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The Wigner Function for General Lie Groups and the Wavelet Transform

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Abstract. We build Wigner maps, functions and operators on general phase spaces arising from a class of Lie groups, including non-unimodular groups (such as the affine group). The phase spaces are coadjoint orbits in the dual of the Lie algebra of these groups and they come equipped with natural symplectic structures and Liouville-type invariant measures. When the group admits square-integrable representations, we present a very general construction of a Wigner function which enjoys all the desirable properties, including full covariance and reconstruction formulae. We study in detail the case of the affine group on the line. In particular, we put into focus the close connection between the well-known wavelet transform and the Wigner function on such groups.

I Introduction: the original Wigner function

The notion of a quantum-mechanical phase space, where the evolution of a state can be described by a (quasi-)probability distribution function over classical ('c-number') coordinates of position and momentum $(q, p) \in \mathbb{R}^2$, hinges on the function introduced by Wigner [25] in 1932. Given two wavefunctions $\phi(x)$ and $\psi(x)$, of the space coordinate $x \in \mathbb{R}$, their Wigner function on phase space (q, p) is

$$W^{\mathrm{QM}}(\phi, \psi|q, p; h) = \frac{1}{h}\int_{\mathbb{R}} dx\, \overline{\phi(q - \tfrac{1}{2}x)}\, e^{-2\pi i\, xp/h}\, \psi(q + \tfrac{1}{2}x). \tag{1}$$

Here h is the Planck constant, which fixes the scale (and units) of the two classical coordinates, incorporating the uncertainty principle. When $\phi = \psi$, we write $W^{\mathrm{QM}}(\psi|q, p; h) = W^{\mathrm{QM}}(\psi, \psi|q, p; h)$ and speak of the Wigner function of the state ψ on phase space. While phase-space probability functions for classical systems are non-negative but otherwise arbitrary, a Wigner function is much more restrictive and (except for Gaussian functions) *does* have (usually small) regions where its values are negative. For this reason it is properly called a *quasi*- probability distribution function and named *the Wigner function* for short. Nevertheless, issues of quantum measurement can be discussed adequately in terms of the Wigner function, and it serves well in formulating a picture of quantum mechanics at par with the Schrödinger, Heisenberg and Feynman formulations [16]. More recently, it has served as a fine tool for quantum optics [1], [19], since using it *density matrices* (*i.e.*, 'impure states', or positive trace-class Hilbert-Schmidt operators) can also be effectively plotted in terms of position-momentum or action-angle phase spaces.

The analysis of signals in time and frequency is crucial for radar technology, so quite early analogues of the Wigner function, such as the Woodward self-ambiguity function [28], were extensively studied and applied [23], [8]; in this case, the unit of phase space area h in (1), is set equal to one. Also in the context of monochromatic, paraxial wave optics, the Wigner function is easily used for the analysis of one-dimensional signals such as a code bar [6], since a simple lens arrangement will produce on a screen an intensity field which is closely related to its Wigner function [18]. The two canonically conjugate coordinates in this case are position on the screen and spatial frequency (optical momentum), with the fundamental unit of phase space area being now λ, the wavelength of the light. Recently, the group-theoretical study of various optical and quantum mechanical models, where the observable quantities of a system are the eigenvalues of the generators of a Lie group [27], [21], [5], has suggested a substantial generalization of the concepts of Wigner functions and of phase space.

In this article we develop a fairly rigorous group-theoretic foundation for the study of Wigner functions for a class of Lie groups which admit square-integrable, unitary irreducible representations. Typically these groups are related to the underlying symmetry and dynamics of the physical system, and incorporate its specific geometry. The group has a natural action on the dual space of its Lie algebra, called the *coadjoint action*, and the orbits of this action are possible phase spaces of the system. These *coadjoint orbits* carry natural symplectic structures which make them geometrically similar to classical phase spaces. They carry invariant measures under the group action, which are analogues of the well-known Liouville measure on ordinary phase space. The original Wigner function (1) arises from a square-integrable representation of the *Heisenberg-Weyl* group —the group of the canonical commutation relations. This has pointed the direction to follow toward a generalization of the notion of the Wigner function to other Lie groups [27].

As an application of the general theory, we treat in detail the affine group, whose Wigner function is shown to be closely related to the well-known wavelet transform. This two-parameter group is *non-unimodular*, *i.e.*, its left- and right-invariant Haar measures are distinct. The affine group, being non-unimodular, makes it imperative that the mathematical properties of square-integrable representations be used in order to arrive at an adequate generalization of the Wigner phase space formalism.

In Section II we present a first approximation to the generalization of Wigner's phase space formalism, developed in Refs. [27], [21], and [5], to display the desirable properties of a phase space function. To make this paper self contained, the following three Sections organize the required mathematical fundamentals and thus the notation. Section III recalls the exponential map between a Lie algebra and group, to define the coadjoint action of the group on the dual of the algebra, providing the 'c-number' arguments of the Wigner function, such as those on the right-hand side of Eq. (1). Section IV looks at various natural representations of the group, associated to the adjoint and coadjoint actions. Finally, Section V discusses the class of group representations for which Wigner functions can be built and presents the objects whose Wigner functions are to be determined; these are Hilbert-Schmidt operators ('density matrices'), including 'pure state' wavefunctions.

In Section VI we define the Wigner map and function of Hilbert-Schmidt operators on the Hilbert space carrying the group representation, and look at their properties. In particular, we obtain the formula for the reconstruction of the density matrix from its Wigner function. In Section VII we explain the relationship between the Wigner transform, coherent states of the group and the generalized wavelet transform. Finally, some of the key expressions are written down in coordinate terms, for comparison with known formulae. The specific case of the two-dimensional affine group is examined in Section VIII and its Wigner transform is calculated there as an explicit example and application. In Section IX, we display for the affine group the exact relation between the Wigner map (the Wigner function with one fixed 'mother' wavelet) and the wavelet transform. Section X is devoted to showing how the original Wigner function in (1) can be derived using the general theory as applied to the Heisenberg-Weyl group. We conclude in Section XI with some general comments and indicate some directions for further research.

The Wigner function does not contain *more* information than the original signal or wavefunction. But, in the same way that a musical score shows through notes on the pentagram the essence of a tune better than a complete pressure-wave register of a performance, the information is presented in a form kinder to human comprehension.

II The Wigner operator: a first generalization

In Reference [27] it was proposed that the Wigner function (1) of ϕ, ψ, or of the density matrix $\rho = |\psi\rangle\langle\phi|$, be written as the matrix element of a 'Wigner operator' $\mathcal{W}$, which is a (measurable, operator-valued) function on phase space,

$$W(\phi, \psi \mid X^*) = \langle \phi \mid \mathcal{W}(X^*) \mid \psi \rangle, \qquad W(\rho \mid X^*) = \mathrm{Tr}[\rho\, \mathcal{W}(X^*)], \tag{2}$$

where X^* is an element of the dual of the Lie algebra whose components in a chosen basis provide the 'c-number' arguments of the Wigner function.

II.1 The Wigner operator

The Wigner operator is, roughly speaking, the Fourier transform of the (Hilbert space representation of) the group elements $g \in G$, previously written in terms of the coordinates $\vec{\xi} = \{\xi_i\}_{i=1}^n$ of X^* as

$$\mathcal{W}(\vec{\xi}) = \int_{R_G} d\mu(g[\vec{x}\,])\, e^{-i\vec{\xi}\cdot\vec{x}}\, g[\vec{x}\,], \qquad g[\vec{x}\,] = \exp(i\vec{x} \cdot \vec{X}\,), \tag{3}$$

where $\vec{X} = \{X_n\}_{n=1}^N$ is a basis of generators (consisting of Hilbert space operators) for the Lie algebra and the square brackets indicate that the arguments are the *polar* coordinates of the group, with domain $\vec{x} \in R_G \subset \mathbb{R}^N$. The measure $d\mu(g)$ is the left invariant Haar measure (also the right invariant Haar measure, since in this expression the group had been assumed to be unimodular). For the Heisenberg-Weyl algebra [26], Eqs. (2) and (3) yield the Wigner function (1), with the common Schrödinger operators $\vec{X} = \{Q, P, \Lambda\}$ and $\Lambda = h\mathbf{1}$, in the irreducible representation labelled by $h \in \mathbb{R} - \{0\}$; the corresponding c-number coordinates are $\vec{\xi} = \{q, p, h\}$. With abuse of notation [24] one may think of this operator as $(2\pi)^3\, \delta(Q-q)\, \delta(P-p)\, \delta(\Lambda - h)$.

The following subsections highlight the basic properties of the construction (2)–(3) which show that it is a proper generalization of the original Wigner function of quantum mechanics [16], [14].

II.2 Sesquilinearity and reality

In a unitary representation of the group, the Wigner operator (3) is (formally) self-adjoint. Hence the Wigner functions (1) and (2) are sesquilinear in their Hilbert space arguments, *i.e.*,

$$\begin{aligned} W(\phi, a\psi_1 + b\psi_2 \mid X^*) &= aW(\phi, \psi_1 \mid X^*) + bW(\phi, \psi_2 \mid X^*), &(4)\\ W(\psi, \phi \mid X^*) &= \overline{W(\phi, \psi \mid X^*)}. &(5) \end{aligned}$$

It follows that the Wigner function of a single wave function is real, $W(\psi \mid X^*) = W(\psi, \psi \mid X^*) \in \mathbb{R}$. For a sum of functions (such as Schrödinger-cat states),

$$W(\phi + \psi \mid X^*) = W(\phi \mid X^*) + W(\psi \mid X^*) + 2\,\mathrm{Re}\, W(\phi, \psi \mid X^*). \tag{6}$$

There is *holographic* information in the interference cross-term.

II.3 Covariance

Translations and linear transformations of classical phase space $X^* = (q, p) \mapsto \mathbf{T}(q, p)$, are equivalent to the canonical transforms $\mathcal{T}$ of the wavefunctions, for the original Wigner function [12]. More generally [5], under an automorphism group of the Lie algebra, the form of (3) implies the covariance property

$$W(\psi \mid \mathbf{T}X^*) = W(\mathcal{T}\psi \mid X^*). \tag{7}$$

This formula is important because it ensures the correspondence between classical and quantum phase spaces.

II.4 Overlap and reconstruction

From the Wigner function one can recover the wavefunction (up to an overall phase) or the density matrix by exploiting the overlap and reconstruction relations.

For the traditional case of the Heisenberg-Weyl algebra (1) the overlap condition is

$$\iint_{\mathbb{R}^2} dq\, dp\, W\,(\phi, \psi \mid q, p)\, W\,(v, \omega \mid q, p) = \langle \phi \mid \omega \rangle \,\langle v \mid \psi \rangle. \tag{8}$$

This formula is important because it shows the passage from the Wigner to the Schrödinger or Heisenberg formalisms of quantum mechanics, and holds correspondingly between the Wigner function and wavefunctions on Lie groups. From Eq. (8) also follow various properties of *marginal distributions*, or projections over some of the phase-space coordinates, and of moments [16], e.g.,

$$\int_{\mathbb{R}} dp\, W\,(\phi, \psi \mid q, p) = \overline{\phi(q)}\, \psi(q). \tag{9}$$

III Lie groups, Lie algebras and their duals

In this and the next two sections we collect some relevant notions and results on Lie groups and some of their representations. While most of the concepts and results presented here are known, especially to people working in representation theory, they may not be generally familiar —at least not in the setting in which we need them. Moreover, since the terminology for analogous concepts in quantum optics is often quite different, we go into some detail to explain the mathematical notions and terminology, particularly since the Wigner function has such wide applications in wave and quantum optics.

