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CONTRACTION OF THE FINITE ONE-DIMENSIONAL OSCILLATOR

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The finite oscillator model of $2j+1$ points has the dynamical algebra u(2), consisting of position, momentum and mode number. It is a paradigm of finite quantum mechanics where a sequence of finite unitary models contract to the well-known continuum theory. We examine its contraction as the number and density of points increase. This is done on the level of the dynamical algebra, of the Schrödinger difference equation, the (Kravchuk) wave functions, and the Fourier–Kravchuk transformation between position and momentum representations.

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1. Introduction

Contractions were introduced in physics by Inönü and Wigner[1] as a mathematical expression of the correspondence principle. This principle tells us that, whenever a new physical theory generalizes an old one, there should exist a well-defined limit in which the results of the old theory are recovered. Typical examples of contractions that are important to physics are the $c \to \infty$ nonrelativistic limit of relativistic theories, the $\hbar \to 0$ classical limit of quantum theories, and the limit $R \to \infty$ of the de Sitter radius to Poincaré relativity. In this and a following paper we study the contraction of a discrete, finite one-dimensional oscillator model[2–5] and a radial oscillator[6] model to their well-known "continuous" quantum mechanical limits.

New models must include a parameter in whose limit (zero or infinity) they reproduce the original ("well-known") model; on the other hand, the inverse route to a "precontracted" theory may not be unique. This has been the case of discrete

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quantum mechanics, where several models have been proposed which contract to the common Schrödinger quantum theory; these have been generally based on cyclic subgroups of the Heisenberg–Weyl group and related constructs (see e.g. Ref. 7). Our finite oscillator model[3] uses the well-known formalism of angular momentum, but proposes a new physical interpretation for the generators of the algebra $\mathrm{u}(2) = \mathrm{u}(1) \oplus \mathrm{su}(2) = \mathrm{so}(2) \oplus \mathrm{so}(3)$ as position, momentum and energy, within a definite unitary irreducible representation $j \in \{0, \frac{1}{2}, 1, \dots\}$, to describe systems with $2j + 1$ observable values. Because the u(2) algebra is compact, the spectra of all operators will be intrinsically discrete and finite; coherent states exist,[4] and a covariant Wigner quasiprobability distribution function is defined.[8] A notorious difference between the previous models and the present one (which was originally introduced for signal processing by optical means[3]), is that in the former the position observable is cyclic, so the two points at the ends of the basic interval are actually first neighbors, while here the points $\pm j$ are properly the endpoints of a finite interval.

In Sec. 2 we recall the essentials of the u(2) finite oscillator model: its Lie algebra realization, the position and mode (energy) bases, the wave functions and their Schrödinger *difference* equation, and the oscillator evolution Green function. The contraction of u(2) to the oscillator algebra is addressed in Sec. 3. In Sec. 4 we contract the finite oscillator (*Kravchuk*) functions to the Hermite functions of the quantum oscillator. This derivation is new in the sense that it does not rely on the limit of the recursion relations, but proceeds directly from function properties; the ${}_2F_1$-hypergeometric representation is not suited for this limit, but a recent relation between Wigner little-d and ${}_3F_2$-functions[9] permits the direct proof. In Sec. 5 we show that the finite oscillator Green functions (finite Fourier–Kravchuk transforms) contract to the ordinary Fourier transform. The previous three contractions correspond to three limits of the little-d Wigner functions $d^j_{m,m'}(\frac{1}{2}\pi)$ when one (j), two (j, m), or three (j, m, m') indices grow to infinity. Some concluding comments are appended in Sec. 6.

2. One-Dimensional Finite Oscillator

The observables of an oscillator are position, momentum and energy. Their values are the points of the spectrum of three operators, Q, P and H, that should close into a Lie algebra. The time evolution is given by the Hamilton equations $[H, Q] = -iP$ and $[H, P] = iQ$, while the third commutator, $[Q, P]$, is left undetermined.[10,11] The ordinary quantum oscillator is obtained when this commutator is $[Q, P] = i\hbar\hat{1}$ (where $\hat{1}$ is the identity operator), so one has the *oscillator* dynamical Lie algebra $H_4 = \mathrm{span}\{H, Q, P, \hat{1}\}$. The finite oscillator model on the other hand,[2,3,8] is characterized through the (non-standard) commutator $[Q, P] = i(H - j - \frac{1}{2}) = iJ_3$, and thus endowed with the dynamical algebra u(2), in the unitary irreducible representation space labeled by $j = \frac{1}{2}N$, of fixed dimension $N + 1$, $N \in \{0, 1, 2, \dots\}$.

