Parseval relation,

$$(f,g)_{\mathcal{S}^D}:=\int_{\mathcal{S}^D}dx\,f(x)^*g(x)\tag{26}$$

$$=\sum_{\ell=0}^{\infty}N^{(D)}(p)\int_{\mathcal{S}^{D-1}}d\mathbf{n}\,\tilde{\tilde{f}}(p\mathbf{n})\,\tilde{g}(p\mathbf{n})\tag{27}$$

$$=\sum_{\ell=0}^{\infty}N^{(D)}(p)\int_{\mathcal{S}^{D-1}}d\mathbf{n}\,\tilde{f}(p\mathbf{n})^*\tilde{\tilde{g}}(p\mathbf{n})^*.\tag{28}$$

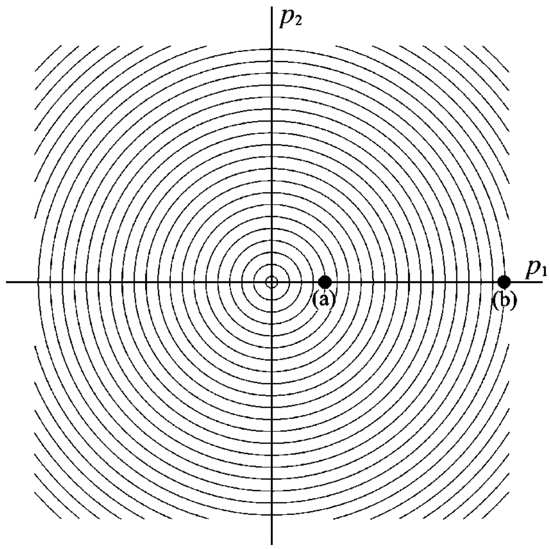

FIG. 2. Momentum space and the Sherman–Volobuyev functions $\Phi^{(2)}_{\mathbf{p}}(x)$. The momentum space $\mathbf{p}=p\,\mathbf{n}$ is composed by concentric circles $\mathbf{n}(\theta)\in\mathcal{S}^1$, of radii $p\in\{\frac{1}{2},\frac{3}{2},\ldots\}/R$ (for $\ell\in\{0,1,\ldots\}$). The two functions of Fig. 1 are shown for $\ell=5$ and 20, with direction $\mathbf{n}$ along the 1-axis.

It will help intuition to consider again the case $D=2$ for wave fields on the ordinary two-sphere $\mathcal{S}^2$, where $x_0^2+x_1^2+x_2^2=R^2$. While in Fig. 1 we show the complex Sherman–Volobuyev functions, Figure 2 schematizes their representation in the momentum space $\mathbf{p}=p\mathbf{n}$ of concentric circles with discrete radii $p\leftrightarrow\ell$ and direction $\mathbf{n}\in\mathcal{S}^1$. The length of the momentum vector $p=|\mathbf{p}|$ is associated to the number ℓ of wavelengths around the meridian indicated by $\mathbf{n}$, which is normal to all wave fronts. There is an evident covariance between the SO(2) rotations of the circles of momentum and rotations of the configuration-space sphere around its x_0 axis.

E. Covariance properties

Because the basis of Sherman–Volobuyev functions (12) and their duals (13) depends on the scalar product $\mathbf{n}\cdot\mathbf{x}$, they will be covariant in $\mathbf{x}$ and $\mathbf{n}$ under rotations $\mathbf{R}\in\mathrm{SO}(D)$ of the sphere $\mathcal{S}^D$ within its equatorial disk $\mathbf{x}\in\mathcal{D}^D$, *viz.*,

$$T(\mathbf{R}):\Phi^{(D)}_{p\mathbf{n}}(x_0,\mathbf{x}):=\Phi^{(D)}_{p\mathbf{n}}(x_0,\mathbf{R}^{-1}\mathbf{x})=\Phi^{(D)}_{p\mathbf{R}\mathbf{n}}(x_0,\mathbf{x}), \tag{29}$$

and similarly for the dual $\overline{\Phi}^{(D)}_{p\mathbf{n}}(x_0,\mathbf{x})$'s.

We now analyze further the transformation properties of the Sherman–Volobuyev plane-wave-like basis under $\mathrm{SO}(D+1)$ rotations *out of* the equatorial disk $\mathbf{x}\in\mathcal{D}^D$ (i.e., mixing x_0 and components of $\mathbf{x}$), and the covariant transformations of the sphere of momentum directions $\mathbf{n}\in\mathcal{S}^{D-1}$. Under these transformations, the direction vector $\mathbf{n}$ of momentum may become complex, as we now show. Indeed, the functions (12) can be written as the power ℓ of a scalar product between one complex and one real $(D+1)$-vectors (Refs. 9 and 10) also indicated by $\cdot$:

$$x_0+i\mathbf{n}\cdot\mathbf{x}=\begin{pmatrix}1\\ i\mathbf{n}\end{pmatrix}\cdot\begin{pmatrix}x_0\\ \mathbf{x}\end{pmatrix}=:\nu\cdot x,\quad \mathbf{n}\cdot\mathbf{n}=1,\quad \nu\cdot\nu=0, \tag{30}$$

To find the transformation of the Sherman–Volobuyev function set under rotations of the ambient-space vectors $x\in\mathcal{S}^D$ in the plane of x_0 and a unit vector $\mathbf{m}\in\mathcal{S}^{D-1}$ in the equatorial subspace of the sphere, we decompose the position vectors as $\mathbf{x}=\mathbf{x}_{\parallel\mathbf{m}}+\mathbf{x}_{\perp\mathbf{m}}$, into their components parallel and perpendicular to the direction of $\mathbf{m}$. The latter are invariant under all rotations of the x_0-$\mathbf{m}$ plane, that we indicate by $\mathbf{R}_\mathbf{m}\in \mathrm{SO}(D+1)$. Then we can write $x=(x_0,\mathbf{x}_{\|\mathbf{m}},\mathbf{x}_{\perp\,\mathbf{m}})^\top$ and $\mathbf{n}=(\mathbf{n}_{\|\mathbf{m}},\mathbf{n}_{\perp\,\mathbf{m}})^\top$, so the action of a rotation by $\alpha\in\mathcal{S}^1$ on ambient space will be

$$T(\mathbf{R}_\mathbf{m}(\alpha)):\begin{pmatrix} x_0\\ \mathbf{x}_{\|\mathbf{m}}\\ \mathbf{x}_{\perp\,\mathbf{m}}\end{pmatrix}=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} x_0\\ \mathbf{x}_{\|\mathbf{m}}\\ \mathbf{x}_{\perp\,\mathbf{m}}\end{pmatrix}. \tag{31}$$