III.1 Exponential map; adjoint action

Let G be a Lie group generated by a Lie algebra $\mathfrak{g}$. The Lie algebra is a linear vector space, and the *exponential* map from $\mathfrak{g}$ to G,

$$\exp X = g, \qquad X \in N_0 \subset \mathfrak{g}, \quad g \in V_e \subset G, \tag{10}$$

is a topological *homeomorphism* between an open set N_0 around the origin 0 in $\mathfrak{g}$, and an open set V_e around the identity element e in G. We denote the inverse map from G to $\mathfrak{g}$ by

$$\log g = X, \qquad X \in N_0 \subset \mathfrak{g}, \quad g \in V_e \subset G. \tag{11}$$

It is well known [15] that the neighbourhood V_e in G can be chosen to be symmetric, *i.e.*, if $g \in V_e$, then also $g^{-1} \in V_e$. In this paper we shall also assume that V_e can be chosen to be an open dense set in G, such that its complement has (Haar) measure zero. (A group such as $\mathrm{Sp}(2n, \mathbb{R})$ does not have this property.)

There is a natural action of the group G on its Lie algebra $\mathfrak{g}$, called the *adjoint action*, Ad. For $g, g_o \in V_e \subset G$ such that $g_o\, g\, g_o^{-1} \in V_e$, this action maps $X \in \mathfrak{g}$ to $Y = \mathrm{Ad}_{g_o} X \in \mathfrak{g}$ through

$$\exp(\mathrm{Ad}_{g_o} X) = \exp Y = g_o(\exp X) g_o^{-1}. \tag{12}$$

If G is a matrix group, so that $\mathfrak{g}$ also consists of matrices, then

$$\mathrm{Ad}_{g_o} X = g_o\, X\, g_o^{-1}. \tag{13}$$

III.2 Dual space, coadjoint action

The Lie algebra $\mathfrak{g}$ is a vector space, and hence has a *dual* denoted $\mathfrak{g}^*$, also a vector space, consisting of all linear functionals on $\mathfrak{g}$. Let $\langle X^*; X\rangle$ denote the dual paring between $X^* \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. In other words, $\langle X^*; X\rangle$ is simply the value $X^*(X)$ of the functional X^*, computed on the vector X. If we introduce a basis $\{X_i\}_{i=1}^n$ in $\mathfrak{g}$ (n being its dimension), and the dual basis $\{X^{*i}\}_{i=1}^n$ in $\mathfrak{g}^*$ (*i.e.*, $\langle X^{*i}; X_j\rangle = \delta^i_j$), then for $X = \sum_{i=1}^n x^i X_i$ and $X^* = \sum_{i=1}^n \xi_i X^{*i}$, with x_i and ξ_i real numbers, we get

$$\langle X^*; X\rangle = \sum_{i=1}^n x^i \xi_i = \vec{x}\cdot\vec{\xi}. \tag{14}$$

Also in these coordinates, the (translation invariant) Lebesgue measures on $\mathfrak{g}$ and $\mathfrak{g}*$ assume respectively the forms

$$dX \to d\vec{x} = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n, \quad dX^* \to d\vec{\xi} = d\xi_1 \wedge d\xi_2 \wedge \ldots \wedge d\xi_n. \tag{15}$$

The *coadjoint action* $\mathrm{Ad}^\sharp_{g_o}$ of a group element $g_o \in V_e \subset \mathfrak{g}$ on a vector $X^* \in \mathfrak{g}^*$ is defined by the relation

$$\langle \mathrm{Ad}^\sharp_{g_o} X^*; X\rangle = \langle X^*; \mathrm{Ad}_{g_o^{-1}} X\rangle. \tag{16}$$

This is the dual action on $\mathfrak{g}^*$ induced by the adjoint action. In terms of the bases introduced in $\mathfrak{g}$ and $\mathfrak{g}^*$ as above, if the adjoint action Ad_{g_o} has a matrix representation, then the representation of the coadjoint action $\mathrm{Ad}^\sharp_{g_o}$ is given by the inverse transposed matrix.

III.3 Coadjoint orbits and invariant measures

If X_0^* is a fixed element of $\mathfrak{g}^*$, the set of all elements of $\mathfrak{g}^*$ of the type $\mathrm{Ad}^\sharp_g X_0^*$, $g \in G$, is its *orbit* under G, denoted $\mathcal{O}^*$. Orbits under the coadjoint action are *symplectic manifolds*, *i.e.*, smooth surfaces in $\mathfrak{g}^*$, generally of lower dimension than $\mathfrak{g}^*$, which have a structure similar to classical phase spaces. Moreover, these orbits $\mathcal{O}^*$ come naturally equipped with measures $d\Omega(X^*)$ (analogues of the Liouville measure on ordinary phase space), which are invariant under the coadjoint action of G [17]. Namely,

$$d\Omega(X^*) = d\Omega(\mathrm{Ad}^\sharp_g X^*), \quad X^* \in \mathcal{O}^*, \tag{17}$$

for all $g \in G$. An orbit $\mathcal{O}^*$ of this type is called a *coadjoint orbit*. Clearly, two such orbits are either distinct or else they coincide entirely. The collection of all coadjoint orbits exhausts $\mathfrak{g}^*$ and we may write

$$\bigcup_{\lambda\in J} \mathcal{O}^*_\lambda = \mathfrak{g}^*, \tag{18}$$

where λ could be a discrete or continuous parameter, or set of parameters that characterize the orbits, and J is the appropriate index set. We denote the invariant measure on the coadjoint orbit $\mathcal{O}^*_\lambda$ by $d\Omega_\lambda$.

In view of (18), any element X^* in the vector space $\mathfrak{g}^*$ belongs to some orbit $\mathcal{O}^*_\lambda$ and hence may be written as X^*_λ, to display the orbit dependence explicitly.

We shall assume the following decomposition of the Lebesgue measure on $\mathfrak{g}^*$:

$$dX^* = d\kappa(\lambda)\; \sigma_\lambda(X^*_\lambda)\; d\Omega_\lambda(X^*_\lambda), \quad X^*_\lambda \in \mathcal{O}^*_\lambda, \tag{19}$$

where κ is the appropriate measure on the parameter space J and σ_λ a positive, non-vanishing function on the orbit $\mathcal{O}^*_\lambda$. Depending on the nature of J, the measure $d\kappa$ can be discrete or continuous, or can have both a discrete and a continuous part. Such a decomposition will hold for all the cases of interest to us, while more generally, it is a statement of a certain regularity condition [11].

IV Invariant Haar measures, the regular and coadjoint representations

Every Lie group carries a left- and a right-invariant Haar measure, $d\mu_\ell$ and $d\mu_r$ respectively. These satisfy

$$d\mu_\ell(g) = d\mu_\ell(g_o g), \qquad d\mu_r(g) = d\mu_r(g g_o), \quad g \in G, \tag{20}$$

for fixed but arbitrary $g_0 \in G$. In general the left and right Haar measures are different (though equivalent in the sense of measures). However, for *unimodular* groups (including compact, abelian, certain semidirect products, etc.) they turn out to be the same, *i.e.*, $d\mu_\ell(g) = d\mu_r(g)$. Note that Naĭmark groups, including solvable groups such as the two-parameter affine group of translations and dilatations, are *not* unimodular.

IV.1 Modular function; left- and right-regular representations

Since generally, the left and right Haar measures are measure-theoretically equivalent, they are related through a *modular* function $\Delta(g)$. This is a positive and real-valued measurable function on G, satisfying

$$d\mu_\ell(g) = \Delta(g)\, d\mu_r(g) = d\mu_r(g^{-1}). \tag{21}$$

The modular function is also a group homomorphism: $\Delta(g_1 g_2) = \Delta(g_1)\Delta(g_2)$, for all $g_1, g_2 \in G$ and $\Delta(e) = 1$. In what follows, we shall only need the left Haar measure $d\mu_\ell$ and so write it simply as $d\mu$. (All conclusions can be formulated equivalently in terms of the right Haar measure.)

Using the Haar measure $d\mu$ one can build two unitary representations of G: the left- and right-regular representations. Consider the Hilbert space $L^2(G, d\mu)$, of all measurable, complex-valued functions f on G, satisfying

$$\int_G |f(g)|^2 \, d\mu(g) < \infty.$$

On this Hilbert space we define a representation of G by unitary operators $U_\ell(g)$, $g \in G$, such that

$$(U_\ell(g)f)(g') = f(g^{-1}g'), \quad f \in L^2(G, d\mu), \tag{22}$$

holding for almost all $g' \in G$ with respect to the measure $d\mu$. This is called the *left-regular representation*; its unitarity is trivially checked using the invariance properties of the measure $d\mu$. Similarly, we define the *right-regular representation* $U_r(g)$ on the *same Hilbert space* $L^2(G, d\mu)$, where it is also unitary, and given by

$$(U_r(g)f)(g') = [\Delta(g)]^{\frac{1}{2}} \, f(g'g), \quad f \in L^2(G, d\mu), \tag{23}$$

again for almost all $g' \in G$ with respect to $d\mu$. It is easy to verify that the left- and right-regular operators $U_\ell(g)$ and $U_r(g)$ commute. Moreover, the two representations are unitarily equivalent by the map M on $f \in L^2(G, d\mu)$ given by $(Mf)(g) = [\Delta(g)]^{-\frac{1}{2}} \, f(g^{-1})$. This map is unitary and a straightforward computation using (21) and the homomorphism properties of the modular function, shows that $MU_\ell(g)M^{-1} = U_r(g)$.

IV.2 Adjoint and coadjoint representations

The adjoint and the coadjoint actions of the group give rise to two interesting unitary representations connected by an integral, generalized *Fourier transform* relation. Consider the two Hilbert spaces $L^2(\mathfrak{g}, dX)$ and $L^2(\mathfrak{g}^*, dX^*)$ of Lebesgue-measurable complex-valued functions on the Lie algebra and its dual, respectively, and which are square-integrable with respect to the corresponding Lebesgue measures. On $L^2(\mathfrak{g}, dX)$ one defines the *adjoint representation* $V(g)$ of $g \in G$ by the operators

$$(V(g)F)(X) = \| \operatorname{Ad}_g \|^{-\frac{1}{2}} \, F(\operatorname{Ad}_{g^{-1}} X), \qquad F \in L^2(\mathfrak{g}, dX), \tag{24}$$

where $\| \operatorname{Ad}_g \|$ is the determinant of the linear transformation Ad_g on $\mathfrak{g}$. The operators $V(g)$ form a unitary representation of G.

Similarly, on $L^2(\mathfrak{g}^*, dX^*)$ one defines the *coadjoint representation* by the operators $V^\sharp(g)$, $g \in G$, given by

$$(V^\sharp(g)\widehat{F})(X^*) = \| \operatorname{Ad}^\sharp_g \|^{-\frac{1}{2}} \, \widehat{F}(\operatorname{Ad}^\sharp_{g^{-1}} X^*), \qquad \widehat{F} \in L^2(\mathfrak{g}^*, dX^*), \tag{25}$$

where $\| \operatorname{Ad}^\sharp_g \|$ is now the determinant of the linear transformation $\operatorname{Ad}^\sharp_g$ on $\mathfrak{g}^*$. Again, the operators $V^\sharp(g)$ form a unitary representation of G.

