Fractional Fourier transformers through reflection

Kurt Bernardo Wolf and Guillermo Krötzsch

Centro de Ciencias Físicas, Universidad Nacional Autónoma de México, Apartado Postal 48-3, Cuernavaca, Morelos 62251, México

Received November 2, 2001; revised manuscript received December 6, 2001; accepted December 10, 2001

We show that an arbitrary paraxial optical system, compounded with its reflection in an appropriately warped mirror, is a pure fractional Fourier transformer between coincident input and output planes. The geometric action of reflection on optical systems is introduced axiomatically and is developed in the paraxial regime. The correction of aberrations by warp of the mirror is briefly addressed. © 2002 Optical Society of America

OCIS codes: 070.2590, 080.1010, 080.2720, 080.2730.

1. INTRODUCTION

Fractional Fourier transforms produced by optical means have been studied for their applications to signal analysis and image processing.[1,2] The construction of such paraxial Fourier transformers through lenses or waveguides has been extensively addressed in the literature, but systems involving reflection have not been subject to similar scrutiny. In this paper we develop a systematic approach to incorporate mirrors with the same purpose. Particularly, we consider configurations where light, after crossing a system, is reflected back through the same system, so that the object and image planes are coincident. At no extra cost we can work in N dimensions, although our immediate interest is in $N = 1$ and 2.

Abstractly, reflection is an antihomomorphism of canonical transformations, linear (paraxial) as well as nonlinear (metaxial). Its definition is addressed in Section 2 on the basis of free propagation and refracting surfaces. We apply the factorization of the refraction map[3,4] to find the action of reflection on the root transformation and characterize systems cum reflection in a warped mirror. In Section 3, reflection is reduced to an antihomomorphic map of matrices in the paraxial regime. In Section 4 we use the modified Iwasawa decomposition of the symplectic groups[5] to build the main statement of this paper, that any optical system cum reflection in an appropriately warped mirror is a fractional Fourier transformer.[6] We present a simple example in Section 5 in which we detail the metaplectic phase obtained with the apparatus. The reflection of aberrations in the metaxial regime is briefly examined in Section 6.

Because paraxial optical systems have geometric and wave realizations (which are 1:2-homomorphic), our presentation of the subject will include the mathematical techniques of group theory. We favor the use of matrices over Fresnel integrals for their simplicity. These integrals (canonical transforms[7]) involve the rather delicate metaplectic sign (phase[5,6]) which stems from the double cover of the wave over the geometric model. When this sign is unimportant, linear matrix algebra is certainly cleaner than multiple (improper oscillating-Gaussian) integrals, especially for numerical purposes.

Warped screens have been used before to correct Fourier transformers, both in the paraxial regime[8,9] to build resonators and in the metaxial regime to correct their aberrations.[10] Warped mirrors in place of screens in a sense duplicate these systems cum reflection. Reflection is a structural feature of the symplectic groups (linear and nonlinear, optical and mechanical) with an interest of its own: It involves similarity with an element **K** used in quantum optics,[11] which is not symplectic. Here we explore reflection as the simplest geometric-optical realization of this structure. Finding the Fresnel integrals for the corresponding wave-optical realization is then a straightforward task. Moreover, as our concluding section shows, the action of reflection defined here can be extended consistently to the metaxial regime in any aberration order. The proof through the root transformation has been algorithmic and holds to order 7, but we are confident in conjecturing its generic validity.

2. OPTICAL ELEMENTS AND REFLECTIONS

Geometric rays are oriented lines in optical space $(\mathbf{q}, z) \in \Re^{N+1}$ and points in phase space $(\mathbf{p}, \mathbf{q}) \in \Re^{2N}$. They are referred to a standard screen, which is the N–dim plane $(\mathbf{q}, 0)$ in optical space; phase space is a $2N$-dim symplectic manifold.[1] Optical systems, indicated by $\mathcal{G}$, can be built with the concatenation of a finite number of optical elements: free spaces and refracting surfaces. In this section we define the operation of reflection $\mathcal{G} \mapsto \overline{\mathcal{G}}$ on these elements and then show that reflection can be defined equivalently on the root transformation,[3,4] and finally we concatenate systems with their reflection in generally warped mirrors.

In Fig. 1 we show free propagation $\mathcal{F}_{n,z}$ by $z \geqslant 0$ in a homogeneous medium of refractive index n and its reflection in a flat mirror, denoted $\overline{\mathcal{F}_{n,z}}$. Since the two maps act identically on all rays and their phase space points, it follows that, as operators,

$$\overline{\mathcal{F}_{n,z}} = \mathcal{F}_{n,z}. \tag{1}$$

(Free flights form a one-parameter commutative semigroup that is invariant under reflections with unit $\mathcal{F}_{n,0} = 1$.)

0740-3232/2002/061191-06$15.00

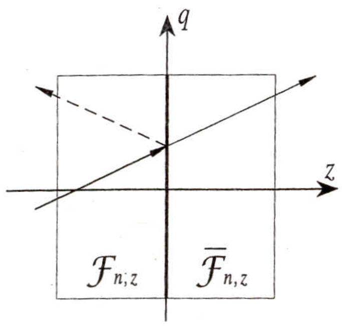

Fig. 1. Reflection of free propagation in a mirror on the **q** plane, seen as free propagation into the mirror world.