The corresponding transformation of the momentum vector $\mathbf{p}=p\mathbf{n}$ will leave the irreducible representation index p invariant, and the action on the unit direction vectors $\mathbf{n}$ which characterize the Sherman–Volobuyev functions can be found from (30), through the $(D+1)$-dimensional inner product form $\nu\cdot x'=\nu'\cdot x$. This yields the transformation of the complex vector $\nu=(1,i\mathbf{n})^\top$ to

$$\nu'(\alpha)=\begin{pmatrix}1\\ i\mathbf{n}_{\|\mathbf{m}}\\ i\mathbf{n}_{\perp\,\mathbf{m}}\end{pmatrix}\cdot\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}=\mu(\mathbf{m},\alpha;\mathbf{n})\begin{pmatrix}1\\ i\mathbf{n}'_{\|\mathbf{m}}\\ i\mathbf{n}'_{\perp\,\mathbf{m}}\end{pmatrix}, \tag{32}$$

with a *multiplier* function (which is independent of x),

$$\mu(\mathbf{m},\alpha;\mathbf{n}):=\cos\alpha+i\mathbf{m}\cdot\mathbf{n}\sin\alpha \tag{33}$$

and a new direction vector

$$\mathbf{n}'=\begin{pmatrix}\mathbf{n}'_{\|\mathbf{m}}\\ \mathbf{n}'_{\perp\,\mathbf{m}}\end{pmatrix}=\frac{1}{\mu(\mathbf{m},\alpha;\mathbf{n})}\begin{pmatrix}(\mathbf{m}\cdot\mathbf{n}\cos\alpha+i\sin\alpha)\mathbf{m}\\ \mathbf{n}_{\perp\,\mathbf{m}}\end{pmatrix}, \tag{34}$$

of real norm $\mathbf{n}'\cdot\mathbf{n}'=1$.

The action of $\mathbf{R}_\mathbf{m}\in\mathrm{SO}(D+1)$ on the Sherman–Volobuyev functions of fixed wave number $p\leftrightarrow\ell$ [recall (11)], and their duals is therefore

$$T(\mathbf{R}_\mathbf{m}(\alpha)):\Phi^{(D)}_{p\mathbf{n}}(x)=\mu(\mathbf{m},\alpha;\mathbf{n})^{\ell}\,\Phi^{(D)}_{p\mathbf{n}'}(x), \tag{35}$$

$$T(\mathbf{R}_\mathbf{m}(\alpha)):\overline{\Phi}^{(D)}_{p\mathbf{n}}(x)=\mu(\mathbf{m},\alpha;\mathbf{n})^{1-D-\ell}\,\overline{\Phi}^{(D)}_{p\mathbf{n}'}(x). \tag{36}$$

The transformations that rotate out of the equatorial subspace thus produce "complex momentum direction vectors." We use quotes around this phrase because the Sherman–Volobuyev functions are already an overcomplete set, and those whose $\mathbf{n}$'s are complex are in any case expressible in terms of the real-$\mathbf{n}$ set, as we shall note below. But formally, the complexification of the direction sphere $\mathbf{n}$ can be a useful tool for intuition. When we separate the real and imaginary parts of $\mathbf{n}'(\alpha)=\mathbf{r}'(\alpha)+i\mathbf{s}'(\alpha)$, we see that

$$\begin{pmatrix}\mathbf{r}'_{\|\mathbf{m}}\\ \mathbf{r}'_{\perp\,\mathbf{m}}\end{pmatrix}=\frac{1}{|\mu(\mathbf{m},\alpha;\mathbf{n})|^2}\begin{pmatrix}\mathbf{n}\cdot\mathbf{m}\mathbf{m}\\ \mathbf{n}_{\perp\,\mathbf{m}}\sin\alpha\cos\alpha\end{pmatrix}, \tag{37}$$

$$\begin{pmatrix}\mathbf{s}'_{\|\mathbf{m}}\\ \mathbf{s}'_{\perp\,\mathbf{m}}\end{pmatrix}=\frac{-1}{|\mu(\mathbf{m},\alpha;\mathbf{n})|^2}\begin{pmatrix}((\mathbf{m}\cdot\mathbf{n})^2-1)\mathbf{m}\sin\alpha\cos\alpha\\ -\mathbf{n}\cdot\mathbf{m}\mathbf{n}_{\perp\,\mathbf{m}}\sin\alpha\end{pmatrix}. \tag{38}$$

Here we note that $\mathbf{r}'\cdot\mathbf{s}'=0$ for all α, and this implies $|\mathbf{r}'|^2-|\mathbf{s}'|^2=1$; this is the surface of a hyperboloid, of signature $(+,-)$ in the D real and D imaginary components. This confines $\mathbf{s}$ to an independent $\mathcal{S}^{D-2}$-sphere.

In dimension D, the complex sphere $\mathcal{C}^{D-1}$ is a homogeneous space for the action of $\mathrm{SO}(D)$, which is determined by its natural action on $\mathcal{S}^D$ through (31)–(34). When we shall discuss in Sec. III C the behavior of the Wigner function under translations (i.e., rotations) of space, the transformations of position and of momentum that are correlated by the map (31)–(34) will define the

Sherman–Volobuyev covariance of the proposed Wigner function. The two lowest-dimensional cases will now be examined to show how the above formalism reduces to the analysis of the well-known Fourier series.

F. The cases $D=1$ (circle) and $D=2$ (sphere)

The $D=1$ case of Sherman–Volobuyev functions (39) on the circle $\chi\in\mathcal{S}^1$ may appear trivial, but it is important to note that we recover the Fourier series that we use in the example of Sec. IV. Momentum space is now the set of points $p=\ell n/R$, $\ell\in\mathcal{Z}_0^+$, and $n\in\mathcal{S}^0=\{-1,+1\}$ is a sign. Moreover, the duals are now the complex conjugate functions,

$$\Phi^{(1)}_{\pm\ell/R}(x)=e^{\pm i\ell\chi}=\bar{\Phi}^{(1)}_{\pm\ell/R}(x)^*. \tag{39}$$

The discrete measure over momentum space $N^{(1)}(p)$ in (16) is, in the $D=1$ case,

$$N^{(1)}(\ell\geqslant 1)=1,\quad N^{(1)}(\ell=0)=\tfrac{1}{2}. \tag{40}$$

In particular the two $\ell=0$ functions $\Phi^{(1)}_{\pm 0}(x)=1$ will sum with the factor $\frac{1}{2}$ from (40) to provide a single $e^{i0\chi}=1$ basis element. This is the full Fourier basis $e^{im\chi}$, with $m=n\ell\in\mathcal{Z}$, reproduced with the correct unit normalization coefficients. The multiplier function in (33) is $\mu(\cdot,\alpha;n)=e^{in\alpha}$ [$n=\operatorname{sign} m\in\{-1,+1\}$, cf. (39)]. Under rotations of the χ-circle therefore, the functions $e^{im\chi}$ are multiplied by the correct phase $e^{in\ell\alpha}$, as follows from (35)–(36). The Sherman–Volobuyev synthesis and analysis (22)–(25) in the $D=1$ case on the circle are given by the well-known Fourier series

$$f(\chi)=\frac{1}{\sqrt{2\pi}}\sum_{m\in\mathcal{Z}}e^{im\chi}\tilde{f}(m),\quad \tilde{f}(m)=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}d\chi\, e^{-im\chi}f(\chi). \tag{41}$$