2.1. *The finite oscillator Lie algebra*

The Lie algebra u(2) is formed by identifying the three commonly designated generators of $\mathrm{su}(2) = \mathrm{span}\{J_1, J_2, J_3\}$, with the observables of

$$\begin{array}{llll} \text{position} & Q = J_1\,, & \text{spectrum:} & q|_{-j}^{j}\,, \\ \text{momentum} & P = -J_2\,, & & p|_{-j}^{j}\,, \\ \text{mode number} & H - \tfrac{1}{2} = J_3 + E_J\,, & & n|_0^N\,, \end{array} \tag{1}$$

where the operator E_J generates the center $\mathrm{u}(1)_c \subset \mathrm{u}(2)$ (commuting with J_k), and takes the value $j\hat{1}$ in the representation space j. The u(2) commutation relations are

$$[J_1, J_2] = iJ_3\,, \quad [J_2, J_3] = iJ_1\,, \quad [J_3, J_1] = iJ_2\,, \quad [E_J, J_k] = 0\,. \tag{2}$$

[The role of E_J is analogous to that of the total-number-of-quanta operator in the u(2) symmetry algebra of the two-dimensional quantum oscillator, where $\mathrm{u}(1)_c$ is conjugate to su(2) [12] within its boson, or metaplectic, representation; but note carefully the physical difference between the symmetry algebra u(2) of the ordinary two-dimensional quantum oscillator, and the dynamical algebra u(2) of our one-dimensional finite oscillator.] The su(2) Casimir operator is $J^2 = J_1^2 + J_2^2 + J_3^2$, so the central operator may be written as $E_J = -\frac{1}{2} + \left(J^2 + \frac{1}{4}\right)^{1/2}$ [with abuse of notation, because such an "operator" does not belong to the enveloping algebra of su(2)].

It is well known[13,14] that in any unitary irreducible representation j of u(2), we can define the eigenbasis of J_3, $|j,m\rangle_3$, $m|_{-j}^{j}$, with its raising and lowering operators $J_\pm = J_1 \pm J_2$, as follows:

$$\begin{aligned} E_J|j,m\rangle_3 &= j|j,m\rangle_3\,, & J^2|j,m\rangle_3 &= j(j+1)|j,m\rangle_3\,, \\ J_3|j,m\rangle_3 &= m|j,m\rangle_3\,, & J_\pm|j,m\rangle_3 &= \alpha^j(\pm m)|j,m\rangle_3\,, \end{aligned} \tag{3}$$

$$\alpha^j(m) := \sqrt{(j+m+1)(j-m)} = \alpha^j(-m-1)\,. \tag{4}$$

The basis $\{|j,m\rangle_3\}_{m=-j}^{j}$ is orthonormal and complete in the representation space of dimension $2j+1$. In this paper we shall work both with the spectrum of the operator J_3, namely $m|_{-j}^{j}$, and with the mode number $n = j+m$, or energy $n + \frac{1}{2}$ of the oscillator.

2.2. *Eigenbases and Kravchuk wave functions*

Any element in the su(2) algebra defines an eigenbasis similar to (3)–(4). In particular, we are interested in the "position" eigenbasis of J_1, which we label $\{|j,q\rangle_1\}_{q=-j}^{j}$, and in the interbasis overlap functions defined by

$$f_m^j(q) := {}_1\langle j,q|j,m\rangle_3 = \left(f_q^j(m)\right)^*. \tag{5}$$

We interpret these functions to be the finite oscillator wave functions of mode number $n = j+m$, at the set of $2j+1$ discrete positions $-j \le q \le j$. And it is easy to verify that overlap functions $f^j_m(q)$ themselves also form a canonical basis in the $(2j+1)$-dimensional complex space,

$$\sum_{m=-j}^{j} \left(f^j_m(q)\right)^* f^j_m(q') = \delta_{q,q'}\,, \qquad \sum_{q=-j}^{j} \left(f^j_m(q)\right)^* f^j_{m'}(q) = \delta_{m,m'}\,. \tag{6}$$