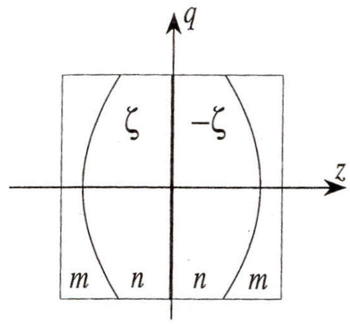

Fig. 2. Reflection of a refracting surface $z = \zeta(\mathbf{q})$ between media m and n is, in the mirror world, a refracting surface $z = -\zeta(\mathbf{q})$ between media n and m.

A refracting surface $z = \zeta(\mathbf{q})$ between two media of refractive indices m and n produces the transformation of phase space denoted $\mathcal{S}_{m\mapsto n;\zeta}$; this is shown in Fig. 2. Its mirror image is also a refracting surface transformation, but of the reflected surface $z = -\zeta(\mathbf{q})$ and between the media n and m (order reversed), namely

$$\overline{\mathcal{S}_{m\mapsto n;\zeta}} = \mathcal{S}_{n\mapsto;-\zeta}\,. \tag{2}$$

(The set of refracting surface transformations is neither a group nor a semigroup, although it includes the identity $\mathcal{S}_{m=n;\zeta} = 1 = \mathcal{S}_{n\mapsto m;0}$; nevertheless, it maps onto itself under reflection.) Equations (1) and (2) are the two elementary requirements that we demand of the reflection map.

The reflection map $\mathcal{G}\mapsto\overline{\mathcal{G}}$ must also satisfy two operational requirements: First, the reflection of reflection returns the original system,

$$\overline{\overline{\mathcal{G}}} = \mathcal{G}, \tag{3}$$

so it must be bijective. And second, reflection is an antihomomorphism for the order of concatenation of optical elements. As shown in Fig. 3, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are two optical systems whose reflections are $\overline{\mathcal{G}_1}$ and $\overline{\mathcal{G}_2}$, then their ordered product will reflect reversing the order of the factors, viz.,

$$\overline{\mathcal{G}_1\mathcal{G}_2} = \overline{\mathcal{G}_2}\;\overline{\mathcal{G}_1}. \tag{4}$$

The reflection of refracting surfaces [Eq. (2)] can be further analyzed by using the factorization theorem proven in Refs. 3 and 4 (see also Refs. 12–16). This states that the phase-space maps due to refraction at a surface ζ can be (locally) factored as

$$\mathcal{S}_{m\mapsto n;\zeta} = \mathcal{R}_{m;\zeta}\mathcal{R}_{n;\zeta}^{-1}, \tag{5}$$

into the root transformation $\mathcal{R}_{m;\zeta}$, which depends only on the refractive index m of the first medium and on the surface ζ, times the inverse of $\mathcal{R}_{n;\zeta}$, which depends exclusively on the index n of the second medium, and the common surface. These transformations are canonical inside a connected region of phase space whose boundary is the set of rays that are tangent to the surface.

The root transformation maps phase space from the standard screen $z = 0$ to the generally warped surface ζ in a medium n.[10] The inverse root transformation brings back the phase-space coordinates from the warped surface ζ to the plane $z = 0$. In particular, for a flat surface $\zeta = z_0$ the root transformation is free flight: $\mathcal{R}_{n;z_0} = \mathcal{F}_{n;z_0}$. Also, when $\zeta(\mathbf{0}) = z_0$, the free flight to the surface can be factored off to the left as $\mathcal{R}_{n;\zeta} = \mathcal{F}_{n,z_0}\mathcal{R}_{n;\zeta-z_0}$. So we need to consider only surfaces whose center is at the origin, $\zeta(\mathbf{0}) = 0$.

Now, using Eqs. (1), (2), (4) and (5), we can write the reflection of a refracting surface transformation in two ways:

$$\overline{\mathcal{S}_{m\mapsto n;\zeta}} = \begin{cases} \overline{\mathcal{R}_{m;\zeta}\mathcal{R}_{n;\zeta}^{-1}} = \overline{\mathcal{R}_{n;\zeta}^{-1}}\;\overline{\mathcal{R}_{m;\zeta}} \\ \mathcal{S}_{n\mapsto m;-\zeta} = \mathcal{R}_{n;-\zeta}\mathcal{R}_{m;-\zeta}^{-1}. \end{cases} \tag{6}$$

Since they depend on different refractive indices, the last two members imply the equality of the root factors. Hence the reflection of the root transformation is

$$\overline{\mathcal{R}_{m;\zeta}} = \mathcal{R}_{m;-\zeta}^{-1}. \tag{7}$$

Both reflection and inversion are antihomomorphisms whose square is the identity. They are distinct and independent, however, because $(\overline{\mathcal{R}_{n,\,\zeta}})^{-1} = \mathcal{R}_{n,-\zeta} = \overline{(\mathcal{R}_{n,\,\zeta}^{-1})}$, and they commute.