For $D\geqslant 2$, the Sherman–Volobuyev basis functions are an overcomplete set. This overcompleteness is transparent in the case $D=2$ of the sphere $\mathcal{S}^2$, where the momentum direction $\mathbf{n}(\theta)$ is parametrized around the circle $\theta\in\mathcal{S}^1$, and $\mathbf{n}(\theta)\cdot\mathbf{n}(\theta')=\cos(\theta-\theta')$—see Fig. 2. For fixed $p\leftrightarrow\ell$, the dimension of the space of functions $f_{(p)}(\theta)$ is $\Delta_\ell^{(2)}=2\ell+1$, where a better known, orthonormal and complete basis is that of solid spherical harmonics $\{\mathcal{Y}_{\ell,m}(x)\}_{m=-\ell}^{\ell}$. In other words, although the momentum circles in Fig. 2 appear continuous, only $2\ell+1$ points on each circle correspond to independent functions. On these circles, the Gegenbauer polynomials in (18) reduce to Chebyshev polynomials of the second kind, $C_\ell^1(\kappa)=U_\ell(\kappa)=\sin[(\ell+1)\theta]/\sin\theta$, and reproduce the well-known Dirichlet kernel,

$$d_p(\mathbf{n},\mathbf{n}')=\frac{1}{2\pi}\sum_{m=-\ell}^{\ell}e^{im(\theta-\theta')}=\frac{\sin[(\ell+\frac{1}{2})(\theta-\theta')]}{2\pi\sin\frac{1}{2}(\theta-\theta')}\xrightarrow[\ell\to\infty]{}\delta(\theta). \tag{42}$$

Since the functions $\Phi^{(2)}_{\mathbf{p}}(x)$ are polynomials of integer degree $\ell=pR-\frac{1}{2}$ in $\mathbf{n}\cdot\mathbf{x}\sim\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$, then any function $f_{(p)}(\theta)$ in this space is fully reproduced by (42), i.e.,

$$\int_{\mathcal{S}^1}d\theta\, d_p(\mathbf{n}(\theta),\mathbf{n}'(\theta'))f_{(p)}(\theta)=f_{(p)}(\theta'). \tag{43}$$

Also visible in the $D=2$ case of Fig. 1 is the covariance of the Sherman–Volobuyev functions under rotations out of the equatorial plane, Eqs. (31)–(38), leading to complex direction vectors $\mathbf{n}=\mathbf{r}+i\mathbf{s}$. The real part $\mathbf{r}\in\Re^2$ of $\mathbf{n}$ here determines the imaginary part $\mathbf{s}$ up to a sign (the two points of $\mathcal{S}^0\subset\Re$). When $\mathbf{n}=\mathbf{r}$ is real, $|\mathbf{r}|^2=1\Rightarrow\mathbf{s}=\mathbf{0}$. The vector $\mathbf{n}$ is complex when and only when $|\mathbf{r}|^2>1$, and then its imaginary part $\mathbf{s}$ has magnitude $|\mathbf{s}|^2=|\mathbf{r}|^2-1$, and lies at right angles to $\mathbf{r}$.

To examine the multiplier function (33), we consider the Sherman–Volobuyev functions $\Phi^{(2)}_{\mathbf{p}}(x)$ on the sphere $x \in \mathcal{S}^2$ [shown in Fig. 1 and Eq. (12)] whose real momentum direction vector is along the 1-axis, $\mathbf{n} = \binom{1}{0}$. When we rotate by α the x-sphere in the 0-1 plane [Eq. (35) with $\mathbf{m}\|\mathbf{n}$, so $\mathbf{m}\cdot\mathbf{n}=1$], then for p and ℓ related by (8), the multiplier factor is

$$\mu(\mathbf{m},\alpha,\mathbf{m}) = e^{i\alpha}, \tag{44}$$

and the transformed Sherman–Volobuyev function will be

$$\mathbf{R}_{\mathbf{m}}(\alpha): \Phi^{(2)}_{p\mathbf{m}}(x) = [(x'_0 + i x'_1)/R]^{\ell} = e^{i\ell\alpha}\Phi^{(2)}_{p\mathbf{m}}(x), \tag{45}$$

i.e., they eigenfunctions of rotations in the direction of the momentum $\mathbf{n}=\mathbf{m}$ [cf. the extreme spherical harmonics $\mathcal{Y}_{\ell,\ell}(x)$ under rotations about x_0]. On the other hand, when the rotation is performed in the 0-2 plane, then instead of (45) we use (35), now with $\binom{0}{1} = \mathbf{m}\perp\mathbf{n} = \binom{1}{0}$, so $\mathbf{m}\cdot\mathbf{n} = 0$, and the multiplier is

$$\mu(\mathbf{m},\alpha,\perp\mathbf{m}) = \cos\alpha. \tag{46}$$

Thus rotated, the Sherman–Volobuyev functions remain plane-wave-like solutions of the Laplace–Beltrami equation,

$$\mathbf{R}_{\mathbf{m}}(\alpha): \Phi^{(2)}_{p(\perp\mathbf{m})}(x) = (x_0\cos\alpha - x_2\sin\alpha + i x_1)^{\ell} = (\cos\alpha)^{\ell}\Phi^{(2)}_{p\mathbf{n}''(\alpha)}(x), \quad \mathbf{n}''(\alpha) = \begin{pmatrix} \sec\alpha \\ i\tan\alpha \end{pmatrix}, \tag{47}$$

whose wave fronts are normal to a maximal circle, which is no longer a sphere meridian, as those in Fig. 1. The real part of $\mathbf{n}''$ points in the same direction as $\mathbf{n}$, but the imaginary part is responsible for displacing the wave train laterally, along the 2-axis. We underline again that when $\mathbf{n}''$ is not real, $\Phi^{(2)}_{p\mathbf{n}''}(x)$ does not belong to the Sherman–Volobuyev function basis [which by itself satisfies (14)–(15)], but to an analytic continuation of their continuous direction label $\mathbf{n}$ to the complex unit circle $\mathcal{C}^1$.

G. Oscillators on the sphere

Free fields on the sphere, whose energy is purely kinetic, are ruled by the Laplace–Beltrami equation (5)–(7). A second energy term is introduced by adding a function $V(\mathbf{x})$ of position,

$$\left(\frac{-1}{2\mu}\Delta_{\mathrm{LB}} + R^2 V(\mathbf{x})\right) f(\mathbf{x}) = R^2 E f(\mathbf{x}). \tag{48}$$

In Schrödinger quantum mechanics this describes a particle of mass $m = \hbar^2\mu$ in a potential $V(\mathbf{x})$.[12–14] In wave optics, the interpretation of the extra term comes from the refractive index anomaly of the medium $n(\mathbf{x}) = \mu - V(\mathbf{x})$, with $\mu \gg V$ and $V^2 \approx 0$.

