JOSA COMMUNICATIONS

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Finite mode analysis through harmonic waveguides

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The mode analysis of signals in a multimodal shallow harmonic waveguide whose eigenfrequencies are equally spaced and finite can be performed by an optoelectronic device, of which the optical part uses the guide to sample the wave field at a number of sensors along its axis and the electronic part computes their fast Fourier transform. We illustrate this process with the Kravchuk transform. © 2000 Optical Society of America [S0740-3232(00)00408-7]

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The analysis of a signal into its orthogonal component modes is a well-known task when the basis of the modes is the set of oscillating exponential functions, which is called Fourier analysis. In this case, an efficient numerical algorithm (the fast Fourier transform) and paraxial optical setups exist that perform the harmonic analysis satisfactorily on digital and continuous analog signals, respectively. When the mode basis is different from these harmonics, the digital or analog analysis becomes less efficient. For the numerical (i.e., digital) evaluation of the discretized Hankel transform, see Ref. 1 and references therein. The task to obtain the Hermite–Gauss and Laguerre–Gauss coefficients of continuous signals was considered recently in Ref. 2.

The question of how to discretize the fractional Fourier transform has brought to the fore several discrete and finite function (or vector) bases that purport to approximate the quantum harmonic-oscillator Hermite functions, since the latter self-reproduce under the fractional Fourier integral transform. The mode analysis of N-point signals in discrete bases is arduous because the transforming matrix elements do not have the simple product–sum property of the oscillating exponentials. There appear more complicated functions,[3–6] including polynomials,[7] whose algorithmic efficiency (with a preconstructed matrix) is $\sim\mathcal{O}(N^2)$.

In this communication we propose an ideal optoelectronic device[8]: a multimodal waveguide with wave-field sensors on a line along the guide, to perform discrete non-Fourier mode analysis in conjunction with the fast-Fourier-transform algorithm. The waveguide must be *harmonic*: an optical medium characterized by a self-adjoint evolution Hamiltonian H, with discrete modes whose eigenfrequencies (eigenenergies) are equally spaced and have a lower bound. Its eigenmodes $\boldsymbol{\phi}_m$ correspond to the eigenvalues $\sim m+\text{constant}$, for $m = 0, 1, 2, \ldots$. In realistic optics, moreover, m will have an upper bound N, so we need not search for more than $N+1$ coefficients, and measurements need not be done with more than $N+1$ field sensors.

Let $\phi_m(n)$ be the (complex) value of the mth mode measured at the nth sensor of a line array across the waveguide—a discrete screen at $z=0$. The sensors must naturally be finite in number and placed within the bounds of the physical waveguide; if the waveguide is symmetric under reflections across the z axis, it is convenient to number them as $n \in \{-\frac{1}{2}N, -\frac{1}{2}N + 1, \ldots, \frac{1}{2}N\}$. We build $\boldsymbol{\phi}_m$ as a column vector of $N+1$ components, which satisfies the orthogonality and completeness properties of an orthonormal basis,

$$\boldsymbol{\phi}_m^{\dagger}\boldsymbol{\phi}_{m'} = \delta_{m,m'}, \qquad \sum_{n=-N/2}^{N/2} \phi_m(n)^* \phi_{m'}(n) = \delta_{m,m'}, \quad (1)$$

$$\sum_{m=0}^{N} \boldsymbol{\phi}_m \boldsymbol{\phi}_m^{\dagger} = \mathbf{1}, \quad \sum_{m=0}^{N} \phi_m(n)^* \phi_m(n') = \delta_{n,n'}, \quad (2)$$

where the dagger indicates adjunction, the asterisk represents complex conjugation, and $\mathbf{1}$ is the unit matrix. This eigenbasis of the harmonic waveguide Hamiltonian is invariant (but for phases) under translations by z along the guide,

$$\begin{aligned} H\boldsymbol{\phi}_m = (m+\kappa)\boldsymbol{\phi}_m \Rightarrow \boldsymbol{\phi}_m(z) &= \exp(-izH)\boldsymbol{\phi}_m \\ &= \exp(-iz\kappa)\exp(-izm)\boldsymbol{\phi}_m, \end{aligned} \quad (3)$$

where κ is the space frequency of the ground mode $\boldsymbol{\phi}_0$ along the z axis.