When we place a flat mirror at the right end of the system $\mathcal{G}$, so that rays after reflection traverse it in the opposite direction seeing it as $\overline{\mathcal{G}}$, we call the compound system between the (coincident) input and output screens the *system cum reflection*:

$$\mathcal{G}^{\mathrm{II}} := \mathcal{G}\overline{\mathcal{G}}. \tag{8}$$

And if we warp the mirror from the $z = 0$ plane to $z = \zeta(\mathbf{q})$, the system cum reflection $\mathcal{G}^{\mathrm{II}}$ will become

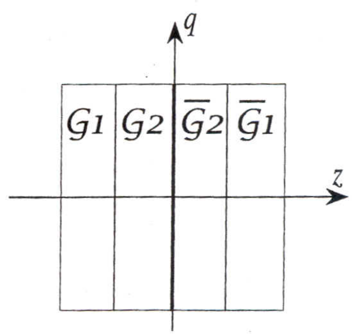

Fig. 3. Reflection is an antihomomorphism: The reflected concatenation of two systems $\mathcal{G}_1$ and $\mathcal{G}_2$ is the product of the reflected systems, $\overline{\mathcal{G}_1}$ and $\overline{\mathcal{G}_2}$, in reversed order.

$$\mathcal{G}^{\mathrm{II}} \mapsto \mathcal{G}^{\mathrm{II}}_{\zeta} := \mathcal{G}\mathcal{R}_{n;\zeta}\overline{\mathcal{G}\mathcal{R}_{n;\zeta}} = \mathcal{G}\mathcal{M}_{n;\zeta}\bar{\mathcal{G}}$$
$$= \mathcal{G}_{\zeta}\overline{\mathcal{G}_{\zeta}}, \qquad \mathcal{G}_{\zeta} := \mathcal{G}\mathcal{R}_{n;\zeta}, \tag{9}$$

where the mirror transformation

$$\mathcal{M}_{n;\zeta} := \mathcal{R}_{n;\zeta}\overline{\mathcal{R}_{n;\zeta}} = \mathcal{R}_{n;\zeta}\mathcal{R}^{-1}_{n;-\zeta}, \tag{10}$$

is interposed between the specular roots of $\mathcal{G}^{\mathrm{II}}$.

3. REFLECTION IN THE PARAXIAL REGIME

In the paraxial regime one replaces optical transformations $\mathcal{G}$ by their linear approximation $G(\mathbf{M})$, represented by matrices $\mathbf{M}$ that act on the coordinates of momentum $\mathbf{p}$ and position $\mathbf{q}$ of rays at the standard screen, $\binom{\mathbf{p}}{\mathbf{q}} \in \mathfrak{R}^{2N}$. The matrices are symplectic,[1] i.e.,

$$\mathbf{M}\mathbf{J}\mathbf{M}^{\mathsf{T}} = \mathbf{J}, \qquad \mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \mathbf{J}^{-1}. \tag{11}$$

In $(N \times N)$-block form, this symplectic condition is written

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$
$$\mathbf{A}\mathbf{B}^{\mathsf{T}},\ \mathbf{A}^{\mathsf{T}}\mathbf{C},\ \mathbf{B}^{\mathsf{T}}\mathbf{D},\ \mathbf{C}\mathbf{D}^{\mathsf{T}}\ \text{symmetric},\ \ \mathbf{A}\mathbf{D}^{\mathsf{T}} - \mathbf{B}\mathbf{C}^{\mathsf{T}} = \mathbf{1}. \tag{12}$$

In particular, it can be shown that $\det \mathbf{M} = 1$, so the inverse exists and

$$\mathbf{M}^{-1} = \mathbf{J}\mathbf{M}^{\mathsf{T}}\mathbf{J} = \begin{bmatrix} \mathbf{D}^{\mathsf{T}} & -\mathbf{B}^{\mathsf{T}} \\ -\mathbf{C}^{\mathsf{T}} & \mathbf{A}^{\mathsf{T}} \end{bmatrix}. \tag{13}$$

The set of $2N \times 2N$ symplectic matrices form a group denoted $\mathrm{Sp}(2N, \mathfrak{R})$. [Many authors write $\mathbf{q}$ atop $\mathbf{p}$ so the elements of the representing matrices are permuted but have the same properties. Also recall that the action of optical systems is through the inverse matrix, $G(\mathbf{M}) : \binom{\mathbf{p}}{\mathbf{q}} = \mathbf{M}^{-1}\binom{\mathbf{p}}{\mathbf{q}}$, so $G(\mathbf{M}_1)\ G(\mathbf{M}_2) = G(\mathbf{M}_1\mathbf{M}_2)$ corresponds to the left-to-right placement of optical elements along the z axis. We shall not need this explicitly, though.]