An SO$(D-1)$-isotropic harmonic oscillator potential on the sphere $\mathcal{S}^D$, depending only on the colatitude angle $\chi \in [0,\pi]$ of (9), can be generalized in many ways. An especially useful model is, as in the hyperbolic case,[1] the Pöschl–Teller potential in D-dimensional configuration space[13] given by

$$V(\mathbf{x}) = \frac{1}{2}\mu\omega^2 R^2 \frac{|\mathbf{x}|^2}{x_0^2} = \frac{1}{2}\mu\omega^2 R^2 \tan^2\chi = \frac{1}{2}\mu\omega^2 R^2(\sec^2\chi - 1). \tag{49}$$

The wave functions of this model are also the Wigner (Clebsch–Gordan) coupling coefficients for the three-dimensional Lorentz algebra so(2,1) between representations belonging to the discrete, lower-bound Bargmann D_k^+ series.[15]

III. WIGNER FUNCTION ON THE SPHERE

Here we construct the Wigner function for the sphere in the same way as for the hyperboloid in Ref. 1, namely generalizing the double-integral form in Eq. (1), replacing the plane waves over $\mathfrak{R}^D$ with the Sherman–Volobuyev functions and their duals over $\mathcal{S}^D$.

A. Definition

With the measure (10) and the functions in (12)–(13), we define the Wigner function on the sphere as

$$W_{\mathcal{S}}(f,g|x,\mathbf{p}):=\frac{1}{(2\pi)^D}\int_{\mathcal{S}^D}d^Dx'\int_{\mathcal{S}^D}d^Dx''\, f(x')^*\Delta^D(x;x',x'')g(x'')\\ \times\frac{1}{2}[\Phi_{\mathbf{p}}^{(D)}(x')\overline{\Phi}_{\mathbf{p}}^{(D)}(x'')+\overline{\Phi}_{\mathbf{p}}^{(D)}(x')^*\Phi_{\mathbf{p}}^{(D)}(x'')^*]. \tag{50}$$

We now describe each of the elements of this definition.

We denote the position argument $x=(x_0,\mathbf{x})$ of the Wigner function by the ambient vector, with the understanding that it is the position on the sphere; contrary to the hyperbolic case, where the surface can be mapped 1:1 on $\mathbf{x}$, the sign of x_0 distinguishes between the two hemispheres [and we prefer not to write $(\sigma,\mathbf{x})$ as in (9)]. As in Ref. 1, the $\Delta^D(x;x',x'')$ which takes the place of the flat Dirac delta $\delta^D(\mathbf{x}-\frac{1}{2}(\mathbf{x}'+\mathbf{x}''))$ in Eq. (1), should guarantee that x be the midpoint of the shortest *geodesic* between x' and x'', and lie on the sphere $\mathcal{S}^D$ of radius R. To this end, we choose any $(D+1)$-vector $y=(y_0,\mathbf{y})\in\mathcal{S}^D$ which is orthogonal to $x=(x_0,\mathbf{x})\in\mathcal{S}^D$, $x\cdot y=0$. Then, we write

$$\begin{aligned} x' &:= x\cos\tfrac{1}{2}\alpha-y\sin\tfrac{1}{2}\alpha\\ x'' &:= x\cos\tfrac{1}{2}\alpha+y\sin\tfrac{1}{2}\alpha \end{aligned}\quad\Rightarrow x=\frac{x'+x''}{2\cos\frac{1}{2}\alpha}, \tag{51}$$

so $|x|=R=|y|\Leftrightarrow|x'|=R=|x''|$ for all $\alpha\in[0,\pi]$ and any y on the $\mathcal{S}^{D-1}$ sphere orthogonal to x. From (51) it also follows that $x\cdot x'=R^2\cos\frac{1}{2}\alpha=x\cdot x''$ and $x'\cdot x''=R^2\cos\alpha$, so x indeed lies at angles $\frac{1}{2}\alpha$ between x' and x'' on the sphere. When the signs of the 0-components match, the binding Δ in (50) that enforces (51) on the equatorial projection disks $\mathcal{D}_{\pm}^D$, can be written as

$$\Delta^D(x;x',x'')=\frac{x_0}{R}\,\delta^D\!\left(\mathbf{x}-\frac{\mathbf{x}'+\mathbf{x}''}{2\cos\frac{1}{2}\alpha}\right). \tag{52}$$

More generally, when we denote by $v_{\perp x}$ the component of $v\in\mathfrak{R}^{D+1}$ which is orthogonal to x, the binding Δ is

$$\Delta^D(x;x',x'')=\delta^D\!\left(\frac{(x'+x'')_{\perp x}}{2\cos\frac{1}{2}\alpha}\right). \tag{53}$$

This distribution has the properties

$$\Delta^D(x;x',x')=\frac{x_0}{R}\,\delta^D(\mathbf{x}-\mathbf{x}'),\qquad\int_{\mathcal{S}^D}d^Dx\,\Delta^D(x;x',x'')=1. \tag{54}$$

Through complex conjugation, we verify that the Wigner function (50) satisfies the necessary property

$$W_{\mathcal{S}}(f,g|x,\mathbf{p})^*=W_{\mathcal{S}}(g,f|x,\mathbf{p}). \tag{55}$$

This is the reason for the factor $\frac{1}{2}[\Phi_{\mathbf{p}}^{(D)}(x')\,\overline{\Phi}_{\mathbf{p}}^{(D)}(x'')+\overline{\Phi}_{\mathbf{p}}^{(D)}(x')^*\,\Phi_{\mathbf{p}}^{(D)}(x'')^*]$; only this combination turns into itself with x' and x'' exchanged, and has the correct contraction limit detailed in Sec. III D. Equation (55) guarantees that, for $f=g$, the Wigner function is real.

By means of this binding Δ and the change of variables in (51), the $2D$-fold integration in the Wigner function (50) reduces to the D-fold integral form

$$\begin{aligned} W_{\mathcal{S}}(f,g|x,\mathbf{p}) = [1/(2\pi)^D]\int_0^\pi (\sin\alpha)^{D-1}\,d\alpha \int_{\mathcal{S}_{\perp x}^{D-1}} d^{D-1}y \\ \times f(x\cos\tfrac{1}{2}\alpha - y\sin\tfrac{1}{2}\alpha)^* g(x\cos\tfrac{1}{2}\alpha + y\sin\tfrac{1}{2}\alpha) \\ \times\tfrac{1}{2}[\Phi_{\mathbf{p}}^{(D)}(x\cos\tfrac{1}{2}\alpha - y\sin\tfrac{1}{2}\alpha)\overline{\Phi}_{\mathbf{p}}^{(D)}(x\cos\tfrac{1}{2}\alpha + y\sin\tfrac{1}{2}\alpha) \\ +\overline{\Phi}_{\mathbf{p}}^{(D)}(x\cos\tfrac{1}{2}\alpha - y\sin\tfrac{1}{2}\alpha)^*\Phi_{\mathbf{p}}^{(D)}(x\cos\tfrac{1}{2}\alpha + y\sin\tfrac{1}{2}\alpha)^*]. \end{aligned} \tag{56}$$