The dual representations $V(g)$ and $V^\sharp(g)$ are unitarily equivalent. They are related by the Fourier transform $\mathcal{F} : L^2(\mathfrak{g}, dX) \to L^2(\mathfrak{g}^*, dX^*)$,

$$(\mathcal{F}F)(X^*) = \widehat{F}(X^*) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathfrak{g}} e^{-i\langle X^* \,;\, X\rangle} F(X)\, dX, \tag{26}$$

which is a unitary map. The unitary equivalence of the two representations:

$$\mathcal{F}V(g)\mathcal{F}^{-1} = V^\sharp(g), \qquad g \in G, \tag{27}$$

is then established using (16) and the facts that $\| \operatorname{Ad}^\sharp_g \| = \| \operatorname{Ad}_g \|^{-1}$ and $d(\operatorname{Ad}_g X) = \| \operatorname{Ad}_g \| \, dX$.

IV.3 Covariant coadjoint representation

We now construct a unitary representation of G on each coadjoint orbit $\mathcal{O}^*_\lambda$. Going back to (17)–(18), for each $\lambda \in J$ we define a Hilbert space $\mathfrak{H}_\lambda = L^2(\mathcal{O}^*_\lambda, d\Omega_\lambda)$, consisting of all complex-valued $d\Omega_\lambda$-measurable functions $\widehat{F}_\lambda$ on the orbit $\mathcal{O}^*_\lambda$, which are square-integrable,

$$\int_{\mathcal{O}^*_\lambda} |\widehat{F}_\lambda(X^*)|^2 \, d\Omega_\lambda(X^*) < \infty.$$

We represent $g \in G$ on the Hilbert space $\mathfrak{H}_\lambda$ by the operator $U^\sharp_\lambda(g)$, where,

$$(U^\sharp_\lambda(g)\widehat{F}_\lambda)(X^*) = \widehat{F}_\lambda(\operatorname{Ad}^\sharp_{g^{-1}} X^*), \quad X^* \in \mathcal{O}^*_\lambda. \tag{28}$$

Because of the invariance of $d\Omega_\lambda$ under the coadjoint action $\operatorname{Ad}^\sharp_g$, this representation is unitary.

Comparing $V^\sharp$ in (25) with $U^\sharp_\lambda$ in (28), we see that although both representations are built on the dual of the Lie algebra $\mathfrak{g}^*$, $U^\sharp_\lambda$ appears to be *more covariant* —in the sense that it has no 'scaling factor' $\| \operatorname{Ad}^\sharp_g \|^{-\frac{1}{2}}$ in it. This is because the measure $d\Omega_\lambda$ is actually invariant under the coadjoint action, while the measure dX^* is (in general) not. On the other hand, the representation $U^\sharp_\lambda$ is restricted to functions on a single orbit $\mathcal{O}_\lambda$ only (λ is fixed), while $V^\sharp$ is defined on functions on the entire dual space $\mathfrak{g}^*$. However, as we now proceed to show, using the factored form (19) of the Lebesgue measure, it is possible to combine all the representations $U^\sharp_\lambda$, $\lambda \in J$, into a single representation $U^\sharp$, which is unitarily equivalent to $V^\sharp$, and which is defined on functions on all of $\mathfrak{g}^*$.

We define the *covariant coadjoint representation* $U^\sharp$ on the set of all the Hilbert spaces $\mathfrak{H}_\lambda$ that carry the representations $U^\sharp_\lambda$, combined into a single *direct integral Hilbert space* $\widetilde{\mathfrak{H}}$ of the component spaces $\mathfrak{H}_\lambda$,

$$\widetilde{\mathfrak{H}} = \int_J^\oplus \mathfrak{H}_\lambda \, d\kappa(\lambda). \tag{29}$$

The elements of $\widetilde{\mathfrak{H}}$ consist of collections of vectors

$$\Phi = \{\widehat{F}_\lambda\}_{\lambda\in J}, \qquad \widehat{F}_\lambda \in \mathfrak{H}_\lambda. \tag{30}$$

When $\Phi = \{\widehat{F}_\lambda\}$ and $\Phi' = \{\widehat{F}'_\lambda\}$ are two such collections, and α, β are complex numbers, then we may also define their linear combination, $\alpha\Phi + \beta\Phi' = \{\alpha\widehat{F}_\lambda + \beta\widehat{F}'_\lambda\}$. In this way we provide a linear vector space structure on the set of all such collections of vectors Φ. Next, using the measure $d\kappa$ appearing in (19), we retain only those vectors Φ which satisfy

$$\|\Phi\|^2 = \int_J \|\widehat{F}_\lambda\|^2 \, d\kappa(\lambda) = \int_J \left[\int_{\mathcal{O}^*_\lambda} |\widehat{F}_\lambda(X^*)|^2 \, d\Omega_\lambda(X^*)\right] d\kappa(\lambda) < \infty. \tag{31}$$

The set of all such vectors forms the Hilbert space $\widetilde{\mathfrak{H}}$ on which Eq. (31) defines a norm.

All the coadjoint representations $U^\sharp_\lambda$ in (28) can now be collected into the one covariant coadjoint representation $U^\sharp$ on $\widetilde{\mathfrak{H}}$ in the manner:

$$U^\sharp(g)\Phi = \{U^\sharp_\lambda(g)\widehat{F}_\lambda\}_{\lambda\in J}, \qquad g \in G, \tag{32}$$

which is unitary by construction. Note that this is indeed defined on all of $\mathfrak{g}^*$, since for any $X^*_\lambda \in \mathfrak{g}^*$ coming from the coadjoint orbit $\mathcal{O}^*_\lambda$, we have

$$(U^\sharp(g)\Phi)(X^*_\lambda) = (U^\sharp_\lambda(g)\widehat{F}_\lambda)(X^*_\lambda) = \widehat{F}_\lambda(\mathrm{Ad}^\sharp_{g^{-1}} X^*_\lambda), \tag{33}$$

by (28).

IV.4 Unitary equivalence of representations

We end this section by showing that the coadjoint representation $V^\sharp$ in (25) and the covariant coadjoint representation $U^\sharp$ above, are unitarily equivalent; this will complete the proof of the equivalence of all three representations: the adjoint V, the coadjoint $V^\sharp$ and covariant coadjoint $U^\sharp$.

Consider the relation (19) between the Lebesgue measure dX^* on the dual $\mathfrak{g}^*$ of the Lie algebra, and the invariant measures $d\Omega_\lambda$ on the coadjoint orbits $\mathcal{O}^*_\lambda$. Again, since $d(\mathrm{Ad}^\sharp_g X^*) = \|\mathrm{Ad}^\sharp_g\| \, dX^*$, and $d\Omega_\lambda(\mathrm{Ad}^\sharp_g X^*_\lambda) = d\Omega_\lambda(X^*)$, it follows that

$$\|\mathrm{Ad}^\sharp_g\| \, \sigma_\lambda(X^*_\lambda) = \sigma_\lambda(\mathrm{Ad}^\sharp_g X^*_\lambda). \tag{34}$$

Thus we introduce finally the linear map $\widetilde{N} : L^2(\mathfrak{g}^*, dX^*) \to \widetilde{\mathfrak{H}}$ intertwining the coadjoint and the covariant coadjoint representations,

$$\widetilde{N}\widehat{F} = \{\widehat{G}'_\lambda\}_{\lambda\in J}, \quad \text{where,} \quad \widehat{G}'_\lambda(X_\lambda) = [\sigma_\lambda(X^*)]^{\frac{1}{2}}\widehat{F}(X^*_\lambda), \quad X^*_\lambda \in \mathcal{O}^*_\lambda. \tag{35}$$

Using (19) and (34) it is then straightforward to check that $\widetilde{N}$ is a unitary map for which

$$\widetilde{N}V^\sharp(g)\widetilde{N}^{-1} = U^\sharp(g). \tag{36}$$

With these preliminaries attended to, we turn our attention in the next two Sections to the actual construction of the Wigner map, the Wigner function and the Wigner operator.

V Square-integrable representations

The existence of the Wigner map hinges on the existence of a class of representations for certain types of groups. These are the so-called square-integrable [2] or discrete-series representations which enjoy an intertwining property with the left-regular representation U_ℓ, characterized in Eq. (22).

V.1 Admissible vectors

Let U be an unitary irreducible representation of G on a Hilbert space $\mathfrak{H}$. We recall that U is *square-integrable* if there exists a non-zero vector $\eta \in \mathfrak{H}$, called an *admissible vector*, such that

$$c(\eta) = \int_G |\langle U(g)\eta|\eta\rangle|^2 \, d\mu(g) < \infty. \tag{37}$$

It is easy to see that if η is admissible, then so is also $U(g)\eta$ for any $g \in G$. In other words, the set $\mathcal{A}$ of all admissible vectors is invariant under U; then, the irreducibility of U implies that either $\mathcal{A}$ is dense in $\mathfrak{H}$, or else $\mathcal{A} = \{0\}$; in the latter case, U is not square-integrable. When G is a unimodular group, the square integrability of U implies that $\mathcal{A} = \mathfrak{H}$, *i.e.*, every vector in $\mathfrak{H}$ is admissible (see, *e.g.*, [13]).

V.2 Orthogonality relations

The matrix elements $\langle U(g)\psi|\phi\rangle$ of a square- integrable representation U satisfy certain useful *orthogonality relations*. Indeed, every square-integrable representation determines a unique positive invertible operator C on $\mathfrak{H}$, whose domain coincides with the set $\mathcal{A}$ of all admissible vectors. Furthermore, for all vectors $\eta_1, \eta_2 \in \mathcal{A}$ and arbitrary vectors $\phi_1, \phi_2 \in \mathfrak{H}$, the following orthogonality relation holds:

$$\int_G \overline{\langle U(g)\eta_2|\phi_2\rangle}\langle U(g)\eta_1|\phi_1\rangle \, d\mu(g) = \langle C\eta_1|C\eta_2\rangle \, \langle\phi_2|\phi_1\rangle. \tag{38}$$

When G is a unimodular group, then C is a positive multiple of the identity, *i.e.*, $C = \lambda I$, for some $\lambda > 0$. For non-unimodular groups, C is an unbounded operator and its domain $\mathcal{A}$ is only dense in $\mathfrak{H}$. This form of the orthogonality relations is well-known; however, for our purposes it will be convenient to use an extended version of these relations [3].

Let $\mathcal{B}_2(\mathfrak{H})$ denote the space of all Hilbert-Schmidt operators on $\mathfrak{H}$. This is the Hilbert space obtained by taking all finite complex combinations of rank-one operators on $\mathfrak{H}$ of the type $\rho = |\psi\rangle\langle\phi|$, $\psi, \phi \in \mathfrak{H}$, and closing the resulting set in the norm $\|\rho\|_{\mathcal{B}} = \{\mathrm{Tr}[\rho^* \rho]\}^{\frac{1}{2}}$, which arises from the scalar product

$$\langle \rho_2 | \rho_1 \rangle_{\mathcal{B}} = \mathrm{Tr}\,[\rho_2{}^* \rho_1]. \tag{39}$$

The orthogonality relations (38) can now be extended to hold between pairs of elements in the Hilbert space $\mathcal{B}_2(\mathfrak{H})$. This is done using the Wigner transform, as we show below.