Because free flight $\mathcal{F}_{n,z}$ in Eq. (1) is invariant under reflection, it follows that so is its linear part, i.e.,

$$\overline{\begin{bmatrix} 1 & 0 \\ z/n\mathbf{1} & \mathbf{1} \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ z/n\mathbf{1} & \mathbf{1} \end{bmatrix}. \tag{14}$$

In the paraxial approximation, the root and refracting surface transformations, $\mathcal{R}_{m;\zeta}$ and $\mathcal{S}_{m\mapsto n;\zeta}$, are represented by symplectic upper-triangular block matrices, called Gaussian matrices. When the quadratic surface is of the form $\zeta(\mathbf{q}) = \frac{1}{2}\mathbf{q}^{\mathsf{T}}\mathbf{Z}\mathbf{q}$ with symmetric $\mathbf{Z}$, the matrix block of the root transformation is $\mathbf{G} = -2m\mathbf{Z}$, and $\mathbf{G} = 2(n - m)\mathbf{Z}$ for the refracting surface. From Eqs. (2) and (7) we see that they are also invariant under reflection:

$$\overline{\begin{bmatrix} \mathbf{1} & \mathbf{G} \\ 0 & \mathbf{1} \end{bmatrix}} = \begin{bmatrix} \mathbf{1} & \mathbf{G} \\ 0 & \mathbf{1} \end{bmatrix}, \qquad \mathbf{G}^{\mathsf{T}} = \mathbf{G}. \tag{15}$$

To find the reflection of a generic paraxial optical system $\mathbf{M}$ from Eqs. (14) and (15) with the antihomomorphic property of Eq. (4), we may construct $\mathbf{M}$ from lenses and free flights,[17,18] writing products of upper- and lower-triangular matrices (the latter with a block that is a positive multiple of unity) and invert the order of product. The result in every case is

$$\mathbf{M} \mapsto \bar{\mathbf{M}} = \overline{\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}} = \begin{bmatrix} \mathbf{D}^{\mathsf{T}} & \mathbf{B}^{\mathsf{T}} \\ \mathbf{C}^{\mathsf{T}} & \mathbf{A}^{\mathsf{T}} \end{bmatrix}. \tag{16}$$

We stated before that reflection is an antihomomorphism, which is distinct from inversion. But in the paraxial approximation, we see from Eqs. (13) and (16) that we can write one in terms of the other through a similarity homomorphism:

$$\bar{\mathbf{M}} = \mathbf{K}\mathbf{M}^{-1}\mathbf{K}, \qquad \mathbf{K} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{K}^{-1}. \tag{17}$$

We note that $\mathbf{K}$ is not a symplectic matrix [it does not satisfy Eq. (11)]; but when $\mathbf{M}$ is symplectic, so is $\bar{\mathbf{M}}$. Paraxial reflection is thus an outer antihomomorphism of $\mathrm{Sp}(2N, \mathfrak{R})$.

The action of the nonsymplectic matrix $\mathbf{K}$ on phase space $\binom{\mathbf{p}}{\mathbf{q}}$ is to reverse the ray momentum vector on the screen, $\mathbf{p} \leftrightarrow -\mathbf{p}$. This is consistent with Fig. 1 when the reflected ray is continued in the $+z$ direction. In a different context, a similar outer antihomomorphism of $\mathrm{Sp}(4, \mathfrak{R})$ has been used in Ref. 11 to distinguish separable from entangled quantum states by their symmetry under $\mathbf{p} \leftrightarrow -\mathbf{p}$ in Wigner's phase space.

In the paraxial regime, systems $\mathbf{M}$ cum reflection [see Eqs. (8) and (16)] are consequently characterized by matrices

$$\mathbf{M}^{\mathrm{II}} = \mathbf{M}\bar{\mathbf{M}} = \begin{bmatrix} \mathbf{A}\mathbf{D}^{\mathsf{T}} + \mathbf{B}\mathbf{C}^{\mathsf{T}} & 2\mathbf{A}\mathbf{B}^{\mathsf{T}} \\ 2\mathbf{C}\mathbf{D}^{\mathsf{T}} & (\mathbf{A}\mathbf{D}^{\mathsf{T}} + \mathbf{B}\mathbf{C}^{\mathsf{T}})^{\mathsf{T}} \end{bmatrix}, \tag{18}$$

where the submatrices on the antidiagonal are symmetric and the two submatrices on the diagonal are the transposes of each other. Systems cum reflection Eq. (18) have the generic form of all reflection-invariant matrices $\bar{\mathbf{M}}^{\mathrm{II}} = \mathbf{M}^{\mathrm{II}}$. Only such matrices can have specular roots $\mathbf{M}$. Specular roots are not uniquely determined, however, because two systems $\mathbf{M}$ and $\mathbf{M}\mathbf{L}$, with $\mathbf{L} = \mathrm{diag}(\mathbf{E}, \mathbf{E}^{\mathsf{T}-1})$, $\det \mathbf{E} \neq 0$, will produce the same system cum reflection, since $\mathbf{L}^{\mathrm{II}} = \mathbf{1}$. These systems $\mathbf{L}$ are pure imagers, which include astigmatic magnifiers and rotators,[5] and form the general linear group in N dimensions, $\mathrm{Gl}(N, \mathfrak{R})$. Therefore, specular roots form right coset manifolds $\mathrm{Sp}(2N, \mathfrak{R})/\mathrm{Gl}(N, \mathfrak{R})$, modulo the subgroup of specular roots of unity.

