Wigner functions for curved spaces. I. On hyperboloids

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We propose a Wigner quasiprobability distribution function for Hamiltonian systems in spaces of constant curvature, in this article on hyperboloids, which returns the correct marginals and has the covariance of the Shapiro functions under SO(D,1) transformations. To the free systems obeying the Laplace–Beltrami equation on the hyperboloid, we add a conic-oscillator potential in the hyperbolic coordinate. As an example, we analyze the one-dimensional case on a hyperbola branch, where this conic-oscillator is the Pöschl–Teller potential. We present the analytical solutions and plot the computed results. The standard theory of quantum oscillators is regained in the contraction limit to the space of zero curvature. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1518139]

I. INTRODUCTION

In Hamiltonian systems which a have flat $\Re^D$ configuration space, among the phase space quasiprobability distribution functions the Wigner function[1] is the only one covariant under Euclidean translations of phase space.[2,3] The present article, and others that will follow it, aim to the construction of Wigner functions on configuration spaces that are conic surfaces, hyperboloids and spheres, which transform under the Lorentz and rotation groups respectively, and which reproduce the traditional Wigner function when the conic contracts to the plane. Quantum motion on spaces of constant curvature is of current interest in various fields of theoretical physics, such as quantum gravity and string theory,[4] noncommutative geometry,[5] and quantum chaos.[6] Hamiltonian systems on conic manifolds have a natural kinetic energy given by the Laplace–Beltrami operator, and, moreover, on these conics also a natural oscillator ''potential'' can be proposed. In one dimension, this oscillator turns out to be one of the Pöschl–Teller potentials.[7]

In this article, subtitled I, we propose a Wigner function on the D-dimensional hyperboloid $\mathcal{H}_+^D$, which generalizes the ordinary Wigner function on flat phase space. It displays the correct marginals, and returns the traditional form of the Wigner function under Inönü–Wigner contraction to the zero-curvature limit. The elements and background for this assertion are contained in Sec. II, including the Shapiro solutions $\Phi_{\mathbf{p}}^{(D)}(\mathbf{x})$ to the Laplace–Beltrami equation.[8] In Sec. III we present our proposed definition of Wigner function on the hyperboloid, and verify the properties of marginality and the contraction limit to flat phase space. Covariance remains an issue because the Wigner function that we propose here follows from the covariance of the basis of wavefunctions, between the argument $\mathbf{x}$ and the index $\mathbf{p}$, as if they were canonically conjugate variables. In this context, we reexamine the interpretation of momentum coordinates.

In Sec. IV we exemplify the D-dimensional theory with a one-dimensional *sui generis* oscillator on one branch of a hyperbola. This example may appear to be trivial, because the hyperbola is in most respects equivalent to a straight line. Nevertheless, the resulting Pöschl–Teller potential is of particular interest because the wavefunctions are also the Clebsch–Gordan (Wigner coupling) coefficients for the three-dimensional Lorentz algebra, $\mathrm{so}(2,1)=\mathrm{sp}(2,\Re)=\mathrm{su}(1,1)$.[9] We display

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the Wigner functions of some Pöschl–Teller wavefunctions; these have not been examined before. Finally, in Sec. V we recapitulate the aim and offer the present outlook of our program.

II. ELEMENTS OF PHASE SPACE AND HYPERBOLOIDS

In his fundamental article,[1] Wigner proposed a distribution function to represent on phase space the wavefunctions of pure and of mixed states in quantum systems. In this section we recall the definition and properties that we shall generalize from flat to conic spaces, using the Laplace–Beltrami operator and the Shapiro functions.

A. Wigner function on flat phase space

In D-dimensional flat configuration space $\mathbf{x}\in\mathfrak{R}^D$, the generalized Dirac basis of plane waves solves the free-space Schrödinger equation, which is identical to the Helmholtz equation

$$-\Delta\,\phi(\mathbf{x})=p^2\,\phi(\mathbf{x}),\quad \phi_{\mathbf{p}}(\mathbf{x})=\exp(i\mathbf{p}\cdot\mathbf{x}),\quad \mathbf{p}\in\mathfrak{R}^D. \tag{1}$$

When we write $p=+(\mathbf{p}\cdot\mathbf{p})^{1/2}$, $\mathbf{n}=\mathbf{p}/p$, and call $\mathbf{p}=p\mathbf{n}$ the momentum or wavenumber vector, the functions $\phi_{\mathbf{p}}(\mathbf{x})$ represent plane waves in the direction of the unit vector $\mathbf{n}\in\mathcal{S}^{D-1}$ in the $(D-1)$-dimensional sphere manifold. In the quantum model with natural units $\hbar=1$, p has units of inverse length; in the wave optical model, p is the wavenumber of light.

The basis of plane wave functions (1) plays many roles: it provides the Fourier transform kernel which bridges the configuration and momentum realizations, it constitutes a basis for representations of the Euclidean group, and it serves for the construction of the $\mathfrak{R}^{2D}$-Wigner function of wavefields $f(\mathbf{x})$, $g(\mathbf{x})$ through the equivalent expressions

$$W_{\mathfrak{R}^D}(f,g|\mathbf{x},\mathbf{p})=\frac{1}{(2\pi)^D}\int_{\mathfrak{R}^D}d^D\mathbf{z}\,f\left(\mathbf{x}-\frac{1}{2}\mathbf{z}\right)^*e^{-i\mathbf{p}\cdot\mathbf{z}}\,g\left(\mathbf{x}+\frac{1}{2}\mathbf{z}\right) \tag{2}$$

$$=\frac{1}{(2\pi)^D}\int_{\mathfrak{R}^D}d^D\mathbf{z}\,f\left(\mathbf{x}-\frac{1}{2}\mathbf{z}\right)^*\times e^{+i\mathbf{p}\cdot(\mathbf{x}-(1/2)\mathbf{z})}\,e^{-i\mathbf{p}\cdot(\mathbf{x}+(1/2)\mathbf{z})}\,g\left(\mathbf{x}+\frac{1}{2}\mathbf{z}\right)$$

$$=\frac{1}{(2\pi)^D}\int_{\mathfrak{R}^D}d^D\mathbf{x}'\int_{\mathfrak{R}^D}d^D\mathbf{x}''\,f(\mathbf{x}')^*\,g(\mathbf{x}'')\times\phi_{\mathbf{p}}(\mathbf{x}')\,\delta^D\left(\mathbf{x}-\frac{1}{2}(\mathbf{x}'+\mathbf{x}'')\right)\phi_{\mathbf{p}}(\mathbf{x}'')^*. \tag{3}$$

This has the well-known properties of being sesquilinear in the functions, real for $f=g$, with the marginal projections $\int_{\mathfrak{R}^D}d\mathbf{p}W=f(\mathbf{x})^*g(\mathbf{x})$, $\int_{\mathfrak{R}^D}d\mathbf{x}W=\tilde{f}(\mathbf{p})^*\tilde{g}(\mathbf{p})$ (the tilde indicates ordinary Fourier transformation, $F:f=\tilde{f}$), and covariant under translations in coordinate and momentum spaces

$$T_{\mathbf{a}}:f(\mathbf{x})=f(\mathbf{x}-\mathbf{a})\Rightarrow W_{\mathfrak{R}^D}(T_{\mathbf{a}}:f,T_{\mathbf{a}}:g|\mathbf{x},\mathbf{p})=W_{\mathfrak{R}^D}(f,g|\mathbf{x}-\mathbf{a},\mathbf{p}), \tag{4}$$

$$\tilde{T}_{\mathbf{b}}:f(\mathbf{x})=e^{i\mathbf{b}\cdot\mathbf{x}}f(\mathbf{x})\Rightarrow W_{\mathfrak{R}^D}(\tilde{T}_{\mathbf{b}}:f,\tilde{T}_{\mathbf{b}}:g|\mathbf{x},\mathbf{p})=W_{\mathfrak{R}^D}(f,g|\mathbf{x},\mathbf{p}-\mathbf{b}), \tag{5}$$

$$F:f(\mathbf{x})=\tilde{f}(\mathbf{x})\Rightarrow W_{\mathfrak{R}^D}(F:f,F:g\,|\mathbf{x},\mathbf{p})=W_{\mathfrak{R}^D}(f,g\,|\mathbf{p},-\mathbf{x}). \tag{6}$$

The last intertwining by the Fourier transform was known when García–Calderón and Moshinsky noticed that the Wigner function is covariant also under the larger group of $\mathrm{Sp}(2D,\mathfrak{R})$ linear canonical transformations of phase space.[10] This is exceptional in the sense that the Heisenberg–Weyl algebra [whose generators are the phase space translations (4) and (5)—and the unit that generates a commuting phase factor] has the outer automorphism group $\mathrm{Sp}(2D,\Re)$. This accident does not occur for Lorentz algebras, so we should not expect similar covariances of the Wigner function under groups larger than $\mathrm{SO}(D,1)$.