The relation between the wave functions in (5) and those used in previous works[3–5] is $f^j_m(q) = \Phi^{(2j)}_{j+m}(q)$, or $\Phi^{(N)}_n(q) = f^{\frac{1}{2}N}_{n-\frac{1}{2}N}(q)$. In Fig. 1 we show these for the lowest,

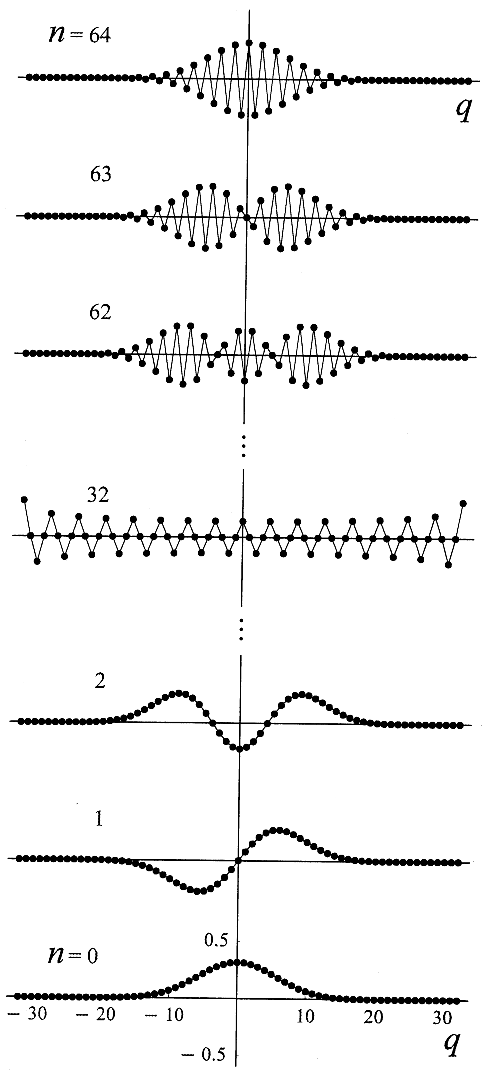

Fig. 1. Finite one-dimensional oscillator Kravchuk functions $f^j_m(q)$ with $j = 32$ ($2j+1 = 65$ points along the q-axis, joined by straight lines for visibility), for $n = j+m = 0, 1, 2, \ldots, 32, \ldots, 62, 63, 64$, from bottom to top.

middle and highest states (cf. Refs. 4 and 5). The ground state ($n = 0$) is the square root of the binomial distribution; the top state ($n = N$) is the previous one with alternating signs between neighboring q's; the resemblance of the lower states to the quantum harmonic oscillator wave functions is evident. In what follows we shall continue to use the indices j and m because they are handier than $N = 2j$ and $n = j + m$ for the purpose of taking limits.

We can establish a useful relation between the previous overlap functions and the Wigner "little-d" functions,

$$d^j_{m,m'}(\theta) := {}_3\langle j, m|e^{-i\theta J_2}|j, m'\rangle_3 = d^j_{m',m}(-\theta) \tag{7}$$

through noting that

$$e^{-i\frac{1}{2}\pi J_2} J_3 e^{i\frac{1}{2}\pi J_2} = J_1 \Rightarrow e^{-i\frac{1}{2}\pi J_2} : |j, m\rangle_3 = |j, m\rangle_1 . \tag{8}$$