Thus let $\{f(n)\}_{-N/2}^{N/2}$ be an arbitrary signal $\mathbf{f}$ measured at the $z=0$ screen. This signal can be analyzed into, and synthesized from, the mode eigenbasis with waveguide coefficients, denoted $\{f_m^w\}_0^N$, by means of a unitary waveguide-transform matrix $\mathbf{W} = \|\phi_m(n)^*\|$, in the following way:

$$\mathbf{f}^{w} = \mathbf{W}\mathbf{f}, \qquad f^{w}_{m} = \sum_{n=N/2}^{N/2} \phi_m(n)^* f(n), \tag{4}$$

$$\mathbf{f} = \mathbf{W}^{\dagger}\mathbf{f}^{w}, \qquad f(n) = \sum_{m=0}^{N} \phi_m(n) f^{w}_{m}. \tag{5}$$

The task is to find the waveguide coefficients $\{f^{w}_{m}\}_0^N$ without having to compute the $(N+1)^2$ products of the components of $\mathbf{f}$ with the elements of the transform matrix $\mathbf{W}$, i.e., to find a faster transform.

The optical part of our ideal apparatus will perform what we can call the fractional Fourier waveguide transform $\mathbf{\Phi}^z$ by propagation along the waveguide axis. First, we note that, owing to Eq. (3), a signal $\mathbf{f}$ in the guide will evolve to $\mathbf{f}(z)$ at distance z, and this is

$$\mathbf{f}(z) = \mathbf{\Phi}^z \mathbf{f},$$

$$f(n,z) = \exp(-iz\kappa)\sum_{m=0}^{N} \exp(-imz)\phi_m(n) f^{w}_{m}$$

$$= \exp(-iz\kappa)\sum_{n'=-N/2}^{N/2} \Phi^z(n,n') f(n'), \tag{6}$$

$$\mathbf{\Phi}^z = \|\Phi^z(n,n')\|,$$

$$\Phi^z(n,n') = \sum_{m=0}^{N} \phi_m(n)\exp(-imz)\phi_m(n')^*. \tag{7}$$

It has the desirable generic properties of group composition, $\mathbf{\Phi}^{z_1}\mathbf{\Phi}^{z_2} = \mathbf{\Phi}^{z_1+z_2}$ modulo 2π (i.e., $\mathbf{\Phi}^{2\pi} = \mathbf{\Phi}^0 = \mathbf{1}$), and unitarity, $(\mathbf{\Phi}^z)^{-1} = \mathbf{\Phi}^{-z} = (\mathbf{\Phi}^z)^\dagger$.

Next, observe that the middle expression of Eq. (6) is also the ordinary finite Fourier transform of $\phi_m(n) f^{w}_{m}$ at the $N+1$ equidistant screens,

$$z_k = 2\pi k/(N+1), \qquad k = 1, 2, \ldots, N+1. \tag{8}$$

Finally, we divide (if nonzero) by $\phi_m(n)$ to obtain the waveguide coefficients,

$$f^{w}_{m} = \frac{\exp(iz_k\kappa)}{\phi_m(n)}\sum_{k=0}^{N} \exp[2\pi imk/(N+1)] f(n, z_k). \tag{9}$$

And so the result of the previous waveguide computation is now subject to the fast-Fourier-transform algorithm. The divisions in Eq. (9) require $N+1$ operations, which add to the $\sim N\log_2 N$ operations of the digital algorithm; the parallel optical part could be considered costless, since it occurs at the speed of light in the medium.

Thus, for every chosen n, a set of sensors placed on $N+1$ screens at $z = z_k$, $k = 1, 2, \ldots, N+1$ along the waveguide—with the same position n—yield the waveguide coefficients f^{W}_{m}, as was the objective. Only when the number of sensors is odd (and thus $n = 0$ is among them) shall we meet the impediment $\phi_m(0) = 0$ for all odd m. Experimentally, the values of $\{\phi_m(n)\}_{m=0}^N$ could be measured by successively exciting the $N+1$ eigenmodes of the waveguide.