B. Laplace–Beltrami operator on the hyperboloid

The purpose of this article is to generalize the expression of the Wigner function (2) and (3) with functions on a D-dimensional space $\mathbf{x}\in\Re^D$ of constant curvature. This manifold can be seen in an "ambient" space of $D+1$ dimensions as a hyperboloid, with vectors $x=(x_0,\mathbf{x})\in\Re^{D+1}$.

Consider the upper sheet of the two-sheeted hyperboloid $\mathcal{H}_+^D\subset\Re^{D+1}$ of hyperbolic radius $R>0$,

$$|x|^2=x_0^2-\mathbf{x}^2=R^2,\quad \mathbf{x}^2=x_1^2+x_2^2+\cdots+x_D^2. \tag{7}$$

In this ambient Minkowski space, the isometry group is the Poincaré group $\mathrm{ISO}(D,1)_+^\uparrow$, in place of the Euclidean group $\mathrm{ISO}(D)_+$ of flat space. The Lie algebra $\mathrm{so}(D,1)$ has then the standard realization

$$M_{j,k}=x_j\partial_{x_k}-x_k\partial_{x_j},\quad M_{0,k}=x_0\partial_{x_k}+x_k\partial_{x_0},\quad j,k=1,2,\ldots,D. \tag{8}$$

The second-order Casimir operator, $\mathcal{C}$, which is an invariant under the group $\mathrm{SO}(D,1)_+^\uparrow$, is $(-R^2$ times) the Laplace–Beltrami operator on $\mathcal{H}_+^D$, namely

$$\frac{1}{R^2}\mathcal{C}=-\Delta_{\mathrm{LB}}=\frac{1}{R^2}\left(\sum_{1\leqslant j<k\leqslant D}M_{j,k}^2-\sum_{1\leqslant k\leqslant D}M_{0,k}^2\right). \tag{9}$$

This operator replaces the Laplacian in the Schödinger equation for hyperbolic curved space. Thus, the Schrödinger equation on this space with a potential $V(\mathbf{x})$ is

$$\left(\frac{-1}{2\mu}\Delta_{\mathrm{LB}}+R^2V(\mathbf{x})\right)f(\mathbf{x})=R^2Ef(\mathbf{x}). \tag{10}$$

In quantum mechanics $\mu=m/\hbar^2$, where m is the particle mass. For application in paraxial wave optics, we recall the interpretation where the extra term characterizes the refractive index anomaly of the medium,

$$n(\mathbf{x})=n_\circ-\nu(\mathbf{x}),\quad n_\circ=n(0),\qquad \begin{array}{c} n_\circ\leftrightarrow\mu,\\ \\ \nu(\mathbf{x})\leftrightarrow V(\mathbf{x}). \end{array} \tag{11}$$

First we consider the case when the potential is identically zero, $V(\mathbf{x})=0$; a nonzero oscillator potential will be introduced in Sec. II E.

For the free case, in the unitary irreducible representation spaces of the D-dimensional Lorentz group belonging to the most degenerate continuous series indicated by p,[11] the operator (9) has a real lower-bound spectrum, as does (1). The wavefunctions of the free system on the hyperboloid are the solutions to the equation

$$\begin{gathered}\Delta_{\mathrm{LB}}f(\mathbf{x})=-\left[\left(\frac{D-1}{2R}\right)^2+p^2\right]f(\mathbf{x})=-\frac{\lambda(\lambda+D-1)}{R^2}f(\mathbf{x}),\\ p\in\Re_0^+=[0,\infty),\quad \lambda=-\frac{1}{2}(D-1)-ipR.\end{gathered} \tag{12}$$

Any wavefield of a given wavenumber p is a solution of this equation.

C. Shapiro functions

A privileged basis for the solutions of the Laplace–Beltrami equation (12) was given by Gel'fand, Graev and Shapiro[8] in the form of D-dimensional plane waves of momentum $\mathbf{p}=p\mathbf{n}$, with positive wavenumber p and in the direction of a unit vector on the sphere $\mathbf{n}\in\mathcal{S}^{D-1}$,

$$\Phi_{\mathbf{p}}^{(D)}(x)=\left(\frac{x_0-\mathbf{n}\cdot\mathbf{x}}{R}\right)^{-(1/2)(D-1)-ipR}=(\cosh\chi-\mathbf{n}\cdot\boldsymbol{\xi}\sinh\chi)^{-(1/2)(D-1)-ipR}, \tag{13}$$

where functions $f(x)$ on the hyperboloid $x\in\mathcal{H}_+^D$ $(x^2=R^2)$ will be denoted, according to convenience, by

$$f(x)=f(x_0,\mathbf{x})=f(\mathbf{x}),\qquad \begin{array}{c} x_0=+\sqrt{R^2+\mathbf{x}^2}=R\cosh\chi\geqslant R,\\ \mathbf{x}=R\boldsymbol{\xi}\sinh\chi\in\mathfrak{R}^D,\quad \chi\in\mathfrak{R}_0^+,\quad \boldsymbol{\xi}\in\mathcal{S}^{D-1}. \end{array} \tag{14}$$

The *Shapiro* functions (13) are a Dirac basis for functions on the hyperboloid, which are orthogonal and complete over $\mathbf{x}$- and $\mathbf{p}$-spaces:

$$\frac{R}{(2\pi)^D}\int_{\mathbf{x}\in\mathfrak{R}^D}\frac{d^D\mathbf{x}}{x_0}\Phi_{\mathbf{p}}^{(D)}(x)^*\,\Phi_{\mathbf{p}'}^{(D)}(x)=N^{(D)}(p)\,\delta^D(\mathbf{p}-\mathbf{p}'), \tag{15}$$

$$\frac{1}{(2\pi)^D}\int_{\mathbf{p}\in\mathfrak{R}^D}\frac{d^D\mathbf{p}}{N^{(D)}(p)}\Phi_{\mathbf{p}}^{(D)}(x)^*\,\Phi_{\mathbf{p}}^{(D)}(x')=\delta^D(x,x'), \tag{16}$$

with the measure and Dirac δ under $\int_{\mathcal{H}_+^D}d^Dx=R\int_{\mathfrak{R}^D}d^D\mathbf{x}/x_0$,

$$N^{(D)}(p)=\left|\frac{\Gamma(ipR)}{\Gamma\left(\frac{1}{2}(D-1)+ipR\right)}\right|^2(pR)^{D-1}, \tag{17}$$

$$\delta^D(x,x')=\frac{x_0}{R}\,\delta^D(\mathbf{x}-\mathbf{x}')=\sqrt{1+\frac{\mathbf{x}^2}{R^2}}\,\delta^D(\mathbf{x}-\mathbf{x}'). \tag{18}$$

In particular, $N^{(1)}(p)=1$, $N^{(2)}(p)=\coth(pR)$, and $N^{(3)}(p)=1$.

The Inönu–Wigner contraction limit of the Lorentz to the Euclidean group $\mathrm{SO}(D,1)_+^\uparrow\to\mathrm{ISO}(D)_+$ is the limit $R\to\infty$ in our expressions for vectors with $x_0\approx R$, $\mathbf{x}^2\ll R^2$, and $\mathbf{p}=p\mathbf{n}$ as before, i.e.,

$$\lim_{R\to\infty}\Phi_{\mathbf{p}}^{(D)}(x)=\lim_{R\to\infty}\left(\frac{x_0-\mathbf{x}\cdot\mathbf{n}}{R}\right)^{-(1/2)(D-1)-ipR}\approx\lim_{R\to\infty}\left(1-\frac{\mathbf{x}\cdot\mathbf{n}}{R}\right)^{-ipR}=\exp(i\mathbf{x}\cdot\mathbf{p}). \tag{19}$$

Correspondingly, $\lim_{R\to\infty}N^{(D)}(p)=1$ and $\delta^D(x,x')\to\delta^D(\mathbf{x}-\mathbf{x}')$.