Hence, from (5),

$$\begin{aligned} f^j_m(q) &= {}_3\langle j, q|e^{+i\frac{\pi}{2}J_2}|j, m\rangle_3 = d^j_{q,m}\left(-\frac{1}{2}\pi\right) = d^j_{m,q}\left(\frac{1}{2}\pi\right) \\ &= \frac{(-1)^{j+m}}{2^j}\sqrt{\binom{2j}{j+m}\binom{2j}{j+q}}\, K_{j+m}\left(j + q; \frac{1}{2}, 2j\right), \end{aligned} \tag{9}$$

where the last expression uses the *symmetric* Kravchuk polynomial $K_n(x; p, N)$,[15] of degree n in x, whose general definition (for $0 \le p < 1$) is

$$K_n(x; p, N) = {}_2F_1(-n, -x; -N; 1/p) . \tag{10}$$

2.3. *Schrödinger difference equation*

For fixed N, the $N + 1$ Kravchuk polynomials (10) are orthonormal with respect to the binomial distribution $\binom{N}{x}$. They satisfy no differential equation, but a *difference* one which relates the values of the polynomial on three real points (not necessarily integer), separated by one unit.[16] From that difference equation follows, through (9), a difference equation for the finite oscillator wave functions $f^j_m(q)$, which is of the Schrödinger form,

$$H^j(q) f^j_m(q) = \left(n + \frac{1}{2}\right) f^j_m(q) , \quad n = 0, 1, \ldots, 2j , \tag{11}$$

with the Hamiltonian difference operator

$$H^j(q) = -\frac{1}{2}[\alpha^j(q)e^{-\partial_q} - (2j + 1) + \alpha^j(-q)e^{\partial_q}] , \tag{12}$$

written in terms of $\alpha^j(q)$ given by (4), and the unit-shift operators $e^{a\partial_q} f(q) = f(q + a)$, for $a = \pm 1$.

We should note that the difference Hamiltonian (12) does not separate into a sum of "kinetic" plus "potential enegy" terms. It would have been difficult to "guess" the discrete form of the finite oscillator Hamiltonian out of a Schrödinger form $\frac{1}{2}(\partial_q^2 + q^2)$. Other discrete oscillator analogs have been proposed, such as that based on the Harper equation,[17] which applies to a finite set of points on a circle, with the second difference operator plus a cosine function in the angular position. Since Harper's Hamiltonian does not have a linear spectrum, this system cannot harbor truly coherent states, however.

2.4. *Fourier–Kravchuk transforms*

Rotations around the 3-axis bring the eigenbasis of J_1 (position) to the eigenbasis of J_2 (−momentum):

$$e^{-i\frac{1}{2}\pi J_3} J_1 e^{i\frac{1}{2}\pi J_3} = J_2 \Rightarrow e^{-i\frac{1}{2}\pi J_3} : |j,m\rangle_1 = |j,m\rangle_2 , \tag{13}$$

and hence the overlap between the two eigenbases is

$$\tilde{f}^j_m(p) := {}_2\langle j,p|j,m\rangle_3 = {}_1\langle j,p|e^{i\frac{1}{2}\pi J_3}|j,m\rangle_3 = e^{i\frac{1}{2}\pi m}\, f^j_m(p)\,. \tag{14}$$

These are also the coordinates of the mode $|j,m\rangle_3$ in (finite) momentum space $p \in \{-j, -j+1, \ldots, j\}$, i.e. the finite oscillator wave functions in momentum representation.

The fractional integral Fourier transform[18] represents rotations of phase space. In our context, its finite counterpart is the Fourier–*Kravchuk* transform introduced in Ref. 3: a rotation of the J_1–J_2 plane by $\frac{1}{2}\pi$ is produced by the Fourier–Kravchuk operator and kernel

$$\mathcal{K} := e^{-i\frac{1}{2}\pi(J_3+E_J)}\,, \qquad K^j_{q,q'} := {}_1\langle j,q|\mathcal{K}|j,q'\rangle_1\,. \tag{15}$$

A closed form for the kernel can be found through an addition theorem for Wigner d-functions,[14]

$$K^j_{q,q'} = \sum_{m=-j}^{j} d^j_{q,m}\left(-\frac{1}{2}\pi\right) e^{-i\frac{1}{2}\pi m} d^j_{m,q'}\left(\frac{1}{2}\pi\right) = e^{i\frac{1}{2}\pi(q'-q)} d^j_{q,q'}\left(\frac{1}{2}\pi\right). \tag{16}$$