V.3 The Wigner transform

Let $\eta \in \mathfrak{H}$ be an admissible vector and consider the vector $\psi = C\eta$, which is in the range of C (*i.e.*, in the domain $\mathcal{D}$ of C^{-1}, dense in $\mathfrak{H}$). Using such vectors we define the *Wigner transform* as the linear map $\widetilde{\mathfrak{W}} : \mathfrak{H} \otimes \mathcal{D}(C^{-1}) \to L^2(G, d\mu)$,

$$(\widetilde{\mathfrak{W}}\rho)(g) = \langle U(g) C^{-1} \psi | \phi \rangle = \mathrm{Tr}[U(g)^* \rho C^{-1}], \tag{40}$$

where $\rho = |\phi\rangle\ \langle\psi| \in \mathcal{B}_2(\mathfrak{H})$ and the star denotes the adjoint. Then for any two such $\rho_i \in \mathcal{B}_2(\mathfrak{H})$, $\ i = 1, 2$ and $\rho_i = |\phi_i\rangle\ \langle\psi_i|$, the orthogonality relations (38) may be reexpressed as:

$$\int_G \overline{(\widetilde{\mathfrak{W}}\rho_2)(g)} (\widetilde{\mathfrak{W}}\rho_1)(g)\, d\mu(g) = \mathrm{Tr}\,[\rho_2{}^* \rho_1] = \langle \rho_2 | \rho_1 \rangle_{\mathcal{B}}. \tag{41}$$

The relation (40) defines a linear transform map $\widetilde{\mathfrak{W}}$ from $\mathfrak{H} \otimes \mathcal{D}(C^{-1})$ [the dense subspace of $\mathcal{B}_2(\mathfrak{H})$ generated by vectors of the form $|\phi\rangle\ \langle\psi|,\ \phi \in \mathfrak{H}$, and ψ in the domain of the operator C^{-1}] into $L^2(G, d\mu)$. Foundations of this map for the Heisenberg-Weyl algebra can be seen in [20], where it is known as the *characteristic function*, and in [14] as the Wigner transform. Our construction finds more basic the Wigner *map* $\mathfrak{W}$ defined in the next Section, and of which $\widetilde{\mathfrak{W}}$ is the (generalized) Fourier transform.

The Wigner transform map $\widetilde{\mathfrak{W}}$ preserves the scalar product, hence it is an isometry; it may be therefore extended by continuity to an isometry valid on all of $\mathcal{B}_2(\mathcal{H})$. We use the same notation for this extended Wigner transform map, now $\widetilde{\mathfrak{W}} : \mathcal{B}_2(\mathfrak{H}) \to L^2(G, d\mu)$. It associates to each Hilbert-Schmidt operator ρ a square-integrable function $f_\rho(g)$ on the group. On a dense set of elements $\rho \in \mathcal{B}_2(\mathfrak{H})$, we can recover the original function f_ρ by the *trace* formula:

$$f_\rho(g) = (\widetilde{\mathfrak{W}}\rho)(g) = \mathrm{Tr}[U(g)^* \rho C^{-1}]. \tag{42}$$

The unitary representation $U(g)$ acting on the Hilbert space $\mathfrak{H}$ can be lifted immediately to a unitary representation $\mathbb{U}_\ell$ on the Hilbert space $\mathcal{B}_2(\mathfrak{H})$. Indeed, we simply define its action on a vector $\rho \in \mathcal{B}_2(\mathfrak{H})$ by ordinary operator product from the left,

$$\mathbb{U}_\ell(g)\rho = U(g)\rho. \tag{43}$$

Now, for any $g \in G$, the operator C satisfies the covariance conditions

$$U(g)^* C U(g) = [\Delta(g)]^{-\frac{1}{2}} C, \qquad C U(g) C^{-1} = [\Delta(g)]^{-\frac{1}{2}} U(g). \tag{44}$$

With this, it is easily verified that the Wigner transform map $\widetilde{\mathfrak{W}}$ intertwines $\mathbb{U}_\ell(g)$ on $\mathcal{B}_2(\mathfrak{H})$ with the left-regular representation U_ℓ on $L^2(G, d\mu)$ which was defined in (22), namely

$$\widetilde{\mathfrak{W}}\, \mathbb{U}_\ell(g) = U_\ell(g) \widetilde{\mathfrak{W}}, \qquad g \in G. \tag{45}$$

VI The Wigner map and function

We now introduce the *Wigner map* $\mathfrak{W}$ which is essentially the Fourier transform of the map $\widetilde{\mathfrak{W}}$ in Subsect. V.3. We shall assume as in Subsect. III.1 that the exponential map (10) relates the open neighbourhood N_0 of $\mathfrak{g}$ to an open dense set V_e in the connected part of the identity of G and such that the complement of V_e has measure zero. We can then use the exponential map $g = e^X$ to introduce local coordinates over the set V_e in G; in the basis X_i, $i = 1, 2, \ldots, n$ we write $X = \sum_{i=1}^n x^i X_i \in \mathfrak{g}$, so the group element $g \in V_e$ will map to $\vec{x} = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$. In these coordinates, the left invariant Haar measure on the group will become

$$d\mu(g) \to m(X)\; dX, \tag{46}$$

where m is a positive Lebesgue-measurable function on N_0, and the relations (41) assume the form

$$\int_{N_0} \overline{(\widetilde{\mathfrak{W}}\rho_2)(e^X)} (\widetilde{\mathfrak{W}}\rho_1)(e^X)\; m(X)\; dX = \mathrm{Tr}\,[{\rho_2}^* \rho_1] = \langle \rho_2 | \rho_1 \rangle_{\mathcal{B}}. \tag{47}$$

VI.1 The Wigner map

We define a linear map $\mathfrak{W}$ from the space of Hilbert-Schmidt operators $\rho \in \mathcal{B}_2(\mathfrak{H})$ to functions of $X^*_\lambda \in \mathcal{O}^*_\lambda$ on the coadjoint orbits, by the Fourier transform-type integral

$$(\mathfrak{W}\rho)(X^*_\lambda) = \frac{[\sigma_\lambda(X^*_\lambda)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^*_\lambda\, ;\, X\rangle} (\widetilde{\mathfrak{W}}\rho)(e^X) [m(X)]^{\frac{1}{2}}\; dX, \tag{48}$$

where σ_λ is the density function in (19). Using this, Eq. (47) and standard properties of the Fourier transform, we immediately establish that $\mathfrak{W}$ maps any Hilbert-Schmidt operator ρ to an element of the direct integral Hilbert space $\widetilde{\mathfrak{H}}$ defined in (29)–(31), and that this map is a linear isometry. We call this map $\mathfrak{W} : \mathcal{B}_2(\mathfrak{H}) \to \widetilde{\mathfrak{H}}$, the *Wigner map*.

VI.2 The Wigner operator and matrix

In the previous literature [27], [21], [5], it was found useful to define a formal operator, called the *Wigner operator*, which acts in the Wigner map (48) extended to the Hilbert space $\mathfrak{H}$ of this Section. It is

$$\mathcal{W}(X^*) = \frac{[\sigma(X^*)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^* \,;\, X\rangle}\, C^{-1}\, U(e^{-X})\, [m(X)]^{\frac{1}{2}}\, dX. \tag{49}$$

Formally, this defines an operator on $\mathfrak{H}$ for almost all $X^* \in \mathfrak{g}^*$ with respect to the Lebesgue measure. For any pair of functions $\phi, \psi \in \mathfrak{H}$, or any Hilbert-Schmidt operator ρ, the Wigner function is

$$W(\phi, \psi \mid X^*) = \langle \phi \mid \mathcal{W}(X^*) \mid \psi \rangle_{\mathfrak{H}}, \qquad W(\rho \mid X^*) = \mathrm{Tr}\,[\rho \mathcal{W}(X^*)]. \tag{50}$$

In the group ISO(2) studied in [21] and SU(2) in [5] and [9], the Hilbert space $\mathfrak{H}$ has a denumerable basis ϕ^λ_m, $m \in J_\lambda$ an integer and a finite number, respectively. Then, it is convenient to define the (infinite) Wigner *matrix* $\mathbf{W}(X^*)$ with diagonal blocks $\mathbf{W}^\lambda(X^*)$, whose matrix elements are

$$W^\lambda_{m,m'}(X^*) = \langle \phi^\lambda_m \mid \mathcal{W}(X^*) \mid \phi^\lambda_{m'} \rangle_{\mathfrak{H}}, \tag{51}$$

and which are reduced to integrals of special functions to be computed.

VI.3 The Wigner function

Introducing the positive Lebesgue-measurable function σ on $X^* \in \mathfrak{g}^*$, which assumes the value $\sigma_\lambda(X^*)$ for all $X^* \in \mathcal{O}^*_\lambda$, we can write Eq. (48) on the whole of $\mathfrak{g}^*$, as

$$\begin{aligned} W(\rho \mid X^*) &= (\mathfrak{W}\rho)(X^*) \qquad\qquad (52)\\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^* \,;\, X\rangle}\, \mathrm{Tr}[U(e^{-X})\rho C^{-1}]\, [\sigma(X^*)\, m(X)]^{\frac{1}{2}}\, dX. \end{aligned}$$

We call this the *Wigner function* of the Hilbert-Schmidt operator ρ, on the dual $\mathfrak{g}^*$ of the the Lie algebra of G (or, more accurately, on its coadjoint orbits $\mathcal{O}^*_\lambda$, $\lambda \in J$). The basic properties of the Wigner map and the Wigner function can immediately be read off their definitions (48) and (52) and compared with the properties listed in Section II, as we now proceed to show.

VI.4 Reality/sesquilinearity

First note that the Wigner map (48) of elements $\rho \in \mathcal{B}_2(\mathfrak{H})$ is linear. Now, let ρ^* be the adjoint of the operator ρ; then, since N_0 is invariant under the interchange $X \to -X$, and since by virtue of (21)

$$\Delta(e^X) m(-X) = m(X), \tag{53}$$

replacing X by $-X$ in the integral in (52) and using (44), we obtain

$$W(\rho \mid X^*) = \overline{W(\rho^* \mid X^*)}. \tag{54}$$

Hence, if ρ is self-adjoint then its Wigner function $W(\rho \mid X^*)$ is real.

For elements of the type $\rho = |\phi\rangle\langle\psi|$, the map $\mathfrak{W}$ can be looked upon as a *sesquilinear* map from $\mathfrak{H} \times \mathfrak{H}$ into $\widetilde{\mathfrak{H}}$, *i.e.*, linear in ϕ, antilinear in ψ and non-degenerate, in the sense that $\mathfrak{W}(|\psi\rangle\langle\psi|) = 0$ if and only if $\psi = 0$. The corresponding Wigner functions, written $W(\psi, \phi \mid X^*)$, hence satisfy Eqs. (5) and (6), and are real (*cf.* Subsect. II.2).

VI.5 Covariance

Here we verify that the covariance property of Subsect. II.3 holds in our new generalized setting. In order to do this, consider the representation $\mathbf{U}_b$ of G, on the Hilbert space $\mathcal{B}_2(\mathfrak{H})$ of Hilbert-Schmidt operators,

$$\mathbf{U}_b(g)\rho = U(g)\rho U(g)^*, \qquad g \in G, \tag{55}$$

where U is the square-integrable representation introduced in Section V.