D. Momentum space for the hyperboloid

The Shapiro functions $\{\Phi_{\mathbf{p}}^{(D)}(\mathbf{x})\}_{\mathbf{p}\in\mathfrak{R}^D}$ in (13) serve as the integral transform kernel between functions of $\mathbf{x}$ on the hyperboloid, $f(\mathbf{x})$, and conjugate functions of $\mathbf{p}$, that has the interpretation of momentum or wavenumber space, and is indicated $\tilde{f}(\mathbf{p})$. Using (14) for $\mathbf{x},\mathbf{p}\in\mathfrak{R}^D$, one writes

$$\tilde{f}(\mathbf{p})=\frac{R}{(2\pi)^{D/2}}\int_{\mathbf{x}\in\mathfrak{R}^D}\frac{d^D\mathbf{x}}{x_0}\Phi_{\mathbf{p}}^{(D)}(\mathbf{x})^*\,f(\mathbf{x}), \tag{20}$$

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}} \int_{\mathbf{p}\in\Re^D} \frac{d^D\mathbf{p}}{N^{(D)}(p)}\, \Phi_{\mathbf{p}}^{(D)}(\mathbf{x})\, \tilde{f}(\mathbf{p}). \tag{21}$$

This *Shapiro transform* has been used as a relativistic analog of the Fourier transform (the physical context here, though, is **not** that of space–time relativity, as we shall clarify below), and is a vector form of one of the two branches of the bilateral Mellin transform.[12] Here the Shapiro transform replaces the traditional Fourier transform in the definition of a momentum space $\mathbf{p}\in\Re^D$, canonically conjugate with respect to this basis, to a configuration space of constant curvature. The corresponding Parseval relation is

$$R\int_{\mathbf{x}\in\Re^D} \frac{d^D\mathbf{x}}{x_0}\, f(\mathbf{x})^*\, g(\mathbf{x}) = (f,\ g)_{\mathcal{H}_+^D} = \int_{\mathbf{p}\in\Re^D} \frac{d^D\mathbf{p}}{N^{(D)}(p)}\, \tilde{f}(\mathbf{p})^*\, \tilde{g}(\mathbf{p}). \tag{22}$$

The manifold of momentum $\mathbf{p}=p\mathbf{n}\in\Re^D$ ($p\in\Re^+$ and $\mathbf{n}\in\mathcal{S}^{D-1}$) can be placed also in a $(D+1)$-dimensional ''ambient'' space, where it occupies the cone $\varpi=(p,\mathbf{p})\in\vee^+$. The momentum thus defined by the Shapiro functions has certain features, however, which do not correspond to those of a standard relativistic momentum vector. If $f(\mathbf{x})$ is a monochromatic wavefield with a definite value of p, this wavenumber will not change under SO$(D,1)$ translations of the hyperboloid (''boosts''), because it is the invariant value of the Casimir operator (9)–(12). Only the direction of momentum, $\mathbf{n}$, can shift over the sphere; it will do so following the well-known Bargmann deformation of the circle,[13] where the colatitude angle ''boosts'' as $\tan\frac{1}{2}\phi \mapsto e^{-\zeta}\tan\frac{1}{2}\phi$ for rapidity $\zeta\in\Re$. Quotation marks are used for ''boost'' because here we mean a translation in the hyperboloid, and not the well-known relativistic acceleration.

E. Oscillators on conics

The Laplace–Beltrami equation (9)–(12) provides the free fields (whose energy is purely kinetic) on the hyperboloid. In Eq. (10) we allowed a potential energy term as in the Schrödinger equation of quantum mechanics, by adding a function of position $V(\mathbf{x})$.[7,14,15] A straightforward and useful generalization of the SO(D)-isotropic harmonic oscillator potential from flat to conic D-dimensional configuration space is[14]

$$V(\mathbf{x}) = \frac{1}{2}\mu\omega^2R^2\frac{|\mathbf{x}|^2}{x_0^2} = \frac{1}{2}\mu\omega^2R^2\tanh^2\chi = \frac{1}{2}\mu\omega^2R^2\,(1-\mathrm{sech}^2\chi), \tag{23}$$

where $\chi\in\Re_0^+$ is the hyperbolic angle coordinate defined in Eqs. (14). This is the Pöschl–Teller ''secant-hyperbolic-squared'' trough.

III. WIGNER FUNCTION ON THE HYPERBOLOID

With the Shapiro basis of wavefunctions of the free system, we construct now our proposed Wigner function following the double-integral form in Eq. (3) for two wavefunctions, $f(x)$ and $g(x)$, by means of integrals on two hyperboloids, $|x'|=R$ and $|x''|=R$.

A. Definition

With the measures in Eqs. (15) and the Shapiro functions in (13), we define the Wigner function on the hyperboloid by

$$\begin{aligned} W_{\mathcal{H}}(f,g|\mathbf{x},\mathbf{p}) = \frac{R^2}{(2\pi)^D} \int_{\mathbf{x}'\in\Re^D} \frac{d^D\mathbf{x}'}{x_0'} \int_{\mathbf{x}''\in\Re^D} \frac{d^D\mathbf{x}''}{x_0''}\, f(x')^*\, g(x'') \\ \times\Phi_{\mathbf{p}}^{(D)}(x')\,\Delta^D(x;x',x'')\,\Phi_{\mathbf{p}}^{(D)}(x'')^*, \end{aligned} \tag{24}$$

where $\Delta^D(x;x',x'')$ takes the place of the Dirac delta $\delta^D(\mathbf{x}-\frac{1}{2}(\mathbf{x}'+\mathbf{x}''))$ on flat space, Eq. (3), and which will be detailed below.

The crucial property that we must require of this "binding-Δ" in (24) is that it should guarantee that x be the midpoint of the *geodesic* between x' and x'', so that all three points lie on the hyperboloid $\mathcal{H}_+^D$. We achieve this in the following way:[16] given $x \in \mathfrak{R}^{D+1}$ in the upper sheet of a two-sheeted hyperboloid, we build any $y \in \mathfrak{R}^{D+1}$ on a one-sheeted hyperboloid $\tilde{\mathcal{H}}^D$ of the same radius R, such that it be Minkowski-orthogonal to x,

$$y=(y_0,\mathbf{y}), \quad |y|^2=y_0^2-\mathbf{y}^2=-R^2, \quad x_0y_0-\mathbf{x}\cdot\mathbf{y}=0. \tag{25}$$

Then, we can express x' and x'' as vectors obtained from x and y as follows:

$$x'=x\cosh\tfrac{1}{2}\tau-y\sinh\tfrac{1}{2}\tau, \quad x''=x\cosh\tfrac{1}{2}\tau+y\sinh\tfrac{1}{2}\tau, \tag{26}$$

where $x',x''\in\mathcal{H}_+^D$ for all $\tau\in\mathfrak{R}$. Also, it is easy to show that

$$x_0'x_0''-\mathbf{x}'\cdot\mathbf{x}''=R^2\cosh\tau, \quad x_0x_0'-\mathbf{x}\cdot\mathbf{x}'=x_0x_0''-\mathbf{x}\cdot\mathbf{x}''=R^2\cosh\tfrac{1}{2}\tau, \tag{27}$$

i.e., the geodesic distance between x' and x'' is $R\tau$, while x is at $\frac{1}{2}R\tau$ from both x' and x''. The arguments x' and x'' in the expression (24) thus emulate the arguments $\mathbf{x}\pm\frac{1}{2}\mathbf{z}$ in (2) with the parameter $\frac{1}{2}\tau$.