The Fourier–Kravchuk transform $\mathcal{K}$ is actually *distinct*, from the common discrete Fourier transform matrix[19] of elements $(2j+1)^{-1/2}\exp[2\pi i q q'/(2j+1)]$. Both are unitary and step-4 idempotent: $\mathcal{K}^4 = \hat{1}$. Yet, we consider the Fourier–Kravchuk kernel transform to be the appropriate discrete analog of the Fourier integral transform, because it stems from the exponential of the Schrödinger Hamiltonian (12), so it multiplies the nth finite oscillator mode eigenstate by $(-i)^n$, which leads to the existence of proper coherent states.[4,20] (Cf. Ref. 7.)

3. Contraction of the Algebra

Many authors, first among them Wigner and Talman,[21] have considered the contraction of su(2) to the one-dimensional Heisenberg–Weyl algebra of three generators (indicated by a hat, and including the unit $\hat{1}$,) $\mathrm{HW}_1 = \mathrm{span}\{\hat{Q}, \hat{P}, \hat{1}\}$, for $\hbar = 1$. Here, all four generators of the dynamical algebra u(2) are subject to contraction to the oscillator algebra $\mathrm{H}_4 = \{\hat{H}, \hat{Q}, \hat{P}, \hat{1}\} \supset \mathrm{HW}_1 = \{\hat{Q}, \hat{P}, \hat{1}\}$.[13]

We perform the following change of basis for the four generators of $\mathrm{u}(2) = \mathrm{span}\{J_k, E_J\}$ in (2), within the irrep j,

$$\begin{pmatrix} Q^{(j)} \\ P^{(j)} \\ H^{(j)} \\ \hat{1} \end{pmatrix} = \begin{pmatrix} j^{-1/2} & 0 & 0 & 0 \\ 0 & j^{-1/2} & 0 & 0 \\ 0 & 0 & 1 & 1+1/2j \\ 0 & 0 & 0 & j^{-1} \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ E_J \end{pmatrix}, \tag{17}$$

Then, the nonzero commutators for new generators are:

$$\begin{aligned} \left[H^{(j)}, Q^{(j)}\right] &= iP^{(j)}\,, \\ \left[H^{(j)}, P^{(j)}\right] &= -iQ^{(j)}\,, \\ \left[Q^{(j)}, P^{(j)}\right] &= i\hat{1} + ij^{-1}H^{(j)}\,. \end{aligned} \tag{18}$$

In the limit $j \to \infty$, the commutation relations (18) become those of H_4,

$$\begin{aligned} \left[H^{(\infty)}, Q^{(\infty)}\right] &= iP^{(\infty)}\,, \\ \left[H^{(\infty)}, P^{(\infty)}\right] &= -iQ^{(\infty)}\,, \\ \left[Q^{(\infty)}, P^{(\infty)}\right] &= i\hat{1}\,. \end{aligned} \tag{19}$$

The limit form of $H^{(j)}$ can be found from the identifications (1) and the value of the Casimir operator J^2,

$$\begin{aligned} j(j+1) = j\big(Q^{(j)2} + P^{(j)2}\big) + \left[H^{(j)} - \left(j + \frac{1}{2}\right)E^{(j)}\right]^2 \Rightarrow H^{(\infty)} \\ = \frac{1}{2}\big(P^{(\infty)2} + Q^{(\infty)2}\big)\,. \end{aligned} \tag{20}$$

The point to note is that while the four independent generators of the dynamical algebra $\mathrm{u}(2)$ of the finite oscillator contract by (17) to the full oscillator algebra H_4, only in the limit does the Hamiltonian generator acquire its standard form as a quadratic function of the Heisenberg–Weyl generators.