The representation $\mathbf{U}_b$ in (55) is clearly unitary. Now, since $d\mu(g_0 g g_0^{-1}) = \Delta(g_0^{-1})\, d\mu(g)$, we easily derive that

$$m(\mathrm{Ad}_g\, X) = \frac{m(X)}{\|\,\mathrm{Ad}_g\,\|\ \Delta(g)}, \qquad X \in \mathfrak{g}, \quad g \in G. \tag{56}$$

Using this relation, $\|\,\mathrm{Ad}_g\,\| = \|\,\mathrm{Ad}_g^\sharp\,\|^{-1}$ and Eq. (34), we find after some routine computations that the Wigner map $\mathfrak{W}$ intertwines the representation $\mathbf{U}_b$ with the covariant coadjoint representation $U^\sharp$, defined in (33), *i.e.*,

$$\mathfrak{W}\mathbf{U}_b(g) = U^\sharp(g)\mathfrak{W}, \qquad g \in G. \tag{57}$$

In terms of the Wigner function [*cf.* Eq. (7)], this is

$$W(U(g)\rho U(g)^* \mid X^*) = W(\rho \mid\ \mathrm{Ad}^\sharp_{g^{-1}}\, X^*), \qquad g \in G, \quad X^* \in \mathfrak{g}^*. \tag{58}$$

VI.6 Overlap and reconstruction formulae

The Wigner map is an isometry since it preserves scalar products: for any two Hilbert-Schmidt operators, ρ_1 and ρ_2, we have

$$\langle \mathfrak{W}\rho_1 \mid \mathfrak{W}\rho_2 \rangle_{\widetilde{\mathfrak{H}}} = \langle \rho_1 \mid \rho_2 \rangle_{\mathcal{B}}. \tag{59}$$

This can be written alternatively as the *overlap condition* [*cf.* Eq. (8)],

$$\int_{\mathfrak{g}^*} \overline{W(\rho_1|X^*)}\, W(\rho_2|X^*)\ [\sigma(X^*)]^{-1}\, dX^* = \mathrm{Tr}[\rho_1^* \rho_2]. \tag{60}$$

It is easy to invert the Wigner map using the overlap formula (60). Indeed, taking $\rho_2 = \rho$ and $\rho_1 = |\psi\rangle\langle\phi|$, with ψ in the domain of C^{-1}, we obtain

$$\int_{\mathfrak{g}^*} \overline{W(|\phi\rangle\langle\psi|\ |X^*)}\ W(\rho|X^*)\ [\sigma(X^*)]^{-1}\ dX^* = \langle\phi|\rho\ \psi\rangle. \tag{61}$$

Using (52) and noting that ϕ and ψ are arbitrary, yields

$$\rho = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathfrak{g}^*} \left[\int_{N_0} e^{-i\langle X^*\ ;\ X\rangle}\ W(\rho|X^*) U(e^X) C^{-1} \left(\frac{m(X)}{\sigma(X^*)}\right)^{\frac{1}{2}} dX \right] dX^*. \tag{62}$$

We shall refer to this relation as the *reconstruction formula*.

VII The Wigner function and wavelet transform

There is a very interesting relation between the Wigner function introduced in the previous Section and the generalized wavelet transform.

VII.1 The wavelet transform

In the square-integrable representation U, any admissible vector $\eta \in \mathcal{A}$, with $c(\eta)$ as in (37), can be used to define the (generalized) *wavelet transform* $f_{\eta,\phi}$ of an arbitrary $\phi \in \mathfrak{H}$:

$$f_{\eta,\phi}(g) = \frac{1}{[c(\eta)]^{\frac{1}{2}}} \langle U(g)\eta|\phi\rangle, \qquad g \in G. \tag{63}$$

The wavelet transform is a square-integrable function on G – an element of the Hilbert space $L^2(G, d\mu)$. In fact, the map $\phi \mapsto f_{\eta,\phi}$ in (63) may be shown to be an isometry [13], *i.e.*,

$$\int_G |f_{\eta,\phi}(g)|^2\ d\mu(g) = \|\phi\|^2. \tag{64}$$

Note that the standard wavelet transform discussed in the literature [10] is a special case of the transform (63) when the group G is the affine group of the real line.

VII.2 Coherent states

It is worthwhile to mention at this point the role of coherent states, where the standard or canonical coherent states belong to the special case of the Heisenberg-Weyl group. The orthogonality relations (38) imply the *resolution of the identity*,

$$\frac{1}{c(\eta)} \int_G |\eta_g\rangle\langle\eta_g|\ d\mu(g) = I, \qquad \eta_g = U(g)\eta, \tag{65}$$

and then the vectors $[c(\eta)]^{-\frac{1}{2}}\ \eta$ are called *coherent states* of the group G in the unitary irreducible representation U (see, *e.g.*, [3], [22]). Thus, the generalized wavelet transform (63) may be called also the *coherent state transform*.

VII.3 The Wigner-wavelet relations

To give explicitly the relation between the Wigner function $W(\rho|X^*)$ in (2) and the wavelet transform $f_{\eta,\phi}(g)$ of $\phi \in \mathfrak{H}$ in (63), for fixed (admissible) $\eta \in \mathcal{A}$ and arbitrary $\phi \in \mathfrak{H}$, consider the Hilbert-Schmidt operators of the form

$$\rho_{\eta,\phi} = \frac{1}{[c(\eta)]^{\frac{1}{2}}} |\phi\rangle\langle\eta| C. \tag{66}$$

Comparing now (63) with (52), we conclude that

$$W(\rho_{\eta,\phi}|X^*) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\langle X^* \,;\, X\rangle} \; f_{\eta,\phi}(e^X) \; [\sigma(X^*) \; m(X)]^{\frac{1}{2}} \; dX. \tag{67}$$

This relation is easily inverted, yielding the wavelet transform in terms of the Wigner function, namely

$$f_{\eta,\phi}(e^X) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathfrak{g}^*} e^{i\langle X^* \,;\, X\rangle} \; W(\rho_{\eta,\phi}|X^*) \; [\sigma(X^*) \; m(X)]^{-\frac{1}{2}} \; dX^*. \tag{68}$$

VII.4 Bases and coordinates in the Lie algebra

As indicated in Section III, one can introduce bases in the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$, in terms of which $X \in \mathfrak{g}$ has the coordinate representation $\vec{x} = (x_1, x_2, \ldots, x_n)$, while $X^* \in \mathfrak{g}^*$ has the coordinates $\vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$; their Lebesgue measures have the forms given in (15). So let $\widehat{N}_0$ be the image, in these coordinates, of the set $N_0 \subset \mathfrak{g}$ (the domain of the exponential map (10), the range V_e of which is assumed to be dense in G with its completment having zero Haar measure). Denote by $\widehat{X}_i$ the Hilbert space operators that represent the basis elements $X_i \in \mathfrak{g}$, *i.e.*, the operators on $\mathfrak{H}$ such that

$$U(e^{X_i}) = e^{-i\widehat{X}_i}. \tag{69}$$

Finally, denote by $\vec{\widehat{X}}$ the vector operator with components $\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_n$.

In these terms the Wigner function (52) and its inverse (62) can be written

$$W(\rho \mid \vec{\xi}\,) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\widehat{N}_0} \mathrm{Tr}\left[e^{i(\vec{\widehat{X}} - \vec{\xi}\,)\cdot\vec{x}} \rho C^{-1}\right] \; [\sigma(\vec{\xi}\,) \; m(\vec{x})]^{\frac{1}{2}} \; d\vec{x}, \tag{70}$$

$$\rho = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathfrak{g}^*} \left[\int_{\widehat{N}_0} e^{-i(\vec{\widehat{X}} - \vec{\xi}\,)\cdot\vec{x}} W(\rho|\vec{\xi}\,) \times C^{-1} \left[\frac{m(\vec{x})}{\sigma(\vec{\xi}\,)}\right]^{\frac{1}{2}} d\vec{x}\right] d\vec{\xi}. \tag{71}$$

Similarly, all other coordinate-free expressions appearing earlier can be written in these coordinates, which are most useful for computational purposes. In particular, the overlap condition (60), for $\rho_1 = |\phi_1\rangle\langle\psi_1|$ and $\rho_2 = |\phi_2\rangle\langle\psi_2|$ becomes

$$\int_{\mathfrak{g}^*} \overline{W(\psi_1, \phi_1|\vec{\xi}\,)}\; W(\psi_2, \phi_2|\vec{\xi}\,)\; [\sigma(\vec{\xi}\,)]^{-1}\; d\vec{\xi} = \langle\phi_1|\phi_2\rangle\langle\psi_2|\psi_1\rangle. \qquad (72)$$

In these coordinates, the covariance condition (7) or (58) assumes the form:

$$W(U(g)\psi, U(g)\phi|\vec{\xi}\,) = W(\psi, \phi|M^T(g)\vec{\xi}\,), \qquad g \in G, \quad \phi, \psi \in \mathfrak{H}, \qquad (73)$$

where $M(g^{-1})^T$ is the matrix of the coadjoint map $\mathrm{Ad}^\sharp_g$.

VIII Wigner functions for the the two-dimensional affine group

In this section we apply the theory presented above to the important particular case of the affine group of the line, G_{aff}, consisting of all transformations of the form $x \mapsto ax + b$, $x \in \mathbb{R}$, with $a > 0$, $b \in \mathbb{R}$. A group element is thus given by a pair $(a, b) \in \mathbb{R}^+_* \times \mathbb{R}$. (Note that $\mathbb{R}^+_*$ denotes the positive real line without the origin.) Group multiplication replicates matrix multiplication when we represent group elements by the matrices

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}. \qquad (74)$$

Wigner functions for this group have been obtained earlier in [7], using different methods. Our analysis reproduces the same results.

VIII.1 Affine algebra and group matrices

The Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ of G_{aff} is generated by the two elements

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad (75)$$

so that the one-parameter subgroups of G_{aff} are

$$e^{(\log a)X_1} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \qquad e^{bX_2} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \qquad (76)$$

Thus, for a general element in the Lie algebra

$$X = x^1 X_1 + x^2 X_2 = \begin{pmatrix} x^1 & x^2 \\ 0 & 0 \end{pmatrix}, \qquad (77)$$

the group element obtained from the exponential map is

$$g = e^X = \begin{pmatrix} e^{x^1} & \frac{x^2}{x^1}(e^{x^1} - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}. \qquad (78)$$

From here follows the inverse map from the group to the algebra,

$$X = \log g = x^1 X_1 + x^2 X_2, \qquad x^1 = \log a, \quad x^2 = \frac{b \log a}{a-1}. \tag{79}$$

Since every $X \in \mathfrak{g}_{\text{aff}}$ is mapped to an element $g \in G_{\text{aff}}$ by the exponential map (78), we identify its domain $\widehat{N}_0$ with the full real plane and use $\vec{x} = (x^1, x^2) \in \mathbb{R}^2$ as the coordinates for the elements of the Lie algebra.