Using (18) and the parameter τ in (27), we propose the binding-Δ in (24) to be

$$\Delta^D(x;x',x'')=\frac{x_0}{R}\,\delta^D\left(\mathbf{x}-\frac{\mathbf{x}'+\mathbf{x}''}{2\cosh\frac{1}{2}\tau}\right). \tag{28}$$

This will yield the correct marginals (to be seen below) due to its properties

$$\Delta^D(x;x',x')=\frac{x_0}{R}\,\delta^D(\mathbf{x}-\mathbf{x}'), \quad R\int_{\mathbf{x}\in\mathfrak{R}^D}\frac{d^D\mathbf{x}}{x_0}\,\Delta^D(x;x',x'')=1. \tag{29}$$

B. Integral forms

The $2D$-fold integral form of the Wigner function in (24) contains Dirac δ's; it can therefore be brought to a $(D+1)$-fold integral noting that the definition of $y\in\tilde{\mathcal{H}}^D$ leaves the freedom of rotating $\mathbf{y}$ around $\mathbf{x}$ on a sphere $\mathcal{S}^{D-1}$. When we change variables from x' and x'' to x and y according to (26), we reduce the integration to $\mathbf{y}$ and τ while keeping Minkowski-orthogonality. The proposed Wigner function (24) then becomes

$$\begin{aligned}W_{\mathcal{H}}(f,g|\mathbf{x},\mathbf{p})=\frac{R^2}{(2\pi)^D}\int_0^\infty(\sinh\tau)^{D-1}\,d\tau\int_{y\in\tilde{\mathcal{H}}^D}d^D\mathbf{y}\;\delta(x_0y_0-\mathbf{x}\cdot\mathbf{y})\\ \times f\left(x\cosh\frac{1}{2}\tau-y\sinh\frac{1}{2}\tau\right)^* g\left(x\cosh\frac{1}{2}\tau+y\sinh\frac{1}{2}\tau\right)\\ \times\Phi_{\mathbf{p}}^{(D)}\left(x\cosh\frac{1}{2}\tau-y\sinh\frac{1}{2}\tau\right)\Phi_{\mathbf{p}}^{(D)}\left(x\cosh\frac{1}{2}\tau+y\sinh\frac{1}{2}\tau\right)^*.\end{aligned} \tag{30}$$

The Dirac δ remaining in (30) can be used to find a third alternative form of the Wigner function. This is obtained with the parametrization of the ambient-space vectors given by

$$x=(x_0,\mathbf{x})=R(\cosh\chi,\boldsymbol{\xi}\sinh\chi), \quad y=(y_0,\mathbf{y})=R(\sinh\omega,\boldsymbol{\eta}\cosh\omega), \tag{31}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are unit vectors on the sphere $\mathcal{S}^{D-1}$ and $\chi,\omega\in\mathfrak{R}_0^+$. The Dirac δ in Eq. (30) is then

$$\delta(x_0 y_0 - \mathbf{x}\cdot\mathbf{y}) = \frac{1}{R^2}\frac{\cosh\Omega}{\cosh\chi}\,\delta(\omega-\Omega), \quad \text{with} \quad \tanh\Omega = \boldsymbol{\xi}\cdot\boldsymbol{\eta}\tanh\chi. \tag{32}$$

The differential $d^D\mathbf{y}$ of the integral in $y\in\widetilde{\mathcal{H}}_+^D$ becomes $R^D(\cosh\omega)^{D-1}d\omega\, d^{D-1}\boldsymbol{\eta}$, so the Wigner function (24) becomes a D-fold integral with the structure of (2), viz.,

$$\begin{aligned} W_{\mathcal{H}}(f,g|\mathbf{x},\mathbf{p}) = {} & \frac{1}{(2\pi)^D}\int_0^\infty (\sinh\tau)^{D-1}\,d\tau \int_{\mathcal{S}^{D-1}} \frac{|\mathbf{y}|^D}{\cosh\chi}\,d^{D-1}\boldsymbol{\eta} \\ & \times f\left(x\cosh\frac{1}{2}\tau - y\sinh\frac{1}{2}\tau\right)^* g\left(x\cosh\frac{1}{2}\tau + y\sinh\frac{1}{2}\tau\right) \\ & \times \Phi_{\mathbf{p}}^{(D)}\left(x\cosh\frac{1}{2}\tau - y\sinh\frac{1}{2}\tau\right)\Phi_{\mathbf{p}}^{(D)}\left(x\cosh\frac{1}{2}\tau + y\sinh\frac{1}{2}\tau\right)^*, \end{aligned} \tag{33}$$

with

$$y = R(\sinh\Omega, \quad \boldsymbol{\eta}\cosh\Omega) = R\,\frac{(\boldsymbol{\xi}\cdot\boldsymbol{\eta}\tanh\chi, \quad \boldsymbol{\eta})}{\sqrt{1-(\boldsymbol{\xi}\cdot\boldsymbol{\eta}\tanh\chi)^2}}. \tag{34}$$

C. Marginal projections

The marginal projections obtained by integrating the proposed Wigner function (24) over momentum and configuration space should yield, respectively, $f(\mathbf{x})^* g(\mathbf{x})$ and $\tilde{f}(\mathbf{p})^*\tilde{g}(\mathbf{p})$ as defined in (20) and (21). The two marginals follow from the orthogonality and completeness relations of the Shapiro functions, Eqs. (15) and (16).

The integration of the Wigner function over $\mathfrak{R}^D$ momentum space with the measure $1/N^{(D)}(p)$ in (17) is

$$\begin{aligned} M_{\mathcal{H}}(f,g|\mathbf{x}) &= \int_{\mathbf{p}\in\mathfrak{R}^D} \frac{d^D\mathbf{p}}{N^{(D)}(p)}\, W_{\mathcal{H}}(f,g|\mathbf{x},\mathbf{p}) \\ &= R^2\int_{\mathbf{x}'\in\mathfrak{R}^D}\frac{d^D\mathbf{x}'}{x_0'}\int_{\mathbf{x}''\in\mathfrak{R}^D}\frac{d^D\mathbf{x}''}{x_0''}\, f(x')^*\, g(x'')\,\Delta^D(x;x',x'')\,\delta^D(x',x'') \\ &= R\int_{\mathbf{x}'\in\mathfrak{R}^D}\frac{d^D\mathbf{x}'}{x_0'}\, f(x')^*\, g(x')\,\Delta^D(x;x',x') = f(\mathbf{x})^*\, g(\mathbf{x}), \end{aligned} \tag{35}$$

where we used (16), (18), and the first property of the binding-Δ in (29).

Similarly, the integration over $\mathfrak{R}^D$ configuration space with the measure R/x_0 in (18) is

$$\begin{aligned} M_{\mathcal{H}}(f,g|\mathbf{p}) &= R\int_{\mathbf{x}\in\mathfrak{R}^D}\frac{d^D\mathbf{x}}{\sqrt{R^2+\mathbf{x}^2}}\, W_{\mathcal{H}}(f,g|\mathbf{x},\mathbf{p}) \\ &= \frac{R^2}{(2\pi)^D}\int_{\mathbf{x}'\in\mathfrak{R}^D}\frac{d^D\mathbf{x}'}{x_0'}\, f(x')^*\,\Phi_{\mathbf{p}}^{(D)}(x')\int_{\mathbf{x}''\in\mathfrak{R}^D}\frac{d^D\mathbf{x}''}{x_0''}\, g(x'')\,\Phi_{\mathbf{p}}^{(D)}(x'')^* = \tilde{f}(\mathbf{p})^*\,\tilde{g}(\mathbf{p}), \end{aligned} \tag{36}$$

where we used the second property of the binding-Δ in (29) and the Shapiro transform (20).