B. Marginal projections

The integral of the Wigner function $W_{\mathcal{S}}(f,g|x,\mathbf{p})$ in (50) over momentum space yields the cross-probability distribution over configuration space, and conversely, integration over the sphere yields a function of momentum shown below. The two marginal distributions derive from the orthogonality and completeness relations of the Sherman–Volobuyev basis and its dual, Eqs. (15)–(14) and (54). They are

$$\begin{aligned} M_{\mathcal{S}}(f,g|x) &= \int_{\mathbf{p}\in\Re^D} \frac{d^D\mathbf{p}}{N^{(D)}(p)} W_{\mathcal{S}}(f,g|x,\mathbf{p}) \\ &= \int_{\mathcal{S}^D} d^Dx' \int_{\mathcal{S}^D} d^Dx''\, f(x')^* g(x'')\Delta^D(x;x',x'')\,\delta^D(x',x'') \\ &= \int_{\mathcal{S}^D} d^Dx'\, f(x')^* g(x')\Delta^D(x;x',x') = f(x)^* g(x), \end{aligned} \tag{57}$$

$$\begin{aligned} M_{\mathcal{S}}(f,g|\mathbf{p}) &= \int_{\mathcal{S}^D} d^Dx\, W_{\mathcal{S}}(f,g|x,\mathbf{p}) \\ &= \frac{1}{2(2\pi)^D}\Bigg[\int_{\mathcal{S}^D} d^Dx'\, f(x')^*\Phi_{\mathbf{p}}^{(D)}(x')\int_{\mathcal{S}^D} d^Dx''\, g(x'')\,\overline{\Phi}_{\mathbf{p}}^{(D)}(x'') \\ &\quad + \int_{\mathcal{S}^D} d^Dx'\, f(x')^*\overline{\Phi}_{\mathbf{p}}^{(D)}(x')^*\int_{\mathcal{S}^D} d^Dx'' g(x'')\Phi_{\mathbf{p}}^{(D)}(x'')^*\Bigg] \\ &= \frac{1}{2}[\tilde{\bar{f}}(\mathbf{p})\tilde{g}(\mathbf{p}) + \tilde{f}(\mathbf{p})^*\tilde{\bar{g}}(\mathbf{p})^*]. \end{aligned} \tag{58}$$

We note that both the momentum representation and its dual appear on equal footing. The Parseval relation (27)–(28) provides the overlap

$$\int_{\mathcal{S}^D} d^Dx\, M_{\mathcal{S}}(f,g|x) = (f,\,g)_{\mathcal{S}^D} = \int_{\mathbf{p}\in\mathcal{S}^D} \frac{d^D\mathbf{p}}{N^{(D)}(p)} M_{\mathcal{S}}(f,g|\mathbf{p}). \tag{59}$$

C. Covariance under SO($D+1$) rotations

Under rotations $\mathbf{R}\in\mathrm{SO}(D+1)$ of the ambient space around the x_0 axis, the basis of Sherman–Volobuyev functions (12)–(13) on the $\mathcal{S}^D$-sphere transform as given by (29). Since the integrations and binding Δ in (53) that appear in the definition (50) are invariant $[\Delta^D(\overline{x};\overline{x}',\overline{x}'')=\Delta^D(x;x',x'')$ for $\overline{\mathbf{x}}=\mathbf{R}^{-1}\mathbf{x}$, etc.], it follows that the proposed Wigner function is covariant under $\mathrm{SO}(D)$ rotations, fulfilling

$$W_{\mathcal{S}}(T(\mathbf{R}):f,T(\mathbf{R}):g|x,\mathbf{p})=W_{\mathcal{S}}(f,g|x_0,\mathbf{R}^{-1}\mathbf{x},\mathbf{R}^{-1}\mathbf{p}). \tag{60}$$

But now consider the rotations of the sphere $\mathcal{S}^D$ out of the equatorial $\mathbf{x}\in\Re^D$ subspace, $\mathbf{R_m}(\alpha)\in\mathrm{SO}(D+1)$, as was done in (35)–(47) for the Sherman–Volobuyev functions and their duals of direction $\mathbf{n}$, and $\ell\leftrightarrow p$ characterizing the invariant wave number [$\mathrm{SO}(D,1)$ irreducible representation] as given by (8), and $\mathbf{n}'$ by (34). The Wigner function (50) is bilinear in $\Phi^{(D)}_{p\,\mathbf{n}}(x)$ and $\overline{\Phi}^{(D)}_{p\,\mathbf{n}}(x)$, and so it will transform with a multiplier factor that is extracted from the integral, as

$$\begin{aligned}W_{\mathcal{S}}(T[\mathbf{R_m}(\alpha)]:f,T[\mathbf{R_m}(\alpha)]:g|x,p\mathbf{n})\\=\mathrm{Re}[(\mu(\mathbf{m},\alpha,\mathbf{n}))^{-D+1}]W_{\mathcal{S}}(f,g|\mathbf{R_m}(\alpha)^{-1}:x,p\mathbf{R_m}(\alpha)^{-1}:\mathbf{n}).\end{aligned} \tag{61}$$

We call (61) the Sherman–Volobuyev covariance of Wigner functions on the sphere. This concept is the analog of that introduced for the hyperbolic case in Ref. 1. Since volume elements of the momentum direction sphere $\mathbf{n}\in\mathcal{S}^D$ are not conserved under rotations $\mathbf{R_m}\in\mathrm{SO}(D+1)$, the multiplier for the Wigner function, $\mu(\mathbf{m},\alpha,\mathbf{n})$ in (33), is necessary to offset this change of measure and ensure the total conservation of probability contained in (59). A new feature that appears in the sphere, however, is that an analytic continuation of the momentum direction is implied by this covariance.

D. Contraction limit

When the radius of the sphere grows and the functions $f(x)$ and $g(x)$ in the Wigner function remain significantly different from zero only within a given area around $x=(R,\mathbf{0})$ that becomes increasingly a flat patch, the Wigner function (56) reduces to the standard Wigner function for flat space, Eq. (1). In (56), the integrand will be significant only when $\mathcal{S}^D$-norms of the vectors fulfill

$$|\mathbf{x}\cos\tfrac{1}{2}\alpha\pm\mathbf{y}\sin\tfrac{1}{2}\alpha|\ll R\Rightarrow\begin{cases}|\mathbf{x}|\cos\tfrac{1}{2}\alpha\ll R & \Rightarrow\sin\chi\ll 1,\\ |\mathbf{y}|\sin\tfrac{1}{2}\alpha\ll R & \Rightarrow\sin\alpha\ll 1,\end{cases} \tag{62}$$

$$\Rightarrow x\approx R(1,\chi\boldsymbol{\xi})^{\mathrm{T}},\quad y\approx R(\chi\boldsymbol{\xi}\cdot\boldsymbol{\eta},\boldsymbol{\eta})^{\mathrm{T}}, \tag{63}$$

where $\boldsymbol{\eta}\in\mathcal{S}^{D-1}$ is a unit vector in the direction of $\mathbf{y}$. The limit (20) and the approximations $\sin\alpha\approx\alpha$ and $\cos\frac{1}{2}\alpha\approx\cos\chi\approx 1$, bring the Wigner function (56) to

$$\begin{aligned}W_{\mathcal{S}}(f,g|x,\mathbf{p})=\frac{R^D}{(2\pi)^D}\int_0^\infty\alpha^{D-1}\,d\alpha\int_{\mathcal{S}^{D-1}}d^{D-1}\boldsymbol{\eta}\\\times f(x_0,\mathbf{x}-\tfrac{1}{2}R\alpha\boldsymbol{\eta})^*\exp(-iR\alpha\boldsymbol{\eta}\cdot\mathbf{p})g(x_0,\mathbf{x}+\tfrac{1}{2}R\alpha\boldsymbol{\eta}).\end{aligned} \tag{64}$$

Changing variables to $\mathbf{z}=R\alpha\,\boldsymbol{\eta}$ and integrations by $\int_{\Re^D}d^D\mathbf{z}=R^D\int_0^\infty\alpha^{D-1}\,d\alpha\times\int_{\mathcal{S}^{D-1}}d^{D-1}\boldsymbol{\eta}$, completes the proof that (64) reduces to (1) in the limit $R\to\infty$.