In the representation space j, the rescaled operators (17) will act on functions $F(\xi) := f(q) = {}_1\langle j, q|f\rangle$ of position $\xi := q/\sqrt{j}$, through shift operators by $1/\sqrt{j}$ [cf. Eqs. (3)–(4)] in the following way

$$Q^{(j)}F(\xi) = \frac{1}{\sqrt{j}}Qf(q) = \frac{q}{\sqrt{j}}\,f(q) = \xi F(\xi)\,, \tag{21}$$

$$P^{(j)}F(\xi) = -\frac{i}{2\sqrt{j}}\left[\alpha^j\big(-\sqrt{j}\xi\big)F\left(\xi + \frac{1}{\sqrt{j}}\right) - \alpha^j\big(\sqrt{j}\xi\big)F\left(\xi - \frac{1}{\sqrt{j}}\right)\right], \tag{22}$$

$$H^{(j)}F(\xi) = -\frac{1}{2j}\left[\alpha^j\big(-\sqrt{j}\xi\big)F\left(\xi + \frac{1}{\sqrt{j}}\right) + \alpha^j\big(\sqrt{j}\xi\big)F\left(\xi - \frac{1}{\sqrt{j}}\right)\right]. \tag{23}$$

As $j \to \infty$ and ξ remains finite, and if we assume that the finite-dimensional Hilbert spaces of functions $F(\xi)$ converge in an appropriate sense[22] to that of square-integrable functions, and moreover that we can expand these functions to three terms as $F(\xi+\epsilon) = F(\xi) + \epsilon F'(\xi) + \frac{1}{2}\epsilon^2 F''(\xi) + \cdots$, then the limit of (22) proceeds through

$$\lim_{j\to\infty} P^{(j)} F(\xi) = -\lim_{j\to\infty} \frac{i\sqrt{j}}{2}\sqrt{1-\frac{\xi^2-1}{j}}\left[F\left(\xi+\frac{1}{\sqrt{j}}\right) - F\left(\xi+\frac{1}{\sqrt{j}}\right)\right]$$
$$= -i\frac{\partial}{\partial\xi}F(\xi)\,. \tag{24}$$

In the limit one obtains thus the standard realization of the momentum operator in the position representation. Done similarly for $H^{(j)}$ in (12), one recovers the quantum harmonic oscillator Schrödinger equation.

4. Contraction of the Finite Oscillator Wave Functions

The energy wave functions of the finite oscillator are solutions to the Schrödinger difference equation, (11)–(12), so the contraction of the u(2) algebra performed in the previous section should in principle imply the $j \to \infty$ limit of the Kravchuk functions (9) to the well-known Hermite wave functions of the ordinary quantum oscillator. Indeed, the original argument[15] for the contraction of Kravchuk to Hermite polynomials was phrased in terms of the limit of the point orthogonality measures: the binomial distribution becomes the Gaussian function when the number of points N grows while their separation decreases by $N^{-1/2}$. But (to our knowledge) the limit of the functions has not yet been proven directly. This may be due to the fact that the usual representation of Kravchuk polynomials in terms of Gauss hypergeometric functions, Eq. (10), is ineligible to serve in this limit.

We now prove directly that the Kravchuk functions $f^j_m(q)$ in (9), for $j \to \infty$ and $n = m + j$ kept finite (so $m \to -\infty$), contract to the quantum harmonic oscillator wave functions $\Psi_n(\xi)$. Instead of the ${}_2F_1$ functions, we use an expression of Ref. 9 for the d-functions in terms of hypergeometric ${}_3F_2$-functions (for q integer),

$$f^j_{n-j}(q) = \frac{(-1)^{\frac{n}{2}+q}}{\sqrt{\pi}j!}\sqrt{(j+q)!(j-q)!}$$
$$\times \begin{cases} \sqrt{\dfrac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(j-\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(j-\frac{n}{2}+1\right)}}\;{}_3F_2\left(\begin{matrix}-q, q, \dfrac{n+1}{2} \\ \dfrac{1}{2}, j+1\end{matrix}\;\middle|\;1\right), & n \text{ even}, \\ \dfrac{2iq}{j+1}\sqrt{\dfrac{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(j-\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(j-\frac{n-1}{2}\right)}}\;{}_3F_2\left(\begin{matrix}-q, q, \dfrac{n}{2}+1 \\ \dfrac{3}{2}, j+2\end{matrix}\;\middle|\;1\right), & n \text{ odd}. \end{cases} \tag{25}$$