Finally, integrating over the whole of phase space with the appropriate measures in the Parseval relation (22), we have the total probability

$$R\int_{\mathbf{x}\in\Re^D}\frac{d^D\mathbf{x}}{x_0}M_{\mathcal{H}}(f,g|\mathbf{x})=(f,g)_{\mathcal{H}_+^D}=\int_{\mathbf{p}\in\Re^D}\frac{d^D\mathbf{p}}{N^{(D)}(p)}M_{\mathcal{H}}(f,g|\mathbf{p}). \tag{37}$$

D. Covariance under rotations and conic translations

Under rotations $\mathbf{R}\in\mathrm{SO}(D)$, wavefunctions $f(x_0,\mathbf{x})$ transform through

$$T(\mathbf{R}):f(x)=f(x_0,\mathbf{R}^{-1}\mathbf{x}). \tag{38}$$

In particular, the basis of Shapiro functions (13) transforms as

$$T(\mathbf{R}):\Phi^{(D)}_{p\,\mathbf{n}}(x_0,\mathbf{x})=\Phi^{(D)}_{p\,\mathbf{n}}(x_0,\mathbf{R}^{-1}\,\mathbf{x})=\Phi^{(D)}_{p\,\mathbf{Rn}}(x_0,\mathbf{x}). \tag{39}$$

Applying rotations $T(\mathbf{R})$ to the wavefields f and g in the Wigner function (24), we next change variables to $\mathbf{x}'=\mathbf{R}\overline{\mathbf{x}'}$ and $\mathbf{x}''=\mathbf{R}\overline{\mathbf{x}''}$ (the ambient x_0-components behave as scalars), then use (39) for $\mathbf{R}^{-1}$, noting that the binding-Δ in (28) is invariant, $\Delta^D(\overline{x};\overline{x'},\overline{x''})=\Delta^D(x;x',x'')$ for $\overline{\mathbf{x}}=\mathbf{R}^{-1}\mathbf{x}$, and so are the measures $d^D\mathbf{x}'=d^D\overline{\mathbf{x}'}$. It thus follows that the Wigner function (24) is covariant under rotations, fulfilling

$$W_{\mathcal{H}}(T(\mathbf{R}):f,T(\mathbf{R}):g|\mathbf{x},\mathbf{p})=W_{\mathcal{H}}(f,g|\mathbf{R}^{-1}\mathbf{x},\mathbf{R}^{-1}\mathbf{p}). \tag{40}$$

Now consider translations by ζ (''boosts'' of rapidity ζ) $\mathbf{B}_{\mathbf{m}}(\zeta)\in\mathrm{SO}(D,1)^{\uparrow}_{+}$ in the direction of unit $\mathbf{m}\in\mathcal{S}^{D-1}$, which transform the ambient space vectors preserving the constant-curvature subspaces $x\in\mathcal{H}^D_+$ for each radius $R>0$. We denote by $\mathbf{x}_{\parallel\mathbf{m}}$ and $\mathbf{x}_{\perp\,\mathbf{m}}$ the projections of $\mathbf{x}$ parallel and perpendicular to the direction of $\mathbf{m}$, so that $\mathbf{x}=\mathbf{x}_{\parallel\mathbf{m}}+\mathbf{x}_{\perp\,\mathbf{m}}$. Then, wavefunctions on the hyperboloid transform as

$$T(\mathbf{B}_{\mathbf{m}}(\zeta)):f\begin{pmatrix}x_0\\ \mathbf{x}_{\parallel\mathbf{m}}\\ \mathbf{x}_{\perp\,\mathbf{m}}\end{pmatrix}=f\begin{pmatrix}x_0\cosh\zeta-\mathbf{m}\cdot\mathbf{x}\sinh\zeta\\ \mathbf{x}_{\parallel\mathbf{m}}\cosh\zeta-x_0\,\mathbf{m}\sinh\zeta\\ \mathbf{x}_{\perp\,\mathbf{m}}\end{pmatrix}. \tag{41}$$

When this transformation is applied to the plane-wave basis of Shapiro functions, their directions $\mathbf{n}$ on the sphere change, and they acquire a multiplier factor:

$$T(\mathbf{B}_{\mathbf{m}}(\zeta)):\Phi^{(D)}_{p\,\mathbf{n}}(x_0,\mathbf{x})=(\cosh\zeta+\mathbf{m}\cdot\mathbf{n}\sinh\zeta)^{-(1/2)(D-1)-ipR}\,\Phi^{(D)}_{p\,\mathbf{n}'}(x_0,\mathbf{x}), \tag{42}$$

where the components of $\mathbf{n}'\in\mathcal{S}^{D-1}$ that are orthogonal and parallel to $\mathbf{m}$ are

$$\mathbf{n}'_{\perp\,\mathbf{m}}=\frac{\mathbf{n}_{\perp\,\mathbf{m}}}{\cosh\zeta+\mathbf{n}\cdot\mathbf{m}\sinh\zeta},\quad \mathbf{n}'_{\parallel\mathbf{m}}=\frac{\mathbf{n}\cdot\mathbf{m}\cosh\zeta+\sinh\zeta}{\cosh\zeta+\mathbf{n}\cdot\mathbf{m}\sinh\zeta}\,\mathbf{m}, \tag{43}$$

within the same $\mathrm{SO}(D,1)^{\uparrow}_{+}$ irreducible representation characterized by the invariant wavenumber, p. If the angle from $\mathbf{m}$ to $\mathbf{n}$ is ϕ, it will transform through the well-known Bargmann SO(2,1) map of the circle.

The expression in the multiplier factor of Eq. (42),

$$\mu(\mathbf{m},\zeta;\mathbf{n})=\cosh\zeta+\mathbf{m}\cdot\mathbf{n}\sinh\zeta, \tag{44}$$

is, not coincidentally, p'/p—if the $(D+1)$-vector $\varpi=(p,\mathbf{p})\in\vee^+$ were allowed to transform as a ''lightlike'' vector in relativity, i.e., without being constrained to its p-sphere. Under the inner product (16), the $\mathrm{SO}(D,1)$ boost with the multiplier (42) is unitary nonetheless, because the measure in $\mathbf{p}$-space is $d^D\mathbf{p}=p^D dp\, d^{N-1}\mathbf{n}$, and while p is invariant, from (43) it follows that $d^{D-1}\mathbf{n}=\mu(\mathbf{m},\zeta;\mathbf{n})^{D-1}d^{D-1}\mathbf{n}'$. This cancels the absolute square of the multiplier (44) in the

Shapiro functions (41). This type of covariance modulo a multiplier function is determined by the Shapiro function basis; we may call Eqs. (41) and (42) the *Shapiro covariance* between the conjugate transformations in $\mathbf{x}$ and $\mathbf{p}$.

When the wavefields in the Wigner function (24) are translated within the hyperboloid by (41), the ambient vectors x are multiplied by a $(D+1)\times(D+1)$ ('boost') matrix $\mathbf{B_m}(\zeta)$, to $x'\mapsto\overline{x'}=\mathbf{B_m}(\zeta)^{-1}x'$ and $x''\mapsto\overline{x''}=\mathbf{B_m}(\zeta)^{-1}x''$. Under this transformation, the measures are again invariant, $d^D\mathbf{x}'/x_0'=d^D\overline{\mathbf{x}'}/\overline{x_0'}$, etc., and so is $\Delta^D(x;x',x'')=\Delta^D(\overline{x};\overline{x'},\overline{x''})$; hence, $\overline{x}=\mathbf{B_m}(\zeta)^{-1}x$ will appear in the first argument of the transformed Wigner function. But the corresponding transformation of $\mathbf{p}$ in each of the two Shapiro functions, Eqs. (42) and (44), yields a multiplier factor. The imaginary exponents of $\mu(\mathbf{m},\zeta;\mathbf{n})$ cancel, and there remains a positive net multiplier factor:

$$W_{\mathcal{H}}(T[\mathbf{B_m}(\zeta)]:f,\,T[\mathbf{B_m}(\zeta)]:g|\mathbf{x},p\,\mathbf{n})=(\mu(\mathbf{m},\zeta;\mathbf{n}))^{-D+1}\,W_{\mathcal{H}}(f,g|\mathbf{B}(\zeta)^{-1}:\mathbf{x},\;p\,\mathbf{B_m}(\zeta)^{-1}:\mathbf{n}),\tag{45}$$

where $\mathbf{n}'=\mathbf{B_m}(\zeta)^{-1}:\mathbf{n}$ is given by (43). Note that in the $D=1$-dimensional case, the multiplier factor is 1.

Covariance of the Wigner function is usually understood in the simple form it has under rotations, as given by (40). Under these transformations, the hyperboloid in the ambient x-space rotates on its axis, and in the momentum plane the circles $\mathbf{n}$ of all radii p rotate in synchrony. Translations within the hyperboloid (45), on the other hand, deform the ambient and projected space vectors, x and $\mathbf{x}$, through (41); momentum space is concurrently squeezed in the direction of the translation so that its points move on constant-p circles and with a common Bargmann deformation of the angle. Since areas are not conserved in momentum space, a multiplier function of $\mathbf{p}$ is necessary for the Wigner function to ensure the total conservation of probability (37).