4. FRACTIONAL FOURIER TRANSFORMS

In this section we show that by properly warping the mirror at the end of an arbitrary paraxial system, the system cum reflection can be made a pure fractional Fourier transformer. We recall from Ref. 6 that in N dimensions, fractional Fourier transforms form the group $U(N)$ of $N \times N$ unitary matrices $\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{1}$ (where $\dagger$ is transpose conjugate), which is the maximal compact subgroup of $\mathrm{Sp}(2N, \mathfrak{R})$, and is represented on phase space by orthosymplectic matrices, (i.e., matrices both orthogonal and symplectic). We build the proof by using the modified

Iwasawa decomposition of Sp(2N, $\mathfrak{R}$) matrices of Ref. 5 into a pure Fourier transformer, a pure imager, and an astigmatic Gaussian matrix:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\,\mathbf{U} & \mathrm{Im}\,\mathbf{U} \\ -\mathrm{Im}\,\mathbf{U} & \mathrm{Re}\,\mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^{\mathrm{T}-1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{G} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}. \tag{19}$$

[This decomposition is global but not unique because rotators $\mathbf{A} = \mathbf{R}(\theta) = \mathbf{D}$, $\mathbf{B} = \mathbf{0} = \mathbf{C}$, $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{1}$, are both pure imagers and pure Fourier transformers.]

Warping the screen to the quadratic shape $z = \zeta(\mathbf{q}) = \mathbf{q}^{\mathrm{T}}\mathbf{Z}\mathbf{q}$ in a medium of refractive index n, multiplies the system on the right by the root transformation $\mathcal{R}_{n,\mathbf{Z}}$, represented by a Gaussian upper-triangular matrix. Hence we can eliminate the Gaussian factor in Eq. (19) when

$$2n\mathbf{Z} = -\mathbf{G} \Rightarrow \mathbf{M}\mathbf{R}_{m,\mathbf{Z}} = \begin{bmatrix} \mathrm{Re}\,\mathbf{U} & \mathrm{Im}\,\mathbf{U} \\ -\mathrm{Im}\,\mathbf{U} & \mathrm{Re}\,\mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^{\mathrm{T}-1} \end{bmatrix}. \tag{20}$$

The system *cum* reflection in the warped mirror is thus a pure $U(N)$–Fourier transformer,

$$\mathbf{M}^{\mathrm{II}}_{n;\mathbf{Z}} = \mathbf{M}\mathbf{R}_{n;\mathbf{Z}}\overline{\mathbf{R}_{n;\mathbf{Z}}\mathbf{M}} = \begin{bmatrix} \mathrm{Re}\,\mathbf{U}\mathbf{U}^{\mathrm{T}} & \mathrm{Im}\,\mathbf{U}\mathbf{U}^{\mathrm{T}} \\ -\mathrm{Im}\,\mathbf{U}\mathbf{U}^{\mathrm{T}} & \mathrm{Re}\,\mathbf{U}\mathbf{U}^{\mathrm{T}} \end{bmatrix}, \tag{21}$$

characterized by the symmetric unitary matrix $\mathbf{U}\mathbf{U}^{\mathrm{T}} \in U(N)$.

5. SIMPLE EXAMPLE

To build a simple prototypical example of a system cum reflection, we consider the axisymmetric "cat's-eye" system of Fig. 4, for which we can use 2 × 2 matrices.[1] The Iwasawa decomposition, Eq. (19), is

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E^{-1} \end{bmatrix} \begin{bmatrix} 1 & G \\ 0 & 1 \end{bmatrix}. \tag{22}$$

We choose the parameters to be simple numbers: the "air" and "eye" refractive indices will be $m = 1$ and $n = \frac{3}{2}$ respectively, and the object plane will be fixed at $z_0 = 6$ units to the left of the optical center of a spherical refracting surface of unit radius $Z = \frac{1}{2}$ in the paraxial approximation. At a variable distance to the right, $z = n\chi = \frac{3}{2}\chi$ ($\chi > 0$), we place the warpable mirror. This system, a lens between two free spaces, is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\chi & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}\chi & \frac{1}{2} \\ -6 + 2\chi & -2 \end{bmatrix}. \tag{23}$$

When the lower-left matrix element of $\mathbf{M}$ vanishes (at $\chi = 3 = \frac{2}{3}z$), the system is an inverting imager (impure, since $b \neq 0$) of magnification -2. The vanishing of the upper-left element (at $\chi = 2 = \frac{2}{3}z$) turns $\mathbf{M}$ into a Fourier transformer (magnifying, slanting, i.e., impure).

By elementary 2 × 2 matrix algebra we find the dependence of the Iwasawa parameters (θ, E, G) on χ,

$$\theta = \arg(a - ic) = \arg[(1 + 6i) - (\tfrac{1}{2} + 2i)\chi], \tag{24}$$

$$E = +(a^2 + c^2)^{1/2} = +(37 - 25\chi + \tfrac{17}{4}\chi^2)^{1/2} > 0, \tag{25}$$

$$G = \frac{ab + cd}{a^2 + c^2} = \frac{50 - 17\chi}{148 - 100\chi + 17\chi^2} = 2n\zeta_2 = 3\zeta_2. \tag{26}$$

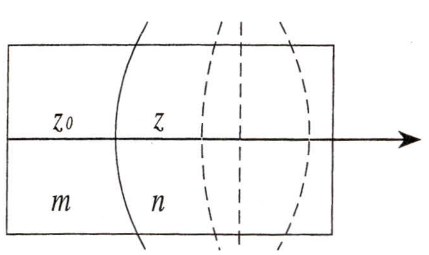

Fig. 4. Example of a pure fractional Fourier transformer composed of two media, m and n, separated by a spherical interface. A movable, warpable mirror is indicated by dashed lines. The distance z_0 is fixed, while for the mirror $z = n\chi > 0$ is adjustable.