VIII.2 Haar measures

The left- and right-invariant measures on G_{aff} are easily computed in the polar coordinates (78),

$$d\mu(g) = d\mu_\ell(g) = \frac{da\, db}{a^2} = \frac{1 - e^{-x^1}}{x^1}\, dx^1\, dx^2, \tag{80}$$

$$d\mu_r(g) = \frac{da\, db}{a} = \frac{e^{x^1} - 1}{x^1}\, dx^1\, dx^2, \tag{81}$$

and the modular function is

$$\Delta(g) = \frac{1}{a}. \tag{82}$$

Writing as in (46), we find

$$\frac{da\, db}{a^2} = m(\vec{x})\, d\vec{x}, \qquad m(x_1, x_2) = \frac{1 - e^{-x^1}}{x^1}. \tag{83}$$

VIII.3 Adjoint and coadjoint action

The adjoint action of the group G_{aff} on an element (77) of the Lie algebra is easily computed to be

$$\mathrm{Ad}_g\, X = gXg^{-1} = \begin{pmatrix} x^1 & -bx^1 + ax^2 \\ 0 & 0 \end{pmatrix}. \tag{84}$$

The matrix of this transformation which acts on (column) vectors $\vec{x} = (x_1, x_2)^\top \in \mathbb{R}^2$ is

$$M(g) = \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix}. \tag{85}$$

On the dual of the Lie algebra, $X^* \in \mathfrak{g}^*$ has coordinates $\vec{\xi} = (\xi_1, \xi_2)^\top$; on this column vector, the coadjoint action is represented by the inverse transpose matrix,

$$M^\sharp(g) = M(g^{-1})^\top = \begin{pmatrix} 1 & ba^{-1} \\ 0 & a^{-1} \end{pmatrix}. \tag{86}$$

The determinants of these matrices are

$$\|\,\mathrm{Ad}_g\,\| = a = \|\,\mathrm{Ad}_g^\sharp\,\|^{-1}. \tag{87}$$

The coadjoint representation of this group in Eq. (25), $V^\sharp$, is carried by the Hilbert space $L^2(\mathbb{R}^2, d\vec{\xi}\,)$ and has the form

$$(V^\sharp(g)\widehat{F})(\vec{\xi}\,) = a^{\frac{1}{2}}\ \widehat{F}(M(g)^\top\vec{\xi}\,) = a^{\frac{1}{2}}\ \widehat{F}(\xi_1 - b\xi_2, a\xi_2). \tag{88}$$

VIII.4 Coadjoint orbits of the affine group

The coadjoint orbits of the affine group are found from the action of the matrices (86) on fixed vectors $\xi \in \mathbb{R}^2$. The following orbit structure emerges:

1. The orbit obtained by acting with the matrices $M(g^{-1})^\top$ on the column vector $(0,1)^\top$,

$$\mathcal{O}^*_+ = \{\vec{\xi}_+ = (\xi_1, \xi_2) \in \mathbb{R}^2 | \xi_2 > 0\} = \mathbb{R} \times \mathbb{R}^+_*. \tag{89}$$

2. The orbit obtained by acting on $(0,-1)^\top$ with the same matrices,

$$\mathcal{O}^*_- = \{\vec{\xi}_- = (\xi_1, \xi_2) \in \mathbb{R}^2 | \xi_2 < 0\} = \mathbb{R} \times \mathbb{R}^-_*. \tag{90}$$

3. Applying the matrices to the column vector $(\alpha, 0)^\top$, for each $\alpha \in \mathbb{R}$ we obtain an orbit that consists of the single point $(\alpha, 0)$; this we denote by $\mathcal{O}^*_\alpha$.

We may thus characterize the foliation of the dual of the Lie algebra $\mathfrak{g}^*_{\text{aff}}$, by the set $J = \{+, -, \mathbb{R}\}$. This we identify with the real plane,

$$\mathbb{R}^2 = \bigcup_{\lambda \in J} \mathcal{O}^*_\lambda. \tag{91}$$

Note that in this classification, the last set of orbits $\mathcal{O}^*_\alpha$ are a set of Lebesgue measure zero in $\mathbb{R}^2$, while the other two orbits $\mathcal{O}^*_\pm$ are open sets in $\mathbb{R}^2$ and their union is dense. Under the coadjoint action, the latter two orbits carry the invariant measures

$$d\Omega_\pm(\vec{\xi}\,) = \frac{d\xi_1\ d\xi_2}{(2\pi)^{\frac{1}{2}}\ |\xi_2|}, \qquad \vec{\xi} \in \mathcal{O}^*_\pm. \tag{92}$$

Comparing with (19), we now define a measure $d\kappa(\lambda)$ on the Borel sets of the set $J = \{+, -, \mathbb{R}\}$ as follows:

$$\kappa(\{\pm\}) = 1, \qquad \kappa(\{E\}) = 0, \tag{93}$$

for any open set $E \subset \mathbb{R}$. Thus the direct integral Hilbert space $\widetilde{\mathfrak{H}}$ of Eq. (29) for the covariant coadjoint representation is now just an orthogonal sum,

$$\widetilde{\mathfrak{H}} = \mathfrak{H}_+ \oplus \mathfrak{H}_-, \quad \text{where} \quad \mathfrak{H}_\pm = L^2(\mathcal{O}^*_\pm, d\Omega_\pm). \tag{94}$$

Elements in $\widetilde{\mathfrak{H}}$ consist of pairs of functions, $\widehat{F} = (\widehat{F}_+, \widehat{F}_-)$, with $\widehat{F}_\pm \in \mathfrak{H}_\pm$.

VIII.5 Covariant coadjoint representation

The covariant coadjoint representation of $G_{\rm aff}$ is carried by $\widetilde{\mathfrak{H}}$; as we defined it in Subsect. IV.3, it has the form

$$(U^{\sharp}(g)\widehat{F})(\vec{\xi}\,) = \widehat{F}(M(g)^{\top}\vec{\xi}\,) = \widehat{F}(\xi_1 - b\xi_2, a\xi_2), \quad \widehat{F} \in \widetilde{\mathfrak{H}}, \tag{95}$$

where ξ_1 is the *translation* parameter in the affine group while ξ_2 is the *scale* parameter.

In order to compute the explicit form of the Wigner function for the affine group, it is necessary to use its unitary irreducible representations; there are only two such representations. To examine them, consider the representation $U(g)$ on the Hilbert space $L^2(\mathbb{R}, dt)$,

$$(U(g)\phi)(t) = a^{-\frac{1}{2}}\, \phi\Big(\frac{t-b}{a}\Big), \quad \phi \in L^2(\mathbb{R}, dt), \quad g = (a,b) \in G_{\rm aff}. \tag{96}$$

This representation is unitary but not irreducible. To isolate its irreducible components we go over to the Fourier-transformed Hilbert space $L^2(\mathbb{R}, d\omega)$, where the representation is

$$(\widehat{U}(g)\widehat{\phi})(\omega) = a^{\frac{1}{2}}\, \widehat{\phi}(a\omega)e^{-ib\omega}, \quad \widehat{\phi} \in L^2(\mathbb{R}, d\omega), \quad g = (a,b) \in G_{\rm aff}. \tag{97}$$

It is now clear that each of the two subspaces of functions defined on the intervals $(0,\infty)$ and $(-\infty, 0)$,

$$\mathfrak{H}^{\pm} = L^2(\mathbb{R}^{\pm}, d\omega), \tag{98}$$

are stable under the action of the $\widehat{U}(g)$, and in fact are irreducible subspaces under this action. We shall denote the restrictions of $\widehat{U}$ to these two subspaces by $\widehat{U}^{\pm}$, respectively. The two subrepresentations are then inequivalent, but both are square-integrable in the sense of Section V.

VIII.6 Wigner functions for $\widehat{U}(g)^{+}$

We shall now derive Wigner functions for the unitary irreducible representation $\widehat{U}^{+}$; analogous results hold in an obvious way for the representation $\widehat{U}^{-}$ as well.

A vector $\widehat{\eta} \in \mathfrak{H}^{+}$ is admissible in the sense of (37), if and only if it satisfies the condition (see, *e.g.*, [10]),

$$\int_0^{\infty} \frac{2\pi}{\omega} |\widehat{\eta}(\omega)|^2 \, d\omega < \infty. \tag{99}$$

This means that $\widehat{\eta}$ must lie in the domain of the positive unbounded operator C, whose action is

$$(C\widehat{\eta})(\omega) = \left[\frac{2\pi}{\omega}\right]^{\frac{1}{2}} \widehat{\eta}(\omega), \qquad \omega \geq 0. \tag{100}$$

Using (92), we identify the density σ appearing in the Wigner function (70) to be

$$\sigma(\vec{\xi}\,) = |\xi_2|, \qquad \vec{\xi} \in \mathcal{O}_+^* \cup \mathcal{O}_-^*. \tag{101}$$

Let $\widehat{\psi} \in L^2(\mathbb{R}^+, d\omega)$ be any vector in the domain of the operator C^{-1} (*i.e.*, $\widehat{\psi} = C\widehat{\eta}$ for some admissible vector $\widehat{\eta}$), and let $\widehat{\phi}$ be an arbitrary element in $L^2(\mathbb{R}^+, d\omega)$. Then, combining (70) with (83), (97), (99) and (101), after some computation we obtain

$$W(\widehat{\psi}, \widehat{\phi} \mid \xi_1, \xi_2) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \overline{\widehat{\psi}\left(\frac{\xi_2 e^{\frac{x}{2}}}{\operatorname{sinch}\frac{x}{2}}\right)}\, \frac{\xi_2\, e^{-i\xi_1 x}}{\operatorname{sinch}\frac{x}{2}}\, \widehat{\phi}\left(\frac{\xi_2 e^{-\frac{x}{2}}}{\operatorname{sinch}\frac{x}{2}}\right)\, dx, \tag{102}$$

which is valid for all $\xi_2 > 0$, and where

$$\operatorname{sinch}(u) = (\sinh u)/u.$$

The above Wigner function was obtained using the *irreducible* representation $\widehat{U}^+$, and it lead to Wigner functions which are supported on the orbit $\mathcal{O}_+^*$. Had we used $\widehat{U}^-$ we would have obtained an analogous function supported on $\mathcal{O}_-^*$. When we use the *reducible* representation $\widehat{U} = \widehat{U}^+ \oplus \widehat{U}^-$ given in (97), for arbitrary $\widehat{\phi} \in L^2(\mathbb{R}, d\omega)$ and $\widehat{\psi} \in L^2(\mathbb{R}, d\omega)$ satisfying

$$\int_{-\infty}^{\infty} \frac{|\omega|}{2\pi} |\widehat{\psi}(\omega)|^2\, d\omega < \infty, \tag{103}$$

we find the Wigner function

$$W(\widehat{\psi}, \widehat{\phi} \mid \xi_1, \xi_2) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \overline{\widehat{\psi}\left(\frac{\xi_2 e^{\frac{x}{2}}}{\operatorname{sinch}\frac{x}{2}}\right)}\, \frac{|\xi_2|\, e^{-i\xi_1 x}}{\operatorname{sinch}\frac{x}{2}}\, \widehat{\phi}\left(\frac{\xi_2 e^{-\frac{x}{2}}}{\operatorname{sinch}\frac{x}{2}}\right)\, dx, \tag{104}$$

which is now valid for all $\vec{\xi} \in \mathbb{R}^2$.