E. Contraction limit

We now show that, when $f(x)$ and $g(x)$ are significantly different from zero only within a small, essentially flat patch of the hyperboloid, the definition of the Wigner function in Eq. (33) reduces to the standard Wigner function for flat space in Eq. (3). In (33), the integrand for $\boldsymbol{\eta}$ [recall Eqs. (31) and (34)] will be significant only when $\mathfrak{R}^D$ norms of the vectors fulfill

$$\left|\mathbf{x}\cosh\frac{1}{2}\tau\pm\mathbf{y}\sinh\frac{1}{2}\tau\right|\ll R\Rightarrow\begin{cases}|\mathbf{x}|\cosh\frac{1}{2}\tau\ll R & \Rightarrow\quad\sinh\chi\ll 1\\ |\mathbf{y}|\sinh\frac{1}{2}\tau\ll R & \Rightarrow\quad\sinh\tau\ll 1\end{cases}\tag{46}$$

$$\Rightarrow x\approx R(1,\chi\,\boldsymbol{\xi}),\quad y\approx R(\chi\,\boldsymbol{\xi}\cdot\boldsymbol{\eta},\boldsymbol{\eta}).\tag{47}$$

Also, using the limit in (19), and approximating $\sinh\tau\approx\tau$ and $\cosh\frac{1}{2}\tau\approx\cosh\chi\approx\cosh\omega\approx 1$, the Wigner function in Eq. (33) reduces to

$$W_{\mathcal{H}}(f,g|\mathbf{x},\mathbf{p})=\frac{R^D}{(2\pi)^D}\int_0^\infty\tau^{D-1}\,d\tau\int_{\mathcal{S}^{D-1}}d^{D-1}\boldsymbol{\eta}\times f\left(x_0,\mathbf{x}-\frac{1}{2}R\tau\boldsymbol{\eta}\right)^*\exp(-iR\tau\,\boldsymbol{\eta}\cdot\mathbf{p})\,g\left(x_0,\mathbf{x}+\frac{1}{2}R\tau\boldsymbol{\eta}\right).\tag{48}$$

Finally, changing variables to $\mathbf{z}=R\tau\,\boldsymbol{\eta}$ and noticing that

$$\int_{\mathfrak{R}^D}d^D\mathbf{z}\cdots=R^D\int_0^\infty\tau^{D-1}\,d\tau\int_{\mathcal{S}^{D-1}}d^{D-1}\boldsymbol{\eta}\cdots,\tag{49}$$

we see that (48) reduces to (2).

F. Special case of one dimension

In the case $D=1$, the Wigner function (33) actually coincides in form with the corresponding one-dimensional standard flat space form (2), as we now proceed to show.

First, notice that the unit vectors $\mathbf{n}$ and $\boldsymbol{\xi}$ in Eq. (13) are now the unit scalars $n,\xi=\pm 1$, and that the Shapiro functions become simple exponentials:

$$\Phi_p^{(1)}(R\xi\cosh\chi,R\xi\sinh\chi)=[\exp(-n\xi\chi)]^{-ipR}=\exp(in\xi pR\chi). \tag{50}$$

We can now let $np\mapsto p$ and $\xi\chi\mapsto\chi$ with $p,\chi\in(-\infty,\infty)$, and recognize that (50) is a 1:1 function of only one variable, $x_1\in\Re$,

$$\Phi_p^{(1)}(x_1)=\exp(i\chi pR), \quad x_1=R\sinh\chi. \tag{51}$$

The arguments $x=(x_0,x_1)$ of the functions f and g in (33) then simplify, in components, to

$$x=\begin{pmatrix}x_0\\x_1\end{pmatrix}\mapsto x\cosh\frac{1}{2}\tau\pm y\sinh\frac{1}{2}\tau=R\begin{pmatrix}\cosh(\chi\pm\frac{1}{2}\eta\tau)\\ \sinh(\chi\pm\frac{1}{2}\eta\tau)\end{pmatrix}. \tag{52}$$

For short, we indicate $f(R\cosh\chi,R\sinh\chi)=f(\chi)$. The unit vector $\boldsymbol{\eta}$ in (33) and (34) also becomes a unit scalar, $\eta=\pm 1$, and the integral extends over $y=\eta R(\sinh\chi,\cosh\chi)$, and $\Omega=\eta\chi$ [see Eq. (32)]. Finally, the integral over $\boldsymbol{\eta}$ reduces to a sum over $\eta=\pm 1$, and for $\tau\mapsto\eta\tau\in(-\infty,\infty)$, the Wigner function (33) becomes

$$W_{\mathcal{H}}(f,g|x,p)=\frac{R}{2\pi}\int_{-\infty}^{\infty}d\tau\, f\left(\chi-\frac{1}{2}\tau\right)^{*}e^{-ipR\tau}g\left(\chi+\frac{1}{2}\tau\right) \tag{53}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dv\,\tilde{f}\left(p-\frac{1}{2}v\right)^{*}e^{+iRvx}\tilde{g}\left(p+\frac{1}{2}v\right). \tag{54}$$

The last expression is the usual flat-space Wigner function in terms of the conjugate wavefunctions on momentum space. Finally, note that for $D=1$, the net multiplier which appears under "boost" transformations in (45) is unity, so standard and Shapiro covariances coincide.

IV. EXAMPLE: OSCILLATOR ON THE HYPERBOLA

We consider the open one-dimensional space which is the upper branch of a hyperbola of fixed radius $R>0$,

$$\mathcal{H}_+^1=\{(x_0,x_1)\in\Re^2|x_0^2-x_1^2=R^2\}, \tag{55}$$

parametrized as usual by the hyperbolic angle $\chi\in\Re$.

A. Laplace–Beltrami operator and the oscillator

When the potential $V(\mathbf{x})$ is a constant (corresponding to a homogeneous optical medium), the $D=1$ Schrödinger equation (10) is the free wave equation, and its Shapiro solutions are simply the oscillating exponentials (51), with energy $E=p^2/2\mu\geqslant 0$.

Since the Laplace–Beltrami operator on the hyperbola (55) is $\Delta_{\mathrm{LB}}=R^{-2}d^2/d\chi^2$, the Schrödinger equation for the conic oscillator (23) is

$$\left(\frac{-1}{2\mu}\frac{d^2}{d\chi^2}-R^2E_0\operatorname{sech}^2\chi\right)f(\chi)=R^2(E-E_0)f(\chi), \quad E_0=\frac{1}{2}\mu\omega^2R^2, \tag{56}$$

$$V(\chi)=\frac{1}{2}\mu\omega^2R^2\frac{x_1^2}{x_0^2}=\sqrt{s(s+1)}\,(1-\operatorname{sech}^2\chi), \tag{57}$$

$$s=-\tfrac{1}{2}+\sqrt{(\mu\omega R^2)^2+\tfrac{1}{4}}. \tag{58}$$

The bound solutions are[17]

$$\begin{aligned}\psi_n^s(\chi)&=\frac{2^{-(s-n)}}{\Gamma(s-n+1)}\sqrt{\frac{(s-n)\,\Gamma(2s-n+1)}{n!}}\operatorname{sech}^{s-n}\chi\;{}_2F_1\left(\begin{matrix}-n,\,2s-n+1\\ s-n+1\end{matrix}\middle|\,\frac{1-\tanh\chi}{2}\right)\\&=\sqrt{\frac{(s-n)\,n!}{\pi\,\Gamma(2s-n+1)}}\,\Gamma\left(s-n+\frac{1}{2}\right)(2\operatorname{sech}\chi)^{s-n}\,C_n^{s-n+1/2}(\tanh\chi),\end{aligned} \tag{59}$$

where n is a non-negative integer bounded by $s+1$, and $C_n^{\alpha}(\xi)$ are the Gegenbauer (or ultraspherical) polynomials[18] for $\alpha>-\frac{1}{2}$. The corresponding quantized values of the energy are quadratic in n, and counted from the lowest level up by

$$E_n^s=\frac{\mu\omega^2R^2}{2}-\frac{1}{2\mu R^2}(n-s)^2,\quad n=0,1,2,\ldots<s+1. \tag{60}$$