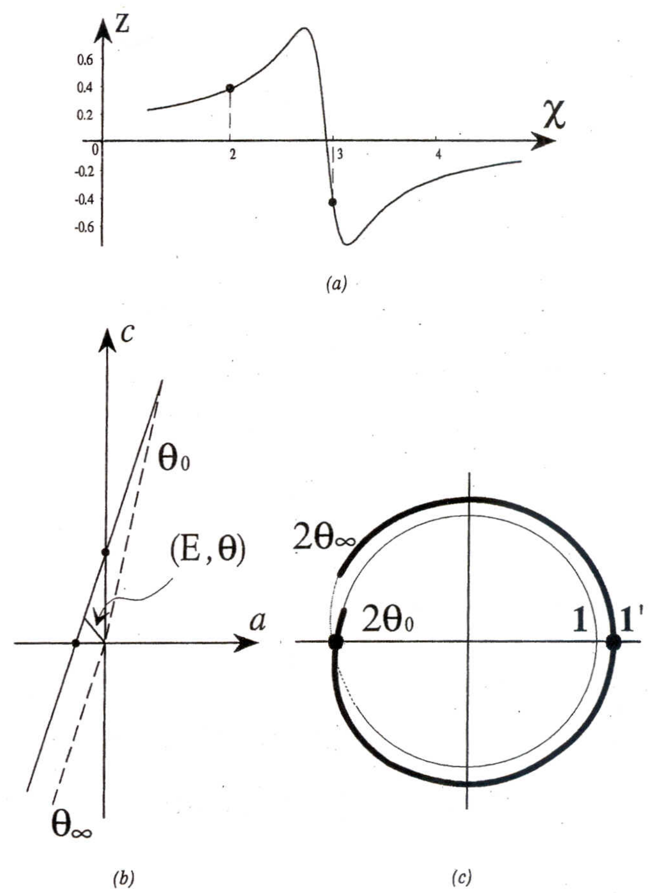

Fig. 5. (a) Mirror warp coefficient $Z(\chi)$ as a function of $\chi = \frac{2}{3}z > 0$ in the Fourier transformer of the Fig. 4. Bullets mark $\chi = 2$, where the one-pass system is an impure Fourier transformer, and $\chi = 3$ where it is an impure inverting imager. (b) Locus of the one-pass systems in the (a, c) plane of matrix elements. (c) Fourier–Iwasawa angle θ counted modulo 4π. The heavy curve shows the range of fractional Fourier transform angles of these systems; the range includes the second metaplectic unit **1**′ but not the identity **1**.

As we move the mirror along z, we should warp its surface to $z = Zq^2$ with $Z = \frac{1}{3}G$ given by Eq. (26), to eliminate the Gaussian factor of the Iwasawa decomposition; the function $Z(\chi)$ is shown in Fig. 5(a). When the angle in the Iwasawa decomposition of $\mathbf{M}$ is θ (an impure Fourier transformer with slant), the system cum reflection $\mathbf{M}^{\mathrm{II}}_{n,\mathbf{Z}}$ will be a pure Fourier transformer of angle 2θ (i.e., of power $4\theta/\pi$). The range of this "Fourier–Iwasawa" angle $\theta(\chi)$ of $\mathbf{M}$ is shown in Fig. 5(b) by the line $[a(\chi), c(\chi)]$ in polar coordinates, with (E, θ) given by Eqs. (25) and (24). The slope of the line is 4, and at the lower limit of the parameter $\chi = 0$, the angle is $\theta(0) = \arctan 6 \approx 80.54°$. The "one-pass" system $\mathbf{M}(z)$ is an impure Fourier transformer at $\theta(2) = \frac{1}{2}\pi$ and an impure imager at $\theta(3) = \pi$. Beyond, the line $[a(\chi), c(\chi)]$ is in the third quadrant; and as $\chi \to \infty$, the Fourier angle becomes $\theta(\infty) = \pi + \arctan 4 \approx 255.96°$.

Paraxially pure fractional Fourier transformers obtained from this system cum reflection thus have a range of angles 2θ between $\approx 162°$ and $\approx 512° \equiv 152°$, an extent of almost 360° as shown in Fig. 5(c). It is useful to keep count of the angle beyond 360° because the symplectic group is doubly covered by the metaplectic group in paraxial wave optics (see Refs. 5 and 6 and references therein). The Fourier transformer cum reflection at $\chi = 2$ is a pure inverting imager, $2\theta(2) = \pi$; and an impure inverting imager cum reflection is a pure upright imager at $2\theta(3) = 2\pi$. As far as geometric optics is concerned, this angle is equivalent to 0 and the latter system is equivalent to the trivial unit system; but in wave optics it is the "second metaplectic unit" at 2π, where wave fields are multiplied by -1.