The contraction limit $j \to \infty$ with the mode number $n = j + m$ fixed and finite, and $q = \sqrt{j}\xi$, so that the points ξ are integers divided by $\sqrt{j}$, is

$$\begin{aligned}\lim_{j\to\infty} & (-1)^q\, j^{\frac{1}{4}} f_n^j(q) \\ &= \frac{(-2)^{n/2}}{\sqrt{\sqrt{\pi}n!}}\, e^{\xi^2} \begin{cases} \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\, {}_1F_1\left(\dfrac{n+1}{2}; \dfrac{1}{2}; -\xi^2\right), & n \text{ even}, \\ 2i\xi\, \dfrac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{1}{2}\right)}\, {}_1F_1\left(\dfrac{n}{2}+1; \dfrac{3}{2}; -\xi^2\right), & n \text{ odd}. \end{cases}\end{aligned} \tag{26}$$

Now, using now the relation between two confluent hypergeometric functions ${}_1F_1(\alpha;\gamma,z) = e^z\, {}_1F_1(\gamma-\alpha;\gamma,-z)$, and comparing with a standard expression for Hermite polynomials in Ref. 23, when $q = \sqrt{j}\xi$, we obtain

$$\begin{aligned}\lim_{j\to\infty} (-1)^q\, j^{\frac{1}{4}} f_{n-j}^j(q) &= \lim_{j\to\infty} (-1)^{n+j} j^{\frac{1}{4}} d_{n-j,q}^j\left(\frac{1}{2}\pi\right) \\ &= \frac{e^{-\xi^2/2}}{\sqrt{\sqrt{\pi}2^n n!}} H_n(\xi) =: \Psi_n(\xi)\,.\end{aligned} \tag{27}$$

The $\Psi_n(\xi)$, $n|_0^\infty$, $\xi \in \Re$, are of course the normalized wave functions of the one-dimensional quantum oscillator.

5. Contraction of the Fourier–Kravchuk Transform Kernel

When $j \to \infty$, we expect the Fourier–Kravchuk transform summation kernel (16) to become the ordinary Fourier transform integral kernel.[5]

To see that this is indeed the case, we can use the previous result (27) and a particular case of Mehler's formula for Hermite polynomials,[23] which is a consequence of the Fourier transform of Hermite functions and their completeness condition,

$$\sum_{n=0}^{\infty} i^n \Psi_n(\xi_1)\Psi_n(\xi_2) = \frac{1}{\sqrt{2\pi}} e^{i\xi_1\xi_2}\,. \tag{28}$$

The contraction limit of (16), when $j \to \infty$ and $m \to -\infty$ such that $j + m = n$ is fixed and finite, and $q = \xi\sqrt{j}$, $q' = \xi'\sqrt{j}$, results now from the limit of the previous section; it is

$$\lim_{j\to\infty} \sqrt{j} K_{q,q'}^j = \lim_{j\to\infty} \sqrt{j} d_{q',q}^j\left(\frac{1}{2}\pi\right) = \frac{1}{\sqrt{2\pi}} e^{i\xi'\xi}\,. \tag{29}$$

6. Conclusions

The finite oscillator model is built with the Lie algebra u(2) [or su(2) for most practical matters], and a new physical interpretation which is distinct from the traditional one of angular momentum theory. In this article we have shown that the finite oscillator contracts to the ordinary quantum oscillator, on the level of the algebra, of the wave functions, and of their Fourier transform.

Based on the theory of the one-dimensional harmonic oscillator, many physical phenomena have been modeled in quantum optics, in signal processing, and other fields apparently unrelated to the original quantum system. In many of these cases, there is the need for a finite counterpart system to account for pixellated objects,[18] or systems (linear or nonlinear) that have both a ground- and a highest-energy state (see e.g. Ref. 20).

Through contraction, su(2) mothers several other three-dimensional Lie algebras. The classical Inönü–Wigner contraction[1] to the Euclidean algebras iso(2) brings the finite oscillator to a two-dimensional Helmholtz system, or to a discrete, one-dimensional infinite system,[24] according to which generator is left unscaled. Further, D-dimensional finite oscillator models have been proposed based on the direct sum of D su(2) algebras or, less obviously, on Lie algebras so$(D+2)$.[6,25] A firm connection with the simplest case examined here will guarantee the correspondence principle for the other models.

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