IV. PÖSCHL–TELLER OSCILLATOR ON THE CIRCLE

We saw in Eqs. (39)–(41) that in the case $D=1$, the Sherman–Volobuyev basis coincides with the Fourier series basis of complex exponential functions on the circle, and that momentum space is a set of equally spaced points on a line,

$$\Phi^{(1)}_p(x_1)=\exp(i\chi pR),\quad x_1=R\sin\chi,\quad \chi\in\mathcal{S}^1,\quad p=m/R,\quad m\in\mathcal{Z}. \tag{65}$$

A. Wigner function on the circle

The Wigner function of wave functions on the circle, Eq. (56), has the same structure as the standard flat-space Wigner function (1) except for the integration ranges. The displaced arguments of the two functions f and g in (56), in the form (65) where $x_1 = R\sin\chi$ and $y_1 = R\sin\eta$, are

$$x\cos\tfrac{1}{2}\alpha \pm y\sin\tfrac{1}{2}\alpha = R\begin{pmatrix}\cos(\chi\pm\frac{1}{2}\eta\alpha)\\ \sin(\chi\pm\frac{1}{2}\eta\alpha)\end{pmatrix}, \tag{66}$$

with $\eta\alpha\in(-\pi,\pi)$. The Wigner function (56), indicating $f(R\cos\chi,R\sin\chi)=f(\chi)$ and $p=m/R$, thus becomes

$$W_{\mathcal{S}}(f,g|x,p) = \frac{R}{2\pi}\int_{-\pi}^{\pi} d\alpha\, f\left(\chi-\frac{1}{2}\alpha\right)^* e^{-im\alpha} g\left(\chi+\frac{1}{2}\alpha\right) \tag{67}$$

$$= \frac{1}{2\pi}\sum_{m',m''=-\infty}^{\infty} \tilde{f}(m')^*\,\mathrm{sinc}\left[\frac{1}{2}(m'+m'')-m\right] e^{i(m''-m')\chi}\tilde{g}(m''), \tag{68}$$

where $\mathrm{sinc}\,\nu := \sin(\pi\nu)/\pi\nu$ is $\delta_{\nu,0}$ when ν is integer, and $(-1)^{\nu-1/2}/\pi\nu$ when ν is half-integer; therefore the double sum in (68) cannot be reduced to a single one except when the coefficients $\tilde{f}(m)$ vanish for a given parity of m. Finally, we recall that for $D=1$ the multiplier function (61) for rotations of the circle is unity.

B. Oscillator on the circle

We now consider the oscillator on the circle which obeys the $D=1$ case of the Schrödinger equation (48) with the Pöschl–Teller potential given in Eq. (49), and written

$$V(\chi) = \sqrt{r(r-1)}(\sec^2\chi - 1), \qquad r := \tfrac{1}{2} + \tfrac{1}{2}\sqrt{(2\mu\omega R^2)^2+1}. \tag{69}$$

This potential exhibits two inpenetrable barriers at $\chi = \pm\frac{1}{2}\pi$ on $\mathcal{S}_1$. We thus expect two independent solutions in the two disconnected open intervals $\chi\in(-\frac{1}{2}\pi,\frac{1}{2}\pi)$ and $\chi\in(\frac{1}{2}\pi,\frac{3}{2}\pi)$.

Changing variables and placing the potential (69) into the Schrödinger equation on the circle (48), one obtains the Pöschl–Teller equation,[16]

$$\frac{d^2\psi}{d\theta^2} + [4\varepsilon - r(r-1)(\sec^2\theta + \csc^2\theta)]\psi = 0, \quad \theta := \tfrac{1}{2}\chi \pm \tfrac{1}{4}\pi \in (0,\tfrac{1}{2}\pi); \quad \varepsilon := 2\mu R^2(E + \tfrac{1}{2}\mu\omega^2R^2). \tag{70}$$

Writing $\chi^{\pm} = 2\theta \mp \frac{1}{2}\pi \in (-\frac{1}{2}\pi,\frac{1}{2}\pi)$ and $\psi^{\pm}(\chi) = \psi(\chi^{\pm}) = \psi(\theta)$, the solutions to this equation are

$$\psi_n^{r,\pm}(\chi) = 2^{2r}\sqrt{\frac{n!(n+r)}{\pi\Gamma(n+2r)}}\,\Gamma(r)\left(\frac{1}{2}\sin 2\theta\right)^r C_n^r(\cos 2\theta)$$

$$= \Theta(\pm\cos\chi)\sqrt{\frac{n!(n+r)}{2\pi\Gamma(n+2r)}}\,\Gamma(r)|2\cos\chi|^r C_n^r(\sin\chi), \tag{71}$$

where $\Theta(x)$ is the Heaviside function that determines the well in which the particle is confined, so that $\psi_n^{r,-}(\chi) = \psi_n^{r,+}(\chi+\pi)$. In what follows we assume the particle is in $(-\frac{1}{2}\pi,\frac{1}{2}\pi)$ and disregard the index $\pm$. The spectrum of values of ε is quantized in the quadratic series $(n+r)^2$, so the energy values are

$$E_n^r = \frac{(r+n)^2}{2\mu R^2} - \frac{1}{2}\mu\omega^2 R^2 = \frac{1}{2\mu R^2}\left(n^2 + 2r\left(n+\frac{1}{2}\right)\right). \tag{72}$$

C. Contractions to the square box and oscillator in flat space

It is interesting to consider two limiting cases of the Pöschl–Teller potential on the two half-circles in Eqs. (71) and (72). The first is the limit of weak potentials $\omega\to 0$ (so $r\to 1$), and the second is the analog of the previous contraction, now from the circle to the line.

In the limit of weak potential barrier, one could *prima facie* expect that the Pöschl–Teller eigenstates (71) may reduce to the free eigenstates (39) on the circle. This is not the case however, as can be seen by setting $r=1$ in Eqs. (71) and using the property[17] that $\cos\chi\, C_n^1(\sin\chi)$ is $\cos[(n+1)\chi]$ for n even, and $\sin[(n+1)\chi]$ for n odd,

$$\psi_n^1(\chi) = \Theta(\cos\chi)\sqrt{\frac{2}{\pi}}\begin{cases}\cos[(n+1)\chi], & n \text{ even},\\ \sin[(n+1)\chi], & n \text{ odd},\end{cases} \qquad E_n^1 = \frac{(n+1)^2}{2\mu R^2}. \tag{73}$$

The energies (72) for the limit states form a quadratic sequence characteristic of a square well with impenetrable barriers at $\chi = \pm\pi/2$. This, rather than the free circle, is the limit $r\to 1$ of the Pöschl–Teller potential.