6. ON REFLECTION AND CORRECTION OF ABERRATIONS

To complement the study of reflection, we examine briefly its action on the aberrations of axisymmetric optical systems in the context of the factored-product expansion of canonical transformations.[12–16,19] (We should warn the reader that we work with Hamilton–Lie aberration coefficients rather than with the traditional Seidel coefficients.[20]) In the metaxial regime of aberration order $2k - 1$ (and rank $k = 2, 3, \ldots$), optical systems are realized by operators

$$\mathcal{G}\{\mathbf{A};\mathbf{M}\}_{(k)} := \mathcal{G}\{\text{aberration part; paraxial part}\}, \tag{27}$$

$$= \exp\{A_k, \circ\} \times \cdots \times \exp\{A_3, \circ\} \times \exp\{A_2, \circ\} \times \mathcal{G}(\mathbf{M}),$$

where $\mathcal{G}(\mathbf{M})$ is the paraxial part seen in Section 3 and the aberration part is factored into exponentiated Poisson–Lie operators $\{A_j, \circ\} := (\partial_{\mathbf{q}} A_j) \cdot \partial_{\mathbf{p}} - (\partial_{\mathbf{p}} A_j) \cdot \partial_{\mathbf{q}}$, of functions $A_j(\mathbf{p}, \mathbf{q})$ of homogeneous degree $2j$ in the coordinates of phase space, which are also polynomials of degree j in the axisymmetric monomials $|\mathbf{p}|^2$, $\mathbf{p} \cdot \mathbf{q}$, and $|\mathbf{q}|^2$. When these operators are concatenated, the factored-product series can be consistently truncated to rank k; one thus obtains a finite-parameter group of nonlinear canonical transformations of phase space up to the corresponding degree.[21]

The paraxial part can also be put in exponentiated Poisson–Lie form, with a linear polynomial in the axisymmetric monomials. In particular, $\exp\{\frac{1}{2}z|\mathbf{p}|^2, \circ\}$ generates paraxial free flight whose matrices are Eq. (14), $\exp\{\frac{1}{2}G|\mathbf{q}|^2, \circ\}$ does the same for the Gaussian matrices Eq. (15) of refraction, and $\exp\{\alpha\mathbf{p} \cdot \mathbf{q}, \circ\}$ generates pure magnification corresponding to the diagonal matrices $\operatorname{diag}[\exp(\alpha), \exp(-\alpha)]$. The first two are invariant under reflection, while the last changes the sign of α. It is surmised thus that under reflection, optical system (27) maps such that its factors undergo

$$\exp\{A_j, \circ\} \mapsto \overline{\exp\{A_j, \circ\}} = \exp\{\bar{A}_j, \circ\},$$

$$\bar{A}_j(|\mathbf{p}|^2, \mathbf{p} \cdot \mathbf{q}, |\mathbf{q}|^2) = A_j(|\mathbf{p}|^2, -\mathbf{p} \cdot \mathbf{q}, |\mathbf{q}|^2), \tag{28}$$

and multiply in the inverse order. The action of reflection given by Eqs. (28) is consistent with the use of the nonsymplectic matrix $\mathbf{K}$ in Eq. (17); the reflection of system (27) can be written (for any rank k) also as

$$\overline{\mathcal{G}\{\mathbf{A}; \mathbf{M}\}} = \mathcal{G}\{\mathbf{0}; \mathbf{K}\}\mathcal{G}\{-\mathbf{A}; \mathbf{M}\}^{-1}\mathcal{G}\{\mathbf{0}; \mathbf{K}\}. \tag{29}$$

The necessity of the minus sign in front of the aberration part in Eq. (29) derives from the reflection invariance of metaxial free flight (with its spherical aberration coefficients). Using the symbolic computation program mexLIE,[22,23] we have verified that the ensuing reflection of the root transformation indeed reproduces Eq. (7) to aberration order 7. This minus sign is important because it negates any possible homomorphism between reflection and inversion beyond the paraxial regime.

The reflection map [expressions (28)] classifies aberrations by their parity; pure comas and distortions[19] exhibit odd (negative) parity. Hence when the system cum reflection has a paraxial part $\mathbf{M}^{\mathrm{II}} = \pm\mathbf{1}$, its third- and fifth-order comas and distortions vanish. Beyond, the compounding of the lower-order even aberrations will yield generally nonzero values to the higher-order odd ones; this is an inevitable artifact of any factored-product expansion. The authors have verified that fractional Fourier transformers, when built as systems cum reflection, can be corrected beyond the paraxial regime by warping the mirror to a polynomial shape $\zeta(\mathbf{q}) = \zeta_2|\mathbf{q}|^2 + \zeta_4|\mathbf{q}|^4 + \cdots$. We have found that one parameter ζ_{2j} at each aberration order allows the iterative correction of one or a linear combination of aberration coefficients, in essentially the same way as in a previous paper.[24] We do not detail this analysis further here because it lacks the immediacy of our results in the paraxial approximation.