VIII.7 Affine covariance

It is easily verified that the Wigner function (102) satisfies the correct covariance condition (7)–(73)–(86),

$$W(\widehat{U}(g)\widehat{\psi},\ \widehat{U}(g)\widehat{\phi} \mid \vec{\xi}\,) = W(\widehat{\psi}, \widehat{\phi} \mid \xi_1 - b\xi_2,\ a\xi_2). \tag{105}$$

Comparing with (57) and (95), we see that the Wigner map intertwines the representation $\mathbf{U}_b(g)\rho = \widehat{U}(g)\rho\widehat{U}(g)^*$ with the covariant coadjoint representation $U^\sharp(g)$ in (95). The overlap condition (72) is also straightforward to verify.

VIII.8 Marginality relations of the affine Wigner function

Although the Wigner functions obtained above are defined on all of $\mathbb{R}^2$, they should be regarded as functions defined on the orbits $\mathcal{O}^*_+ \cup \mathcal{O}^*_-$. This is because we would like to think of the Wigner functions as phase space distributions, and these orbits have the structure of symplectic manifolds with invariant measures, and hence are classical phase spaces. The interpretation of the Wigner function as a distribution over the phase space $\mathcal{O}^*_+ \cup \mathcal{O}^*_-$ is further supported by the following observations.

Consider the affine Wigner function (104) of one wavefunction, $W(\widehat{\psi} \mid \vec{\xi}\,) = W(\widehat{\psi}, \widehat{\psi} \mid \vec{\xi}\,)$. Integrating this over the coadjoint orbits with respect to the invariant phase space measure (92), we obtain the full projection to a positive number,

$$\begin{aligned}\int_{\mathcal{O}^*_+ \cup \mathcal{O}^*_-} W(\widehat{\psi} \mid \vec{\xi}\,)\, d\Omega_{\pm}(\vec{\xi}\,) &= \int_{\mathcal{O}^*_+ \cup \mathcal{O}^*_-} W(\widehat{\psi} \mid \vec{\xi}\,)\, \frac{d\xi_1\, d\xi_2}{(2\pi)^{\frac{1}{2}}\, |\xi_2|} \\ &= \int_{-\infty}^{\infty} |\widehat{\psi}(\omega)|^2\, d\omega = \|\widehat{\psi}\|. \qquad (106)\end{aligned}$$

For an arbitrary density matrix ρ, the result is

$$\int_{\mathcal{O}^*_+ \cup \mathcal{O}^*_-} W(\rho \mid \vec{\xi}\,)\, d\Omega_{\pm}(\vec{\xi}\,) = \operatorname{Tr} \rho. \qquad (107)$$

Therefore, though the Wigner function, even for a pure state, is not in general positive, its phase space integral has the proper measurement-theoretic interpretation as the squared norm of the state.

The well-known projection or marginality properties satisfied by the original Wigner function discussed in Section I, cannot be expected to hold in the affine case. We do have however, a similar relation when we project (integrate) over the translation parameter ξ_1 of the affine group, to find the marginal distribution over the scale parameter ξ_1, namely

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} W(\widehat{\psi} \mid \vec{\xi}\,)\, \frac{d\xi_1}{|\xi_2|} = |\widehat{\psi}(\xi_2)|^2. \qquad (108)$$

In the scale parameter ξ_2 however, the marginality relation has a more complicated form. Indeed, a straightforward manipulation of integrals leads to the relation

$$\int_{-\infty}^{\infty} W(\widehat{\psi} \mid \vec{\xi}\,)\, \frac{d\xi_2}{|\xi_2|} = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\xi_1 x}\, \overline{\widehat{\psi}(\omega e^{\frac{x}{2}})}\, \widehat{\psi}(\omega e^{-\frac{x}{2}})\, dx\, d\omega. \qquad (109)$$

On the other hand, the choice of phase space coordinates (ξ_1, ξ_2), which we have adopted here, is not the only possible one and a different choice could lead to a simpler form for this marginality condition. Unfortunately, there does not seem to exist an obvious "natural" choice of coordinates for general phase spaces.

IX Wavelet transform and Wigner function in the affine group

It is worthwhile to display in detail the relationship between the Wigner functions of the affine group and the wavelet transform, since both the wavelet transform and the Wigner function are used extensively in image reconstruction computations. As pointed out in the general setting in Section VI, Eqs. (67) and (68), there is an intimate connection between the two.

IX.1 Coherent states of the affine group

Consider the (doubly reducible) representation of the affine group given in (97). A *mother wavelet* is any vector (a signal) $\widehat{\eta}$ in the carrier Hilbert space $L^2(\mathbb{R}, d\omega)$ of the representation, which satisfies the admissibility condition [10] of Eqs. (99) and (99),

$$\int_{-\infty}^{\infty} \frac{2\pi}{|\omega|} |\widehat{\eta}(\omega)|^2 \, d\omega < \infty. \tag{110}$$

This implies in particular that $\widehat{\eta}(\omega)$ must vanish at the origin. Now choose a particular mother wavelet, normalized by (37) so that

$$c(\widehat{\eta}) = \int_0^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} db \, |\langle \widehat{U}(a,b)\widehat{\eta}|\widehat{\eta}\rangle|^2 = 1. \tag{111}$$

Using this mother wavelet we define a family of *wavelets*, or equivalently *coherent states of the affine group*,

$$\widehat{\eta}_{a,b} = \widehat{U}(a,b)\widehat{\eta}, \qquad (a,b) \in G_{\text{aff}}. \tag{112}$$

In view of (96), these are simply functions in the inverse Fourier domain which are scaled and translated versions of the mother wavelet, and with the same normalization (111). The resolution of the identity (65) on the Hilbert space $L^2(\mathbb{R}, d\omega)$ now assumes the form

$$\int_0^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} db \, |\widehat{\eta}_{a,b}\rangle\langle\widehat{\eta}_{a,b}| = I. \tag{113}$$

IX.2 Wigner-wavelet relations

Consider an arbitrary signal $\widehat{\phi} \in L^2(\mathbb{R}, d\omega)$ and its *wavelet transform* in the translation and scale parameters a, b of the wavelet family,

$$f_{\widehat{\eta},\widehat{\phi}}(a,b) = \langle \widehat{\eta}_{a,b}|\widehat{\phi}\rangle. \tag{114}$$

Next, for the chosen mother wavelet $\widehat{\eta}$, note that the function

$$(C\widehat{\eta})(\omega) = \frac{2\pi}{|\omega|^{\frac{1}{2}}} \, \widehat{\eta}(\omega), \qquad \omega \in \mathbb{R}, \tag{115}$$

is admissible and hence vanishes at the origin $\omega = 0$. Then, specializing the integral in (67) to the affine group and using (78), we find the relation between the wavelet transform and the Wigner function given by

$$W(C\widehat{\eta},\ \widehat{\phi} \mid \vec{\xi}\,) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\vec{x}\cdot\vec{\xi}}\ f_{\widehat{\eta},\widehat{\phi}}\Big(e^{x^1},\ x^2 e^{\frac{x^1}{2}}\,\mathrm{sinch}\,\tfrac{x^1}{2}\Big)\ \left(\frac{|\xi_2|}{e^{\frac{x^1}{2}}\,\mathrm{sinch}\,\frac{x^1}{2}}\right)^{\frac{1}{2}}\ d\vec{x}. \tag{116}$$

In this expression the choice of the mother wavelet $\widehat{\eta}$ is fixed, and the equation refers to $\widehat{\phi}$ only. It is a routine matter now to invert this relation and to write the wavelet transform in terms of the Wigner function,

$$f_{\widehat{\eta},\widehat{\phi}}\Big(e^{x^1},\ x^2 e^{\frac{x^1}{2}}\,\mathrm{sinch}\,\tfrac{x^1}{2}\Big) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\vec{x}\cdot\vec{\xi}}\ W(C\widehat{\eta},\ \widehat{\phi} \mid \vec{\xi}\,)\Big(e^{\frac{x^1}{2}}\,\mathrm{sinch}\,\tfrac{x^1}{2}\Big)^{\frac{1}{2}}\frac{d\xi_1\,d\xi_2}{|\xi_2|^{\frac{1}{2}}} \tag{117}$$

X The standard Wigner function revisited

We finally go back to the well-known Wigner function in (1) at the beginning of this paper and see how it fits into the same theoretical considerations. (It will actually be necessary to do a somewhat different constuction in this case, since the representation in question will not be square integrable with respect to the entire group.) As mentioned in the Introduction, the Wigner function has its origin in the Heisenberg-Weyl group G_{HW} (of the canonical commutation relations). This group is the central extension of the abelian group of $\mathbb{R}^2$ and is topologically isomorphic to $\mathbb{R}\times\mathbb{R}^2$. Denoting a generic element in G_{HW} by $g = (\theta, \xi, \eta)$, the multiplication rule is,

$$(\theta_1, \xi_1, \eta_1)\ (\theta_2, \xi_2, \eta_2) = (\theta_1 + \theta_2 + \tfrac{1}{2}\,[\eta_1\xi_2 - \eta_2\xi_1],\ \xi_1 + \xi_2,\ \eta_1 + \eta_2). \tag{118}$$

The corresponding Lie algebra $\mathfrak{g}_{\mathrm{HW}}$ is generated by the three elements X_0, X_1, X_2, satisfying the Lie bracket relations $[X_1, X_2] = X_0$ and $[X_i, X_0] = 0,\quad i = 1, 2$. The central element X_0 generates the *phase subgroup* Θ, consisting of group elements of the type $(\theta, 0, 0)$. We shall actually use the three elements X_0, X_1 and $-X_2$ as basis vectors for the Lie algebra, and write its general element as $X = x^0X_0 + x^1X_1 - x^2X_2$. From the relation

$$g_0 g g_0^{-1} = (\theta + \eta_0\xi - \eta\xi_0,\ \xi,\ \eta), \tag{119}$$

we readily derive the matrices of the adjoint and coadjoint actions in this basis. They are

$$M(\theta,\xi,\eta) = \begin{pmatrix} 1 & \eta & \xi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M^\sharp(\theta,\xi,\eta) = \begin{pmatrix} 1 & 0 & 0 \\ -\eta & 1 & 0 \\ -\xi & 0 & 1 \end{pmatrix}, \tag{120}$$

respectively. The coadjoint action on a vector $\vec{\gamma} = (\gamma_0, \gamma_1, \gamma_2)^T$ is,

$$M^\sharp(\theta, \xi, \eta)\vec{\gamma} = \begin{pmatrix} \gamma_0 \\ \gamma_1 - \gamma_0\eta \\ \gamma_2 - \gamma_0\xi \end{pmatrix}. \tag{121}$$

From this we see that the coadjoint orbits of G_{HW} are of the following types:

1. The planes
$$\mathcal{O}^*_\lambda = \{(\lambda, \vec{x})^T \mid \vec{x} \in \mathbb{R}^2\}, \tag{122}$$
one for each $\lambda \neq 0$ and generated from the vector $(\lambda, 0, 0)^T$.

2. The singletons
$$\mathcal{O}^*_{\vec{\lambda}} = \{(0, \vec{\lambda})^T\}, \tag{123}$$
one for each $\vec{\lambda} \in \mathbb{R}^2$ and generated from the vector $(0, \vec{\lambda})^T$.

The invariant measures on the orbits $\mathcal{O}^*_\lambda$ are simply the Lebesgue measures $d\vec{x}$ on the planes.