We were led to this general strategy for discrete and finite sensor arrays following the continuous and infinite mode analysis of Ref. 2 because the finitely sampled harmonic-oscillator wave functions are not orthogonal, as is well known, and because several finite orthonormal bases that broadly resemble the Hermite functions require a fair amount of numerical work.[4,6]

In Ref. 7 Atakishiyev and Wolf proposed the Kravchuk and the fractional Fourier–Kravchuk transforms, which are based on a group-theoretical analysis of the harmonic-oscillator Newton equation.[9] The Kravchuk functions are proper functions of a continuous argument that form a finite orthonormal eigenbasis of a difference-operator Hamiltonian[10]; they contain the Kravchuk polynomials, which are orthogonal with respect to the binomial distribution on $N+1$ points[11]; in a well-defined limit $N \to \infty$ they converge pointwise to the Hermite functions. When this interpolating wave field advances along the waveguide, it undergoes the fractional Fourier–Kravchuk transform; this has a claim to provide a physically meaningful approximation to real waveguides because it correctly reproduces the behavior of its coherent states.[12,13] The analysis involves the ordinary rotation group, and the Kravchuk functions are classified according to spin $l = \frac{1}{2}N$. This Kravchuk basis, and its finite Fourier transform, have been used recently to study the phase-space picture of discrete, finite quantum mechanics, through a proper discrete (action-angle) Wigner function.[14]

The elements of the irreducible unitary representation matrices $\mathbf{d}^l(\theta) = \|d^l_{\mu,\mu'}(\theta)\|$ of rotations around the 2-axis by θ satisfy composition and there hold discrete orthonormality and completeness relations,

$$\sum_{\mu'=-l}^{l} d^l_{\mu,\mu'}(\theta)^* d^l_{\mu'',\mu'}(\theta) = d^l_{\mu,\mu''}(0) = \delta_{\mu,\mu''}. \tag{10}$$

These are the well-known Wigner little d's [Ref. 15, Eq. (3.65)], which naturally match the orthogonality and completeness relations Eqs. (1) and (2) for $N = 2l$. For $\theta = \frac{1}{2}\pi$, they are the (real, symmetric) Kravchuk functions defined by Refs. 7 and 12,

$$\phi_m(n) = (-1)^{l-m} d^l_{n,l-m}\left(\frac{1}{2}\pi\right)$$

$$= \frac{1}{2^{l-m}}\left[\binom{2l}{l+n} \bigg/ \binom{2l}{m}\right]^{1/2} k_m(l+n, 2l), \tag{11}$$

$$k_m(x, 2l) = \frac{(-1)^m}{2^m}\binom{2l}{m}\, {}_2F_1(-m, -x; -2l; 2), \tag{12}$$

where $k_m(x, 2l)$ are the Kravchuk polynomials.[11] When n is taken to be a continuous variable denoted x, the Kravchuk functions [Eq. (11)] are well defined in the interval $-N-1 \leqslant x \leqslant N+1$, with branch-point zeros at the ends. The Kravchuk transform is performed with the matrix $\mathbf{W} = \|\phi_m(n)\|$; since its elements are real, this Kravchuk matrix is orthogonal.

The optoelectronic device we describe here will find the Kravchuk mode coefficients [Eq. (4)]. The optical part of the device propagates the signal along the axis of the waveguide, where it undergoes a fractional Fourier–Kravchuk transform with the generating function [Eq. (7)]. [This Kravchuk kernel has in fact a closed expression in terms of a nonsymmetric Kravchuk polynomial and trigonometric functions that we need not write here; see Ref. 7, Eq. (5.3).] The wave field is sampled along the line of sensors n and fed into the electronic part, which performs the fast Fourier transform. We recall that the Kravchuk transform can be performed with the Feinsilver–Schott algorithm,[12,16] which is disarmingly simple but not fast ($\sim N^3$) and also through matrix multiplication ($\sim N^2$) with closed analytic expressions. Or it can be performed with the ideal optoelectronic device built here, where the optical component is costless and the electronic algorithm is fast ($\sim N \log_2 N$).

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