As a check on our concepts, we verify that the contraction limit $R\to\infty$ of this system, when the radius of the hyperbola grows without bound, is the harmonic oscillator on flat space. Since the coefficient s correspondingly grows as $s\approx\mu\omega R^2$, the linear-quadratic spectrum of energies in Eq. (60) becomes the linear spectrum of the quantum harmonic oscillator $E_n=\omega(n+\frac{1}{2})$. To implement this limit on the wavefunctions (59), it is convenient to use the following forms for the Gegenbauer polynomials in $\xi=\tanh\chi$:

$$C_n^{\alpha}(\xi)=\begin{cases}(-1)^{(1/2)\,n}\dfrac{\Gamma(\alpha+\frac{1}{2}n)}{(\frac{1}{2}n)!\,\Gamma(\alpha)}\,{}_2F_1\left(\begin{matrix}-\frac{1}{2}n,\,\frac{1}{2}n+\alpha\\ \frac{1}{2}\end{matrix}\middle|\,\xi^2\right), & n \text{ even},\\[2ex] (-1)^{(1/2)(n-1)}\dfrac{\Gamma(\alpha+\frac{1}{2}(n+1))}{(\frac{1}{2}(n-1))!\,\Gamma(\alpha)}2\xi\,{}_2F_1\left(\begin{matrix}-\frac{1}{2}(n-1),\,\frac{1}{2}(n+1)+\alpha\\ \frac{3}{2}\end{matrix}\middle|\,\xi^2\right), & n\text{ odd}.\end{cases} \tag{61}$$

Then, for $\alpha=s-n+\frac{1}{2}\to\infty$, the hypergeometric polynomials simplify: for n even, ${}_2F_1(-\frac{1}{2}n,\alpha;\frac{1}{2};\xi^2)\approx{}_1F_1(-\frac{1}{2}n;\frac{1}{2};\alpha\xi^2)=H_n(\sqrt{\alpha}\xi)$, and similarly for n odd. Replacing this into Eq. (59), with $\xi=\tanh\chi\approx\sinh\chi=x_1/R$ and $\cosh^{-s+n}\chi\approx\exp(-s\tanh^2\frac{1}{2}\chi)$, we obtain the harmonic oscillator wavefunctions on flat space,

$$\frac{1}{\sqrt{R}}\psi_n^s(\chi)\approx\frac{1}{\sqrt{2^n n!\,\sqrt{\pi/\mu\omega}}}e^{-\mu\omega x_1^2/2}H_n(\sqrt{\mu\omega}\,x_1). \tag{62}$$

The factor $\sqrt{R}$ restores the proper normalization on the x_1 axis.

In addition to the bound states, the sech-trough Pöschl–Teller potential also has free states with energy above the asymptotic value of the potential $\lim_{\chi\to\pm\infty}V(\chi)=\frac{1}{2}\mu\omega^2R^2$. These scattering solutions contain associated Legendre polynomials of imaginary upper index:

$$\psi_p(\chi)=\frac{|\Gamma(1-ip)|}{2\pi}P_\sigma^{ip}(\tanh\chi),\qquad \begin{aligned}p&=R\sqrt{2\mu E-\mu^2\omega^2R^2}>0,\\ \sigma&=\tfrac{1}{2}\pm\sqrt{\omega^2\mu^2R^4+\tfrac{1}{4}}.\end{aligned} \tag{63}$$

These wavefunctions are Dirac-orthonormal.

B. Momentum representation of the wavefunctions

The bound wavefunctions of the Pöschl–Teller sech-trough potential in the momentum representation, $\tilde{\psi}_n^s(p)$, are found through the ordinary Fourier transform [Eqs. (20) and (21) for $D=1$ and (51)] of the wavefunctions $\psi_n^s(\chi)$ found in Eq. (59). The result is

$$\begin{aligned}\tilde{\psi}_n^s(p)&=\sqrt{\frac{R}{2\pi}}\int_{-\infty}^{\infty}d\chi\exp(-ipR\chi)\,\psi_n^s(\chi)\\ &=\frac{R}{2}\sqrt{\frac{\Gamma(2s-n+1)}{\pi(s-n)n!}}\frac{|\Gamma(\frac{1}{2}(s-n-ipR))|^2}{\Gamma(s-n)^2}\,{}_3F_2\left(\left.\begin{matrix}-n,\,2s-n+1,\frac{1}{2}(s-n-ipR)\\ s-n+1,s-n\end{matrix}\right|1\right)\end{aligned} \tag{64}$$

$$\begin{aligned}&=\frac{(-i)^nR}{2\sqrt{\pi}}\frac{\sqrt{(s-n)\,n!\,\Gamma(2s-n+1)}}{\Gamma(s)\,\Gamma(s+1)}\left|\Gamma\left(\frac{1}{2}(s-n-ipR)\right)\right|^2\\ &\quad\times R_n\left(-\frac{1}{2}ipR;\frac{1}{2}(s-n),\frac{1}{2}(s-n),\frac{1}{2}(s-n),\frac{1}{2}(s-n)+1\right),\end{aligned} \tag{65}$$

where $R_n(z;\alpha,\beta,\gamma,\delta)$ are the continuous Hahn polynomials.[19] On the other hand, the unbound solutions (63) are not square-integrable, so their Fourier transform must be performed allowing for the phase difference between asymptotic incoming and outgoing waves, as determined by the scattering properties of this Pöschl–Teller potential. We shall not further detail the free states here; they can be found in Ref. 9 among the coupling coefficients between the $D^+\times D^-\rightarrow\Sigma D^++\int C$ irreducible representations series of SO(2,1).

C. Wigner function for the oscillator eigenstates on the hyperbola

On the one-dimensional hyperbola $\mathcal{H}_+^1$, the Wigner function (24) collapses to (53) and (54), its usual form in Eqs. (2) and (3) for $D=1$.

We are interested in the single-function form $W(f|\mathbf{x},\mathbf{p})\equiv W(f,f|\mathbf{x},\mathbf{p})=W(\tilde{f},\tilde{f}|\mathbf{p},-\mathbf{x})$ for the Pöschl–Teller wavefunctions, whose explicit form is in Eq. (59) for $\psi_n^s(x)$, and in Eq. (64) for $\tilde{\psi}_n^s(p)$; we find the latter more amenable to analytic solution. We change the integration to a contour along the imaginary axis, and find

$$\begin{aligned}W(\psi_n^s|\chi,p)=\frac{R^2}{8\pi^2}\frac{s-n}{n!\,\Gamma(2s-n+1)}\sum_{m,l=0}^{n}\frac{(-n)_m\,(-n)_l}{\Gamma(s-n+m)\,\Gamma(s-n+l)}\\ \times\frac{\Gamma(2s-n+m+1)\,\Gamma(2s-n+l+1)}{\Gamma(s-n+m+1)\,\Gamma(s-n+l+1)}\frac{1}{l!\,m!}I_n^s(\chi,p),\end{aligned} \tag{66}$$

where

$$I_n^s(\chi,p) = -\frac{4i}{R}\int_{-i\infty}^{i\infty} dz\, \Gamma\left(\frac{1}{2}(s-n)-\frac{1}{2}ipR-z\right)\Gamma\left(\frac{1}{2}(s-n)+\frac{1}{2}ipR+z+m\right)$$
$$\times e^{-4\chi z}\,\Gamma\left(\frac{1}{2}(s-n)+\frac{1}{2}ipR-z\right)\Gamma\left(\frac{1}{2}(s-n)-\frac{1}{2}ipR+z+l\right). \tag{67}$$

The last integral can be computed in the complex plane straightforwardly, and leads to a pair of complex conjugate ${}_2F_1$-functions of $e^{-4\chi}$. We thus finally have

$$W(\psi_n^s|\chi,p) = \frac{2R\,(s-n)\,n!\,e^{-2\chi(s-n)}}{\pi\Gamma(2s-n+1)}\sum_{m,l=0}^{n}\frac{\Gamma(2s-n+m+1)\,\Gamma(2s-n+l+1)}{\Gamma(s-n+l+1)\,\Gamma(s-n+m+1)\,\Gamma(s-n+l)}$$
$$\times\frac{(-1)^{l+m}}{l!\,m!\,(n-m)!\,(n-l)!}\,\mathrm{Re}\left\{\Gamma(ipR)\,\Gamma(s-n+l-ipR)\,e^{2ipR\chi}\right.$$
$$\left.\times{}_2F_1\left(\left.\begin{matrix}s-n+m,s-n+l-ipR,\\ 1-ipR\end{matrix}\right| e^{-4\chi}\right)\right\}. \tag{68}$$

We note that this expression is suitable for numerical computation only for $\chi>0$ because it converges fast, but it holds everywhere analytically, with the reflection symmetries $W(\psi_n^s|\chi,p) = W(\psi_n^s|-\chi,p) = W(\psi_n^s|\chi,-p)$.