Corresponding to each one of the non-degenerate orbits $\mathcal{O}^*_\lambda$, there is a unitary irreducible representation U^λ of G_{HW} carried by the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}, dx)$:

$$(U^\lambda(\theta, \xi, \eta)\phi)(x) = e^{i\lambda\theta}\ e^{i\lambda\eta(x-\frac{\xi}{2})}\ \phi(x - \xi). \tag{124}$$

Since we may also write,

$$U^\lambda(\theta, \xi, \eta) = U^\lambda(e^X) = e^{i\lambda(\theta I + \eta Q - \xi P)}, \qquad X = x^0X_0 + x^1X_1 - x^2X_2, \tag{125}$$

with

$$(Q\phi)(x) = x\phi(x) \quad \text{and} \quad (P\phi)(x) = -\frac{i}{\lambda}\frac{\partial\phi(x)}{\partial x}, \tag{126}$$

the Hilbert space generators corresponding to X_0, X_1 and X_2 are seen to be I, Q and P, respectively, with the further identification, $x^0 = -\theta$, $x^1 = -\eta$ and $x^2 = -\xi$. Let us consider the case $\lambda = 1$ (equivalent to setting $\hbar = 1$) and simply write U for the corresponding representation. Also, we shall write $U(0, \xi, \eta) = U(\xi, \eta)$. This representation is not square-integrable with respect to the whole group G_{HW}, but only with respect to the homogeneous space $G_{\mathrm{HW}}/\Theta \simeq \mathcal{O}^*_\lambda \simeq \mathbb{R}^2$ [2, 4]. However, it is possible to adapt the theory of square integrable representations, as outlined in Section V, to this situation [4, 14]. Basically, we work with the operators $U(\xi, \eta)$, which give a *multiplier representation* of $\mathbb{R}^2$ and which admit the following orthogonality relations:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \overline{\langle U(\xi, \eta)\phi_1|\psi_1\rangle}\langle U(\xi, \eta)\phi_2|\psi_2\rangle\ d\xi\, d\eta = \overline{\langle\phi_1|\phi_2\rangle}\langle\psi_1|\psi_2\rangle, \tag{127}$$

for arbitrary vectors ϕ_1, ϕ_2, ψ_1 and ψ_2 in the Hilbert space. The operator C in (38) is in this case $(2\pi)^{\frac{1}{2}}\ I$. Since the phase subgroup Θ has now been factored out, the

Wigner transform has to be defined using the remaining two generators Q and P and will be a function on the coadjoint orbit $\mathcal{O}^*_{\lambda=1} \simeq \mathbb{R}^2$. (These considerations will be made more rigorous in a subsequent publication, where we intend to deal with Wigner functions obtained from group representations which are square integrable only with respect to a homogeneous space. It will turn out that we shall have to extend the general theory to include certain types of *reducible* square integrable representations.)

It is also clear that both the densities σ and m, appearing in the expression for the Wigner function in (70), are constants in this case and we set them equal to unity. The Wigner function, for arbitrary Hilbert-Schmidt operators ρ, now assumes the form,

$$W(\rho \mid \gamma_1, \gamma_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\gamma_1 \eta - \gamma_2 \xi)} \operatorname{Tr}[e^{-i(Q\eta - P\xi)} \rho] \, d\xi \, d\eta. \tag{128}$$

Taking $\rho = |\psi\rangle\langle\phi|$, for wave functions $\phi, \psi \in \mathfrak{H}$, writing (q, p) for (γ_1, γ_2) and simplifying the resulting expression, we easily obtain

$$W(\phi, \psi \mid q, p) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixp} \, \overline{\phi(q - \tfrac{1}{2}x)} \, \psi(q + \tfrac{1}{2}x) \, dx, \tag{129}$$

which is exactly the same expression as in (1).

XI Conclusion

The object which is crucial to our construction of the Wigner function is a square-integrable group representation [2]. Such a representation belongs to the discrete series of the group, and not every group has a representation in this series. The groups studied thus far, the Heisenberg-Weyl group [27], the Euclidean group [21], and the spin group [5], have these representations, are unimodular, and enjoy several other simplifying properties, such having global polar coordinates.

The affine group is the simplest example where one of these properties — unimodularity— is transcended. We have refined the definitions of the Wigner operator and function given in the previous literature so that the affine case is included cogently, and we have compared the results on wavelets richly contained in the literature. We have found that indeed there is a close relation between the Wigner function and the wavelet transform: they are essentially Fourier transforms of each other. This has been noted before in the case of the Heisenberg-Weyl group, where the Wigner and the radar Woodward ambiguity functions [28] are also Fourier transforms [23]. The important advantage of the Wigner function (52) is that it is defined on a coadjoint orbit. This ensures its interpretation as a (quasi-)distribution on a phase space.

The fact that coherent states which satisfy a resolution of the identity of the type (65) are also associated to square integrable representations, gives the link between generalized wavelet transforms and generalized Wigner functions. In a following publication we shall indicate how the construction of this paper can be extended to certain other types of representations that are not square-integrable, such as the continuous series of Sp(2,$\mathbb{R}$) representations.

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References

[1] G.S. Agarwal, Entropy, the Wigner distribution function and the approach to equilibrium of systems of coupled harmonic oscillators, *Phys. Rev.* **A3** (1971) 828–831.

[2] S.T. Ali, J-P. Antoine and J-P. Gazeau, Square integrability of group representations on homogeneous spaces. I. Reproducing triples and frames, *Ann. Inst. H. Poincaré* **55** (1991) 829–855; II. Coherent and quasi-coherent states. The case of the Poincaré group, *Ann. Inst. H. Poincaré* **55** (1991) 857-890.

[3] S.T. Ali, J-P. Antoine, J-P. Gazeau and U.A. Mueller, Coherent states and their generalizations: A mathematical overview, *Reviews Math. Phys*. **7** (1995) 1013–1104; also, S.T. Ali, J.-P. Antoine and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations*, (Springer-Verlag, New York, 2000).

[4] S.T. Ali, A general theorem on square-integrability: Vector coherent states, *J. Math. Phys.* **39** (1998) 3954-3964.

[5] N.M. Atakishiyev, S.M. Chumakov, and K.B. Wolf, Wigner distribution function for finite systems, *J. Math. Phys.* **139** (1998) 6244–6261.

[6] M.J. Bastiaans, The Wigner distribution function applied to optical signals and systems, *Opt. Comm.* **25** (1978) 26–30; M.J. Bastiaans, Wigner distribution functions and its application to first-order optics, *J. Opt. Soc. Am.* **69** (1979) 1710–1716.

[7] J. Bertrand and P. Bertrand, Représentations temps-fréquence des signaux, *C.R. Acad. Sc. Paris* **299**, Série I (1984) 635-638; A class of Wigner functions with extended covariance properties, *J. Math. Phys.* **33** (1992) 2515–2527.

[8] J. Bertrand and P. Bertrand, Microwave imaging of time-varying radar targets, *Inverse Problems*, **13** (1997) 621–645.

[9] S.M. Chumakov, A. Frank and K.B. Wolf, Finite Kerr medium: macroscopic quantum superposition states and Wigner functions on the sphere, *Physical Review* **A60** (1999) 1817–1823.

[10] I. Daubechies, *Ten Lectures on Wavelets*, (SIAM, Philadelphia, 1992).

[11] H. Führ and M. Mayer, Continuous wavelet transforms from cyclic representations: A general approach using Plancherel measure, Univ. München preprint (July 1998).

[12] G. García-Calderón and M. Moshinsky, Wigner distribution functions and the representation of canonical transformations in quantum mechanics, *J. Phys. A* **13** (1980) L185–L188.

[13] A. Grossmann, J. Morlet and T. Paul, Integral transforms associated to square integrable representations. I. General results, *J. Math. Phys* **26** (1985) 2473–2479.

[14] D.M. Healy Jr. and F.E. Schroeck Jr., On informational completeness of covariant localization observables and Wigner coefficients, *J. Math. Phys.* **36** (1995) 453–507; Franklin E. Schroeck, Jr., *Quantum Mechanics on Phase Space* (Kluwer Academic Publ., Boston, 1996).

[15] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, (Academic Press, Inc, San Diego, Ca, 1978).

[16] M. Hillery, R.F. O'Connel, M.O. Scully, and E.P. Wigner, Distribution functions in physics: fundamentals, *Phys. Rep.* **106** (1984) 121–167; Y.S. Kim and M.E. Noz, *Phase Space Picture of Quantum Mechanics*, (World Scientific, Singapore, 1991); H.-W. Lee, Theory and application of the quantum phase-space distribution functions, *Phys. Rep.* **259** (1995) 147–211; Vladimir I. Man'ko, Conventional quantum mechanics without wave function and density matrix. In: *Proceedings of the 1998 Latin American School of Physics* (American Institute of Physics, New York, 1999).

[17] A.A. Kirillov, *Elements of the Theory of Representations*, (Springer-Verlag, Berlin, 1976).

[18] A. Lohmann, The Wigner function and its optical production, *Opt. Comm.* **42** (1980) 32–37.

[19] S. Mancini, V.I. Man'ko, and P. Tombesi, Reconstructing the density operator by using generalized field quadratures, *Quantum Semiclass. Opt.* **7** (1995) 615; S. Mancini, V.I. Man'ko and P. Tombesi, *Phys. Lett. A* **213** (1996) 1–6; S. Mancini, V.I. Man'ko, and P. Tombesi, *Found. Phys.* **27** (1997) 81.

[20] J.E. Moyal, Quantum mechanics as a statistical theory, *Proc. Cambridge Phil. Soc.* **45** (1949) 99–124.

[21] L.M. Nieto, N.M. Atakishiyev, S.M. Chumakov and K.B. Wolf, Wigner distribution function for Euclidean systems, *J. Phys. A* **31** (1998) 3875–3895.

[22] A. Perelomov, *Generalized Coherent States and their Applications*, (Springer-Verlag, Heidelberg, 1986).

[23] W. Schempp, Analog radar signal design and digital signal processing – a Heisenberg nilpotent Lie group approach. In *Lie Methods in Optics*. Ed. by J. Sánchez-Mondragón and K.B. Wolf. Springer Lecture Notes in Physics, Vol. 250 (Springer Verlag, Heidelberg, 1986); W. Schempp, *Harmonic Analysis on the Heisenberg Nilpotent Lie Group, with Applications to Signal Theory*, Pitman Research Notes in Mathematics, Vol. 147 (Longman Scientific and Technical, London, 1986); W. Schempp, *Magnetic Resonance Imaging: Mathematical Foundations and Applications*, (John wiley & Sons, New York, 1997).

[24] Remark by R. Simon (Institute of Mathematical Science, Madras, India).

[25] E. Wigner, On the quantum correction for thermodynamic equilibrium, *Phys. Rev.* **40** (1932) 749–759.

[26] K.B. Wolf, The Heisenberg–Weyl ring in quantum mechanics. In *Group Theory and its Applications, III*, Ed. by E.M. Loebl (Academic Press, New York, 1975), pp. 189–247.

[27] K.B. Wolf, Wigner distribution function for paraxial polychromatic optics, *Opt. Comm.* **132** (1996) 343–352.

[28] P.M. Woodward, *Probability and Information Theory, with Applications to Radar*, (Artech House, Dedham MA, 1980).

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