ACKNOWLEDGMENTS

The authors thank R. Simon of The Institute of Mathematical Sciences, Chennai, India, for the key remark on the matrix $\mathbf{K}$ during his stay at the Centro Internacional de Ciencias, Cuernavaca. We acknowledge the support of Direccíon General de Asuntos del Personal Académico–Universidad Nacional Autonoma de Mexíco project IN-112300 Optica Matemática.

K. B. Wolf may be reached by e-mail at bwolf @fis.unam.mx

REFERENCES AND NOTES

1. D. Mendlovic and H. M. Ozaktas, "Fourier transforms of fractional order and their optical implementation," J. Opt. Soc. Am. A **10**, 1875–1881 (1993).
2. H. M. Ozaktas, Z. Zalevsky, and M. Alper Kutay, *The Fractional Fourier Transform* (Wiley, Chichester, UK, 2001).
3. M. Navarro-Saad and K. B. Wolf, "Factorization of the phase-space transformation produced by an arbitrary refracting surface," J. Opt. Soc. Am. A **3**, 340–346 (1986).
4. K. B. Wolf and G. Krötzsch, "La transformación raíz de superficies refractantes y espejos," Rev. Mex. Fís. **37**, 540–554 (1991).
5. R. Simon and K. B. Wolf, "Structure of the set of paraxial optical systems," J. Opt. Soc. Am. A **17**, 342–355 (2000).
6. R. Simon and K. B. Wolf, "Fractional Fourier transforms in two dimensions," J. Opt. Soc. Am. A **17**, 2368–2381 (2000).
7. M. Moshinsky, "Canonical transformations in quantum mechanics," SIAM J. Appl. Math. **25**, 193–212 (1973).
8. H. M. Ozaktas and D. Mendlovic, "Fractional Fourier transform as a tool for analyzing beam propagation and spherical mirror resonators," Opt. Lett. **19**, 1678–1680 (1985).
9. H. M. Ozaktas and D. Mendlovic, "Fractional Fourier optics," J. Opt. Soc. Am. A **12**, 743–751 (1995).
10. E. J. Atzema, G. Krötzsch and K. B. Wolf, "Canonical transformations to warped surfaces: correction of aberrated optical images," J. Phys. A **30**, 5793–5803 (1997).
11. R. Simon, "Peres–Hodorecki separability criterion for continuous variable systems," Phys. Rev. Lett. **84**, 2726–2729 (2000).
12. A. J. Dragt, "Lie algebraic theory of geometric optics and optical aberrations," J. Opt. Soc. Am. **72**, 372–379 (1982).
13. S. Steinberg, "Lie series, Lie transformations, and their applications" in *Lie Methods in Optics*, J. Sánchez-Mondragón and K. B. Wolf, eds., Vol. 250 of Lecture Notes in Physics, (Springer Verlag, Heidelberg, Germany 1986), Chap. 3, pp. 45–102.
14. A. J. Dragt, E. Forest, and K. B. Wolf, "Foundations of a Lie algebraic theory of geometrical optics," in *Lie Methods in Optics*, J. Sánchez-Mondragón and K. B. Wolf, eds., Vol. 250 of Lecture Notes in Physics, (Springer Verlag, Heidelberg, Germany 1986), Chap. 4, pp. 105–158.
15. O. Castaños, E. López Moreno and K. B. Wolf, "Canonical transforms for paraxial wave optics," in *Lie Methods in Optics*, J. Sánchez-Mondragón and K. B. Wolf, eds., Vol. 250 of Lecture Notes in Physics, (Springer Verlag, Heidelberg, Germany 1986), Chap. 5, pp. 159–182.
16. A. J. Dragt, "Elementary and advanced Lie algebraic methods with applications to accelerator design, electron microscopes, and light optics," Nucl. Instrum. Methods Phys. Res. A **258**, 339–354 (1987).
17. E. C. G. Sudarshan, N. Mukunda, and R. Simon, "Realization of first-order optical systems using thin lenses," Opt. Acta **32**, 855–872 (1985).
18. K. B. Wolf and G. Krötzsch, "El problema de las tres lentes," Rev. Mex. Fís. **47**, 291–298 (2001).
19. K. B. Wolf, "Symmetry-adapted classification of aberrations," J. Opt. Soc. Am. A **5**, 1226–1232 (1988).
20. See, e.g., H. A. Buchdahl, *Optical Aberration Coefficients* (Dover, New York, 1968).
21. K. B. Wolf, "Nonlinearity in aberration optics," in *Symmetries and Non-linear Phenomena*, Proceedings of the International School on Applied Mathematics, Centro Internacional de Física, Paipa, Colombia, D. Levi and P. Winternitz eds., (World Scientific, Singapore, 1988), pp. 376–429.
22. K. B. Wolf and G. Krötzsch, "Group-classified polynomials of phase space in higher-order aberration expansions," J. Symb. Comput. **12**, 673–695 (1991).
23. K. B. Wolf and G. Krötzsch, "MEXLIE 2, A set of symbolic computation functions for geometric aberration optics," Manuales IIMAS-UNAM No. 10 (Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, Cuernavaca, Morelos 62251, México, 1995).
24. K. B. Wolf and G. Krötzsch, "Metaxial correction of fractional Fourier transformers," J. Opt. Soc. Am. A **16**, 821–830 (1999).