The second limit of interest is the contraction $r\to\infty$ of the Pöschl–Teller potential on the circle to the harmonic oscillator on flat space,

$$r \gg 1 \Leftarrow r \sim \mu\omega R^2,$$

$$(1-z^2)^{r/2} \sim \exp(-rz^2/2) \quad \text{for } z^2 < 1.$$

Then, Eq. (71) becomes

$$\begin{aligned}\psi_n^r(\chi) &= \Theta(\cos\chi)\sqrt{\frac{n!(n+r)\Gamma^2(r)}{2^n\sqrt{\pi}\,\Gamma(r+\frac{1}{2}n)\Gamma(r+\frac{1}{2}[n+1])}}\,|\cos\chi|^r C_n^r(\sin\chi)\\ &\sim \sqrt{\frac{n!}{\sqrt{2\pi}}}(2r)^{1/4-n/2}e^{-\frac{1}{2}r\sin^2\chi}C_n^r(\sin\chi)\\ &\sim \frac{1}{\sqrt{n!2^n\sqrt{\pi/r}}}e^{-\frac{1}{2}r\sin^2\chi}H_n(\sqrt{r}\sin\chi)\\ &= \frac{\sqrt{R}}{\sqrt{n!\,2^n\sqrt{\pi/\mu\omega}}}e^{-\mu\omega x_1^2/2}H_n(\sqrt{\mu\omega}x_1).\end{aligned} \tag{74}$$

In the last expression we replaced $z=\sin\chi = x_1/R$, and again, these are the energy eigenstates of the harmonic oscillator in flat space. The energies of these limit states, from (72), now exhibit the linear harmonic oscillator spectrum $E_n = \omega(n+\frac{1}{2})$. The $\sqrt{R}$ factor compensates the normalization on x_1.

D. Wave functions in momentum representation

The momentum representation of the wave functions $\psi_n^r(\chi)$ can be found from the Fourier series coefficients $\tilde{\psi}_n^{r,\pm}(m)$ in (41) of the functions $\psi_n^{r,\pm}(\chi)$ in (71). It is convenient to expand the Gegenbauer polynomials as

$$C_n^r(\sin\chi)=\frac{e^{in\pi/2}}{[\Gamma(r)]^2}\sum_{j=0}^{n}(-1)^j\frac{\Gamma(j+r)\Gamma(n-j+r)}{j!(n-j)!}e^{i(2j-n)\chi}. \tag{75}$$

The integral can be then performed[17] and yields the momentum representation of the wave functions of $p=m/R$ in terms of the hypergeometric ${}_3F_2$ function of unit argument, as

$$\begin{aligned}\tilde{\psi}_n^r(m)&:=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}d\chi\,\psi_n^r(\chi)e^{im\chi}\\&=\frac{e^{in\pi/2}}{\sqrt{8\pi}}\sqrt{\frac{n+r}{n!\Gamma(n+2r)}}\frac{\Gamma(n+r)\Gamma(r+1)}{\Gamma(\frac{1}{2}(r+m+n)+1)\Gamma(\frac{1}{2}(r-m-n)+1)}\\&\quad\times{}_3F_2\left(\left.\begin{array}{c}-n,\ -\frac{1}{2}(r+m+n),\ r\\ -r-n+1,\ \frac{1}{2}(r-m-n)+1\end{array}\right|1\right).\end{aligned} \tag{76}$$

Because the wave functions $\psi_n^r(\chi)$ vanish on one half-circle, it turns out that it is sufficient to determine the coefficients for m even; in fact, any periodic function $f(\chi)$ vanishing in the interval $(\frac{1}{2}\pi,\frac{3}{2}\pi)$ will have its odd-m coefficients determined by the even-m ones through the relation

$$\tilde{f}(2m+1)=(-1)^m\sum_{k\in\mathcal{Z}}\frac{(-1)^k}{\pi(m-k+\frac{1}{2})}\tilde{f}(2k). \tag{77}$$

E. Wigner function for the Pöschl–Teller states

The Wigner function (50) in the case $D=1$ for two functions f,g on the circle $\chi\in\mathcal{S}_1$ was written in Eqs. (67)–(68). For the energy eigenstates $\psi_n^r(\chi)$ of the Pöschl–Teller potential given in (71), the Wigner functions can be computed numerically; we have not been able to find a closed expression for them. They are plotted in Fig. 3 along with their marginal projections, for $n=0,1,5,10$.

V. CONCLUDING REMARKS

We have defined the analog of the Wigner function of Ref. 1 for the case of a spherical configuration space. We have observed remarkably different properties between the hyperbolic and spherical cases. First, unlike the Shapiro functions of the former, the Sherman–Volobuyev functions of real momentum are an overcomplete set; a dual basis is thus required and this implies the existence of two dual momentum representations. Further, a coordinate translation which displaces the poles causes the momentum of a Sherman–Volobuyev function to become complex. As a consequence, the covariance of the momentum representation(s) as well as the that of Wigner function under this type of translation are meaningful only as an analytic continuation of the momentum direction vector. The appearance of a multiplier is analogous to the hyperbolic case in Ref. 1.

These features derive from the definition of momentum afforded by the Shapiro and the Sherman–Volobuyev plane-wave-like solutions of the Laplace–Beltrami equation on the hyperbolic and spherical manifolds, and are reflected by the Wigner function introduced here. In trying to fit the definition (50) and the corresponding one for the hyperbolic case in Ref. 1, into the existing plethora of Wigner functions defined in Refs. 3–7, 18 and others found in the literature, it seems increasingly clear that the concept of a Wigner function is not unique. Perhaps a working definition of such a class of functions $W(f,g|x,p)$ should include only (cf. Ref. 19) sesquilinearity in the wave fields $(\sim f(x')^*g(x''))$, a symmetric correlation between their arguments x',x'' to a point x in the manifold [determined by a Dirac-type $\Delta(x;x',x'')$], and a complete (or overcomplete) basis (or generalized basis) $\{\Phi_p(x)\}$ which will provide p as conjugate coordinate for a momentum manifold to complete phase space. The minimal properties to be expected of such