The Wigner functions, together with their marginal projections $|\psi_n^s(x)|^2$ and $|\tilde{\psi}_n^s(p)|^2$, are shown in Fig. 1, for $n=0,1,2,3$, and for the potential depth parameter $s=4$ and 30. It can be appreciated that the Wigner function of the most tightly bound states resemble the familiar Gaussian-bell form of the harmonic oscillator ground state. According to (60), for $s=4$ there are only five bound states ($n=0,\ldots,4$), and as the energy of the state approaches the binding energy, the wavefunction stretches in space with ever-smaller momentum in a neighborhood of the classical turning point. The contraction limit can be appreciated in the $s=30$ column, corresponding to a large binding energy; the Wigner functions for the eigenstates in Eq. (59) approach the familiar Laguerre–Gaussian form that corresponds to the Wigner function of the harmonic oscillator wavefunctions on flat space.

V. CONCLUDING REMARKS

We have generalized the Wigner quasiprobability distribution function by replacing the oscillating exponential functions of the standard version, which are Dirac solutions of the free Schrödinger equation on flat space, by the Shapiro functions, because they are solutions to the Laplace–Beltrami equation on a simply connected hyperbolic space, while respecting the midpoint condition through an appropriate Dirac-like $\Delta^D(x;x',x'')$. The role of the Fourier transform in the standard version is transfered by the Shapiro functions to a Mellin-like transform between the position and momentum coordinates of phase space. Indeed, the relation between what were called position and momentum variables is actually defined by the argument and index of the basis of Shapiro functions. Thus built, the proposed Wigner function is covariant under the group SO(D,1) of motions of the hyperbola, with the hyperbolic translations extracting a multiplier factor. The correct marginals are found and the contraction limit to flat space returns the standard Wigner function.

The transformations of the momentum direction under translations of the hyperbola are the (unique) action of SO(D,1) on the sphere $\mathcal{S}^D$.[13] There are several models of Hamiltonian systems where momentum is restricted to a sphere, such as geometric and Helmholtz (monochromatic) optics.[20,21] In the first model, a "Descartes sphere" of momentum vectors $\mathbf{p}$, which is of radius $|\mathbf{p}|=n(\mathbf{x})$ (the refractive index), is associated to each point $\mathbf{x}$ in space. This sphere of momentum vectors has been subject to Bargmann's "boost" transformation in Ref. 22 (which is unique for Lorentz groups acting on spheres) and here given by (43), with a canonically conjugate transformation of the ray positions at a plane screen. The resulting phenomenon has been called relativ-

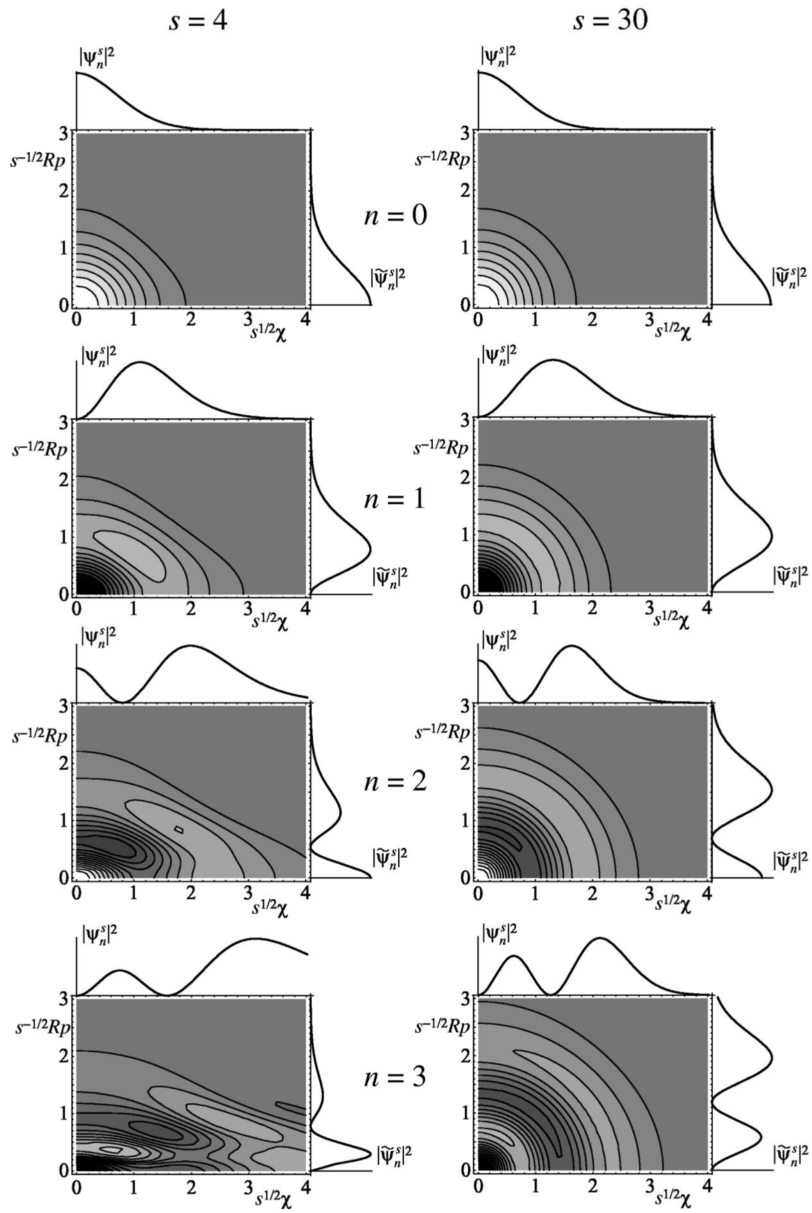

FIG. 1. Wigner functions of the Pöschl–Teller eigenfunctions $\psi_n^s(\chi)$ on a quadrant of phase space (axes are position $\chi\sqrt{s}$ and momentum $pR/\sqrt{s}$; the quadrants have reflection symmetry across the axes). Rows are numbered by the mode $n=0,1,2,3$. Left: $s=4$ (so only five states, $n=0,\ldots,4$, are bound); right: $s=30$ (so states are bound up to $n=30$). White is the maximum, black is the minimum; the shade at the upper right corner corresponds to zero. From each Wigner function we project up the marginal distribution of position $|\psi_n^s(\chi)|^2$, and right the marginal of momentum $|\tilde{\psi}_n^s(p)|^2$.

istic coma, because the aberration in the images projected on moving screens in vacuum is comatic. The Doppler effect is of course absent, so this transformation cannot be called physically relativistic in the Helmholtz case, which involves a configuration space with a nonlocal metric. Indeed, the difference between the various models consists in their space of positions; here, it is the hyperboloid. A Wigner function on spheres will be examined in part II of this title; it is expected to clarify further the use of function bases to define conjugate variables as a simile or substitute for phase space.

The context of this work has a wider significance as a model for phase space with noncommutative geometry, although here the noncommutativity is restricted to momentum space only. In one dimension this argument does not apply, but the example of the one-dimensional Wigner function of Pöschl–Teller wavefunctions is of interest on its own for the traditional quantum mechanical model, and also for the paraxial propagation of light along shallow nonharmonic waveguides whose index of refraction has a sech^2 profile, as given by Eq. (11). One of the manifestations of higher symmetry is the appearance of ''closed-form'' wavefunctions expressed in terms of well-known (and some not-so-well-known) special functions. Generally, the Fourier transforms and Wigner functions of such wavefunctions are again known special functions because symmetry, when it occurs, is displayed in phase space.

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