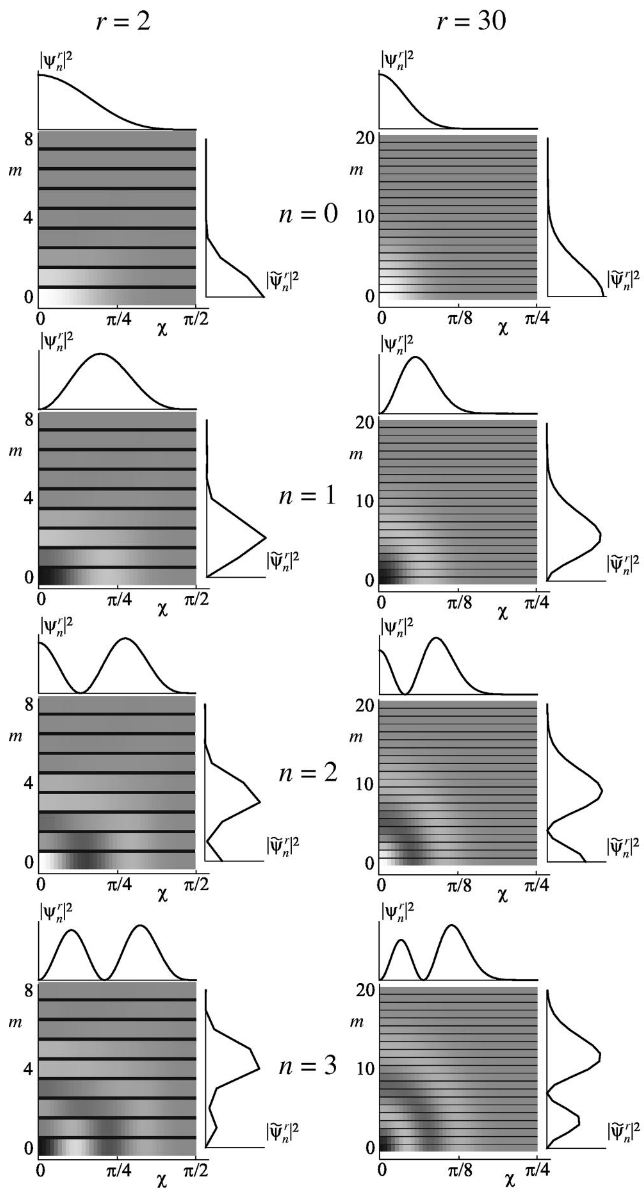

FIG. 3. Wigner functions of the Pöschl–Teller eigenstates $\psi_n^r(\chi)$, on rows of mode $n=0,1,2,3$, for values of the parameter r of the sphere [Eq. (69)], $r=2$ (left) and $r=30$ (right); we show a quadrant of position $x_1=R\sin\chi$, $\chi\in[0,\frac{1}{2}\pi]$ and momentum/angular momentum $p=m/R$ [Eqs. (65)]. The quadrants have reflection symmetry across the axes. White is the maximum, black is the minimum; the shade at the upper right corner corresponds to zero. The marginal projection $|\psi_n^r(\chi)|^2$ is plotted at top, and $|\widetilde{\psi}_n^r(m)|^2$ is plotted to the right.

Wigner functions should include the correct marginals, a useful form of covariance between the wave fields and the phase space coordinates, and a natural contraction limit to flat space returning the traditional Wigner function.

To test Wigner function models, it is also important to have a number of basic systems, such as the harmonic or Pöschl–Teller potentials, or Coulomb systems, that should substantiate intuition and the usefulness of the representation. A practical example could be the description of surface waves on sperical bubbles. Let us not forget that the Wigner function does not provide more information than the wave fields do (in fact, overall phases are lost), but displays this information in a manner that should be more amenable to our understanding.

ACKNOWLEDGMENTS

We thank the support of the Dirección General de Asuntos del Personal Académico, Universidad Nacional Autónoma de México (DGAPA–UNAM) grant IN112300 *Optica Matemática*. M.A.A. and K.B.W. thank Professor Harrison Barrett for the hospitality at the Optical Sciences Center (Tucson, Arizona), and Dana Clarke for introducing to us the objective of an interesting research project. G.S.P. acknowledges the Consejo Nacional de Ciencia y Tecnología (México) for a Cátedra Patrimonial Nivel II, and the Armenian National Science and Engineering Foundation for Grant No. PS124–01.

[1] M. A. Alonso, G. S. Pogosyan, and K. B. Wolf, J. Math. Phys. **43**, 5857 (2002).
[2] I. M. Gel'fand, M. I. Graev, and I. I. Pyatetskiĭ-Shapiro, *Representation Theory and Automorphic Functions* (Saunders, Philadelphia, 1969).
[3] E. Wigner, Phys. Rev. **40**, 749 (1932).
[4] G. S. Agarwal, Phys. Rev. A **24**, 2889 (1981); J. C. Várilly and J. M. Gracia-Bondía, Ann. Phys. (N.Y.) **190**, 107 (1989); J. P. Dowling, G. S. Agarwal, and W. P. Schleich, Phys. Rev. A **49**, 4101 (1994).
[5] K. B. Wolf, N. M. Atakishiyev, and S. M. Chumakov, *Photonic Quantum Computing*, Proceedings of SPIE, Vol. 3076, pp. 196–206, 1997; N. M. Atakishiyev, S. M. Chumakov, and K. B. Wolf, J. Math. Phys. **39**, 6247 (1998).
[6] S. M. Chumakov, A. B. Klimov, and K. B. Wolf, Phys. Rev. A **61**, 034101 (2000).
[7] S. T. Ali, N. M. Atakishiyev, S. M. Chumakov, and K. B. Wolf, Ann. Henri Poincare **1**, 685 (2000).
[8] K. B. Wolf, M. A. Alonso, and G. W. Forbes, J. Opt. Soc. Am. A **16**, 2476 (1999); M. A. Alonso, *ibid.* **18**, 910 (2001).
[9] T. O. Sherman, Trans. Am. Math. Soc. **209**, 1 (1975).
[10] I. P. Volobuyev, Teor. Mat. Fiz. **45**, 421 (1980).
[11] V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, Problems Elementary Particle At. Nucl. Phys. **2**, 635 (1972).
[12] C. Grosche, G. S. Pogosyan, and A. N. Sissakian, Phys. Part. Nucl. **27**, 244 (1996); Fortschr. Phys. **43**, 453 (1995).
[13] P. W. Higgs, J. Phys. A **12**, 309 (1979); H. I. Leemon, *ibid.* **12**, 489 (1979).
[14] E. Schrödinger, Proc. R. Irish Acad. **46**, 9 (1941); **46**, 183 (1941); **47**, 53 (1941).
[15] V. Bargmann, Ann. Math. **48**, 568 (1947); D. Basu and K. B. Wolf, J. Math. Phys. **24**, 478 (1983), Eqs. (6.9)–(6.12); A. Frank and K. B. Wolf, Phys. Rev. Lett. **52**, 1737 (1984); J. Math. Phys. **26**, 973 (1985).
[16] S. Flügge, *Practical Quantum Mechanics* (Springer-Verlag, Berlin, 1994).
[17] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 2.
[18] See, e.g., Feature Issue, *Wigner Distributions and Phase Space in Optics*, edited by G. W. Forbes, V. I. Man'ko, H. M. Ozaktas, R. Simon, and K. B. Wolf, J. Opt. Soc. Am. A (2000).
[19] H.-W. Lee, Phys. Rep. **259**, 147 (1995); C. Brif and A. Mann, Phys. Rev. A **59**, 971 (1999).